Dynamic Linear Economies with Social Interactions*

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Abstract

Social interactions arguably provide a rationale for several important socio-economic phenomena, from smoking and other risky behavior in teens to peer effects in school performance. We study *social interactions* in *dynamic* economies. For these economies, we provide existence (Markov Perfect Equilibrium in pure strategies), ergodicity, and welfare results. We characterize several equilibrium properties of policy functions, spatial correlations, and social multiplier effects. Most importantly, we study formally the issue of the identification of social interactions, emphasizing the restrictions imposed by dynamic equilibrium conditions with respect to economies populated by myopic agents and economies in which spatial correlation is induced by selection. In this respect we find that identification can be obtained but only in non-stationary economies.

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1 Introduction

Agents interact in markets as well as socially—that is, in the various socioeconomic groups they belong to. Models of social interactions are designed to capture in a simple abstract way socioeconomic environments in which markets do not mediate all of agents' choices. In such environments agents' choices are determined by their preferences as well as by their interactions with others—that is, their positions in a predetermined network of relationships, e.g., a family, a peer group, or more generally any socioeconomic group.

Social interactions arguably provide a rationale for several important phenomena, Peer effects, in particular, have been indicated as one of the main empirical determinants of risky behavior in adolescents. Relatedly, peer effects have been studied in connection with education outcomes, obesity, friendship and sex, labor market referrals, neighborhood and employment segregation, criminal activity, and several other socioeconomic phenomena.[1]

The large majority of the existing models of social interactions are static; or, when dynamic models of social interactions are studied, it is typically assumed that agents are myopic and their choices are subject to particular behavioral assumptions.[2] In this paper, we contribute to this literature by studying social interactions in *dynamic* economies. In most applications of interest, in fact, social interactions are affected or constrained by relevant state variables. Indeed, peer effects act differently on individuals in different (non freely-reversible) states: belonging to a social group whose members are actively engaging in criminal activities influences differently agents with/out previous criminal experience; social links with female peers with an active job market occupation has a different effect on young women whose mother did/did not work in regards to their decision to enter/not enter the labor market; and so on. Furthermore, several forms of risky behavior among adolescents induced by social interactions involve substance abuse and hence (the fundamentally dynamic) issues of addiction and habits. We shall show that dynamic equilibrium considerations have fundamental effects on the properties of economies with social interactions.

We focus our attention on *linear economies*, in which each agent's preferences display preferences for *conformity*, that is, preferences which incorporate the desire to conform to the choices of agents in a reference group are quadratic.[3] More specifically, in our economy, each agent's

[1]See Brock and Durlauf (2001b), Glaeser and Scheinkman (2001), Glaeser and Scheinkman (2003), Moody (2001) for surveys; see also the *Handbook of Social Economics*, Benhabib, Bisin, and Jackson (2011).

[2]Exceptions include an example on female labor force participation in Glaeser and Scheinkman (2001), Binder and Pesaran (2001) on life-cycle consumption under social Interactions, Blume (2003) on social stigma, Brock and Durlauf (2010) and De Paula (2009) on duration models, Ioannides and Soetevent (2007) on endogenous local and global interactions, where agents best respond to lagged decisions, and the theoretical analysis of Bisin, Horst, and Özgür (2006).

[3]Dynamic linear models of course have appealing analytical properties; Hansen and Sargent (2004) study this class of models systematically, exploiting the tractability of linear control methods. While the class of economies

preferences are hit by random preference shocks over time. Each agent interacts with agents in his social reference group, in the sense that each agent's instantaneous preferences depend on the current choices of agents in his social reference group, as a direct externality. Each agent's instantaneous preferences also depend on the agent's own previous choice, representing the inherent costs to dynamic behavioural changes due e.g., to irreversibility and/or habits. When agents' reference groups overlap, each agent's *optimal* choice depends on the previous choices and current preference shocks of all the other agents in the economy, as long as they are observable. We allow for complete and incomplete information with respect to preference shocks. Requiring that the social and informational structure of each agent satisfy a symmetry condition, we restrict our analysis to *symmetric Markov perfect equilibria.* Agents' choices at equilibrium are determined by *linear* policy (best reply) functions. More specifically, e.g., in infinite-horizon economies, a symmetric Markov perfect equilibrium (MPE) is represented by a symmetric policy function, g, which maps an agent's current choice at time t, linearly in each agent's past choices, x^b_{t-1}, in each agent's contemporaneous idiosyncratic preference shock, θ^b_t, and in the mean preference shock, $\overline{\theta}$:

$$g\left(x_{t-1},\theta_t\right)=\sum_{b\in\mathbb{A}}c^b\,x^b_{t-1}+\sum_{b\in\mathbb{A}}d^b\,\theta^b_t+e\,\overline{\theta}$$

For these economies, we provide some fundamental theoretical results: (Markov perfect) equilibria in pure strategies exist (for finite economies they are unique) and they induce an ergodic stochastic process over the equilibrium configuration of actions. Furthermore, a stationary ergodic distribution exists. We also derive a recursive algorithm to compute equilibria. The proof of the existence theorem, in particular, requires some subtle arguments. In fact, standard variational arguments require bounding the marginal effect of any infinitesimal change dx^a on the agent's value function. But in the class of economies we study, the envelope theorem (as e.g., in Benveniste and Scheinkman (1979)) is not sufficient for this purpose, as dx^a affects agent a's value function directly and indirectly, through its effects on all agents $b\in\mathbb{A}\backslash a$'s choices, which in turn affect agent a's value function. The marginal effect of any infinitesimal change dx^a is then an infinite sum of endogenous terms. In our economy, however, we can exploit the linearity of policy functions to represent a symmetric MPE by a fixed point of a recursive map which can be directly studied.

Exploiting the linear structure of our economies we can study equilibria in some detail, characterizing the parameters of the policy function as well as a fundamental statistical property of equilibrium, the cross-sectional auto-correlation of actions. Based on this, we obtain a series of results regarding the welfare properties of equilibrium and various comparative dynamics exercises of interest. First of all, we show that, since social interactions are modelled in this paper as a preference externality, equilibria will not be efficient in general. We also characterize the form

we study in this paper allows for a countable number of heterogeneous agents and an infinite horizon, giving rise to infinite dimensional systems, some tractability is still maintained.

of the inefficiency: at equilibrium each agent's policy function weights too heavily the agent's own preference shock and previous action and not enough the other agents'. The comparative dynamics exercises illustrate e.g., the equilibrium effects of the strength of social interactions and of the social and informational structure of the economy.

Finally, we exploit our characterization results of the equilibria to address generally the issue of identification of social interactions in our context, with population data. While the empirical literature has often interpreted a significant high correlation of socioeconomic choices across agents, e.g., peers, as evidence of social interactions, in the form, e.g., of preferences for conformity, it is well known at least since the work of Manski (1993) that the empirical study of social interactions is plagued by subtle identification problems. Intuitively, in our economy for instance, the spatial correlation of actions at equilibrium can be due to social interactions or to the spatial correlation of preference shocks. More formally, take two agents, e.g., agent a and agent b. A positive correlation between x_t^a and x_t^b could be due to e.g., preference for conformity. But the positive correlation between x_t^a and x_t^b could also be due to a positive correlation between θ_t^a and θ_t^b. In this last case, preferences for conformity and social interactions would play no role in the correlation of actions at equilibrium. Rather, such correlation would be due to the fact that agents have correlated preferences. Correlated preferences could generally be due to some sort of assortative matching or positive selection, which induce agents with correlated preferences to interact socially. High correlations of substance abuse between adolescent friends, for instance, could be due to social interaction or to friendship relations being selected in terms demographic and psychological characteristics.

In the context of our economy, we ask whether the restrictions implied by the dynamic equilibrium analysis help identify social interactions and distinguish them from correlated preferences. We show that the answer is in fact affirmative, but only if the economy is non-stationary, in a precise sense. While correlated actions could be induced by social interactions and/or by selection into social groups, any significant variation over time of the correlation pattern in the population is bound to be due to endogenous changes in the stregth of social interactions induced by non-stationarity of behavior at equilibrium. Consider once again the issue of peer effects in adolescents' substance use. Suppose that the econometrician observes the behavior of a population of students in a school over time (at different grades). A significant high correlation of socioeconomic choices across students in the school could be due to selection in the endogenous composition of the school in terms of unobserved (to the econometrician) correlated characteristics of the agents. Any significant variation in students' behavior through time (grades) must however be due to social interactions. A student whose choice is affected by the choices of his school peers will in fact rationally anticipate how much longer he will interact with them. In particular, his propensity to conform to his peers' actions will tend to decrease over time (grades) and will be the lowest in the final years in the school. This non-stationarity of each student's behavior at equilibrium is the key to the identification of social interactions in our class of economies.

The simplicity of linear models allows us to extend our analysis in several directions which are important in applications and empirical work. This is the case, for instance of general (including asymmetric) neighborhood network structures for social interactions. But our analysis extends also to general stochastic processes for preference shocks and to the addition of global interactions. One particular form of global interactions occurs when each agent's preferences depend on an average of actions of all other agents in the population, e.g. Brock and Durlauf (2001a), and Glaeser and Scheinkman (2003). This is the case, for instance, if agents have preferences for *social status*. More generally, global interactions could capture preferences to adhere to aggregate norms of behavior, such as specific group cultures, or other externalities as well as price effects. Finally, and perhaps most importantly, we extend our analysis to encompass a richer structure of dynamic dependence of agents' actions at equilibrium. In particular we study an economy in which agents' past behavior is aggregated through an accumulated stock variable which carries habit persistence, which can be directly applied e.g., to the issue of teenage substance addiction due to peer pressure at school. With respect to the addiction literature, as e.g., Becker and Murphy (1988), we model the dynamics of addiction considering peer effects not only in a single-person decision problem, but rather in a social equilibrium, allowing for the intertemporal feedback channel between agents across social space and through time.[4] In this context we show that in equilibrium each agent's choice depends on the stock of his neighbors' actions, on their long-term behavioral patterns rather than just on their previous period actions. Also, in non-stationary economies, as the final period approaches, each agent assigns higher weights to his own stock, giving rise to an initiation-addiction behavioral pattern at equilibrium which is consistent with observation, e.g., in Cutler and Glaeser (2007) and DeCicca et al. (2008).

2 Dynamic economies with social interactions

While we develop most of our analysis in the context of linear models, it is useful to set up the general model first, as we do in this section, to be as clear and specific as possible regarding the assumptions we impose on the economy we study.

Time is discrete and is denoted by $t = 1, \ldots, T$. We allow both for infinite economies ($T = \infty$) and economies with an end period ($T < \infty$). A typical economy is populated by a countable set of *agents* $\mathbb{A}$, a generic element of which being denoted by a.[5] Each agent lives for the duration of the economy. At the beginning of each period t, agent a's random preference *type* θ_t^a is drawn

[4] See also Becker et al. (1994), Gul and Pesendorfer (2007), Gruber and Koszegi (2001); see Rozen (2010) for theoretical foundations for intrinsic linear habit formation; see also Elster (1999) and Elster and Skog (1999) for surveys.

[5] We study an economy populated by a countably infinite number of agents where $\mathbb{A} := \mathbb{Z}$, but our analysis applies to economies with a finite number of agents.

from Θ, a compact subset of a finite dimensional Euclidean space $\mathbb{R}^n$. The random variables θ_t^a are independently and identically distributed across time and agents with probability law ν.[6] We assume, with no loss of generality, that the random variable $\theta_t := (\theta_t^a)_{a\in\mathbb{A}}$ is defined, for all t, on the canonical probability space $(\boldsymbol{\Theta}, \mathcal{F}, \mathbb{P})$, where $\boldsymbol{\Theta} := \{(\theta^a)_{a\in\mathbb{A}} : \theta^a \in \Theta\}$. At each period t, agent $a \in \mathbb{A}$ chooses an *action* x_t^a from the set X, a compact subset of a finite dimensional Euclidean space $\mathbb{R}^p$. Let $\mathbf{X} := \{x = (x^a)_{a\in\mathbb{A}} : x^a \in X\}$ be the space of individual action profiles.

Each agent $a \in \mathbb{A}$ *interacts* with agents in the set $N(a)$, a nonempty subset of the set of agents $\mathbb{A}$, which represents agent a's social reference group. The map $\mathbb{A} : N \to 2^{\mathbb{A}}$ is referred to as a *neighbourhood correspondence* and is assumed exogenous. Agent a's instantaneous preferences depend on the current choices of agents in his reference group, $\{x_t^b\}_{b\in N(a)}$, representing social interactions as direct preference externalities. Agent a's instantaneous preferences also depend on the agent's own previous choice, x_{t-1}^a, representing inherent costs to dynamic behavioural changes due e.g., to habits. In summary, agent a's instantaneous preferences at time t are represented by a continuous utility function

$$\left(x_{t-1}^a, x_t^a, \{x_t^b\}_{b\in N(a)}, \theta_t^a\right) \mapsto u\left(x_{t-1}^a, x_t^a, \{x_t^b\}_{b\in N(a)}, \theta_t^a\right)$$

Agents discount expected future utilities using the common stationary discount factor $\beta \in (0, 1)$.

Let $x^{t-1} = (x_0, x_1, \ldots, x_{t-1})$ and $\theta^{t-1} = (\theta_1, \ldots, \theta_{t-1})$ be the choices and type realizations upto period $t-1$, where $x_0 \in \mathbf{X}$ is the initial configuration. Before each agent's time t choice, x^{t-1} is observed by all agents and the current value of the random variable θ_t realizes. Agent $a \in \mathbb{A}$ observes $I_a\theta_t := \{\theta_t^b : b \in I(a)\}$, where $I(a) \subset \mathbb{A}$ is his *information set*. Similarly, let $I_a\theta^{t-1} = (I_a\theta_1, \ldots, I_a\theta_{t-1})$. We call an economy one with *complete information* if $I(a) = \mathbb{A}$, and one with *incomplete information* if $I(a) \subsetneq \mathbb{A}$.[7] After time t choices are made, $x_t = (x_t^b)_{b\in\mathbb{A}} \in \mathbf{X}$ becomes common knowledge and the economy moves to time $t+1$.

A *strategy* for an agent a is a sequence of measurable functions $x^a = (x_t^a)$, where for each t, $x_t^a : \mathbf{X}^t \times \boldsymbol{\Theta}^t \to X$. Agents' strategies along with the probability law for types induce a stochastic process over future configuration paths. Each agent $a \in \mathbb{A}$'s objective is to choose x^a to maximize

$$E\left[\sum_{t=1}^{T} \beta^{t-1} u\left(x_{t-1}^a, x_t^a, \{x_t^b\}_{b\in N(a)}, \theta_t^a\right) \,\Big|\, (x_0, \theta_1)\right] \tag{1}$$

given the strategies of other agents and given $(x_0, \theta_1) \in \mathbf{X} \times \boldsymbol{\Theta}$.

[6] We use the i.i.d. assumption for clarity, to focus our analysis on equilibrium correlations generated solely by endogenous interactions. In Section 6 we dispose of the i.i.d. assumption and ask whether the correlations induced by social interactions can be identified from those induced by correlated shocks. More details about the general case are contained in Section 7.2.

[7] In the current section we focus on economies with complete information, but the whole theoretical analysis, including the existence and ergodicity results, can be extended to incomplete information; see Supplemental Appendix A

We focus our analysis to *symmetric Markov perfect equilibria*. Agents' strategies are *Markovian* if after any $t-1$-period history (x^{t-1}, θ^t), they depend only on the previous period configuration x_{t-1} and the current type realizations θ_t. To that effect, we require that the social structure satisfies the following symmetry restriction:[8] For all $a, b \in \mathbb{A}$, $N(b) = R^{b-a}N(a)$, where R^{b-a} is the canonical *shift* operator in the direction $b-a$.[9]

Because of symmetry, it is enough to analyze the optimization problem relative to a single reference agent, say agent $0 \in \mathbf{A}$. Assume that the optimal choice of any economic agent $b \in \mathbb{A}$ is determined by a continuous choice function $g : \mathbf{X} \times \mathbf{\Theta} \times \{1, \dots, T\} \to X$ such that for all $t = 1, \dots, T$ and after any history $(x^{t-1}, \theta^t) \in \mathbf{X}^t \times \Theta^t$, his t-th period choice is given by

$$x_t^b\,(g)\left(x^{t-1}, \theta^t\right) = g_{T-(t-1)}(R^b\, x_{t-1}, R^b\, \theta_t)$$

The value of the optimization problem of agent a is then given by[10]

$$V_g^T(R^a\, x_0, R^a\, \theta_1) = \max_{(x_t^a)_{t=1}^T} E\left[\sum_{t=1}^T \beta^{t-1} u\left(x_{t-1}^a, x_t^a, \{x_t^b(g)\}_{b \in N(a)}, \theta_t^a\right)\right]$$

The value function associated with this dynamic choice problem can be shown to satisfy Bellman's Principle of Optimality by standard arguments (see e.g., Stokey and Lucas (1989)) and, hence, can be written in the following recursive form,

$$\begin{aligned} V_g^{T-(t-1)}\left(R^a\, x_{t-1}, R^a\, \theta_t\right) &= \max_{x_t^a \in X} E\Bigg[u\left(x_{t-1}^a, x_t^a, \{x_t^b(g)\}_{b \in N(a)}, \theta_t^a\right) \\ &\quad + \beta\, V_g^{T-t}\left(R^a\left(x_t^a, \left\{x_t^b(g)\right\}_{b \neq a}\right), R^a\, \theta_{t+1}\right)\Bigg] \end{aligned} \qquad (2)$$

for $t = 1, \dots, T$ and for all $(x^{t-1}, \theta^t) \in \mathbf{X}^t \times \Theta^t$.[11] We are now ready to define our equilibrium concept.

Definition 1 *A symmetric Markov Perfect Equilibrium of a dynamic economy with social interactions is a measurable map* $g^* : \mathbf{X} \times \mathbf{\Theta} \times \{1, \dots, T\} \to X$ *such that for all* $a \in \mathbb{A}$, *for all*

[8] Under incomplete information, $I(a) \neq \mathbb{A}$, symmetry of the information structure translates to $I(b) = R^{b-a}I(a)$. Heterogeneity can be incorporated into the probabilistic structure of the types θ_t^a. Also, we can allow for heterogeneity of the network structure across agents by augmenting the strategy spaces to incorporate network structure into individual heterogeneity. We do this in Section 7.1.

[9] That is, $c \in N(a)$ if and only if $c + (b-a) \in N(b)$. The operations of addition and subtraction are legitimate given that we typically let $\mathbb{A} := \mathbb{Z}^d$, the d-dimensional integer lattice.

[10] The preference shocks being serially uncorrelated, we do not need to condition on the value of past realizations. See Section 7.2 for a treatment of persistent shocks.

[11] We have adopted the the convention that $V_g^0(x, \theta) := 0$ for any $(x, \theta) \in \mathbf{X} \times \Theta$.

$t = 1, \ldots, T,$ *and for all* $(x^{t-1}, \theta^t) \in \mathbf{X}^t \times \Theta^t$

$$g^*_{T-(t-1)}\left(R^a\, x_{t-1}, R^a\, \theta_t\right) \in \underset{x^a_t \in X}{\operatorname{argmax}}\; E\left[u\left(x^a_{t-1}, x^a_t, \left\{x^b_t(g^*)\right\}_{b \in N(a)}, \theta^a_t\right) + \beta\, V^{T-t}_{g^*}\left(R^a\left(x^a_t, \left\{x^b_t(g^*)\right\}_{b \neq a}\right), R^a\, \theta_{t+1}\right)\right] \tag{3}$$

Clearly, an MPE is necessarily a *subgame perfect equilibrium*; that is, each agent's continuation strategy is a best response to other agent's continuation strategies after any possible history. Notice also the time notation we use for the Markovian policy: $g^*_{T-(t-1)}$ denotes the first-period equilibrium choice in a $T-(t-1)$-period economy. Since economies are nested, $g^*_{T-(t-1)}$ represents also the t-period equilibrium choice in a T-period economy.

We conclude this section with a few remarks to justify our focus on MPEs. First of all, Markovian strategies are not a restriction for finite-horizon economies: we prove that the unique symmetric subgame perfect equilibrium for any finite-horizon economy is necessarily Markovian. Moreover, in an infinite horizon economy ($T = \infty$), a symmetric MPE is not necessarily stationary. The sequence of unique MPEs for finite horizon economies converges however to a $g^* : \mathbf{X} \times \boldsymbol{\Theta} \to X$ which turns out to be a stationary MPE of the infinite-horizon economy whose properties we focus on. Finally, we refer to Bisin, Horst, and Özgür (2006) for a discussion of non-Markovian equilibria in a related context.

3 Dynamic Linear Economies with Social interactions and Conformity Preferences

We focus our attention on *linear economies* with *conformity preferences.* These are environments in which each agent's preferences incorporate the desire to conform to the choices of agents in his reference group.[12] Conformity preferences arguably provide a rationale for several important social phenomena. The role has been empirically documented, for instance, as a motivation for smoking and other risky behaviour in teens, for engaging in criminal activity, and for the effects of peers in education outcomes.

With the objective of providing a clean and simple analysis of dynamic social interactions in a conformity economy, we impose strong(er than required) but natural assumptions.[13] In particular (i) we restrict preferences to be quadratic, so as to restrict ourselves to a linear economy; (ii) we restrict the neighborhood correspondence to represent the minimal interaction structure allowing

[12] While we model preferences for conformity directly as a preference externality, we intend this as a reduced form of models of behavior in groups which induce indirect preferences for conformity, as e.g., Jones (1984), Cole et al. (1992), Bernheim (1994), Peski (2007).

[13] See Section 7 for possible directions in which the structure and the results we obtain are easily generalized.

for overlapping groups; and (*iii*) we impose enough regularity conditions on the agents' choice problem to render it convex. Formally,

Assumption 1 *A linear conformity economy satisfies the following.*

1. *Let* $\mathbb{A} := \mathbb{Z}$ *represent the countable set of agents. Each agent interacts with his immediate neighbors, i.e., for all* $a \in \mathbb{A}$, $N(a) := \{a-1, a+1\}$.

2. *The contemporaneous preferences of an agent* $a \in \mathbb{A}$ *are represented by the utility function*

$$\begin{aligned} u(x_{t-1}^a, x_t^a, x_t^{a-1}, x_t^{a+1}, \theta_t^a) \quad := \quad & -\alpha_1 (x_{t-1}^a - x_t^a)^2 - \alpha_2 (\theta_t^a - x_t^a)^2 \\ & -\alpha_3 (x_t^{a-1} - x_t^a)^2 - \alpha_3 (x_t^{a+1} - x_t^a)^2 \end{aligned} \tag{4}$$

   where $\alpha_1, \alpha_2,$ *and* α_3, *are positive constants.*

3. *Let* $X = \Theta = [\underline{x}, \bar{x}] \subset \mathbb{R}$, *where* $\underline{x} < \bar{x}$. *Let* v *be absolutely continuous with a positive density*[14], $E[\theta_t^a] = \int \theta_t^a d\nu =: \bar{\theta} \in (\underline{x}, \bar{x})$, *and* $Var(\theta_t^a) = \int \left(\theta_t^a - \bar{\theta}\right)^2 d\nu < \infty$.

Assumption 1-1 requires that the reference group of each agent $a \in \mathbb{A}$ be composed of his immediate neighbors in the social space, namely the agents $a-1$ and $a+1$. The utility function u defined in Assumption 1-2 describes the trade-off that agent $a \in \mathbb{A}$ faces between matching his individual characteristics (x_{t-1}^a, θ_t^a) and the utility he receives from conforming to the current choices of his peers (x_t^{a-1}, x_t^{a+1}). The different values of α_i represent different levels of intensity of the social interaction motive relative to the own (or intrinsic) motive. Assumption 1-2 and 1-3 jointly guarantee that the agents' choice problem is convex. Finally, notice that the requirements that $\alpha_1,\ \alpha_2 > 0$ anchor agents' preferences on their own private types and past choices. It is easy to see that, without such anchor, actions are driven only by social interactions and a large multiplicity of equilibria arises.

3.1 Equilibrium

We provide here the basic theoretical results regarding our dynamic linear social interaction economy with conformity. The reader only interested in the characterization can skip this section, keeping in mind that equilibria exist (for finite economies they are unique) and they induce an ergodic stochastic process over paths of action profiles. Furthermore, a stationary ergodic distribution also exists for the economy. Finally, a recursive algorithm to compute equilibria is derived. Unless otherwise mentioned specifically, the proofs of all statements and other results can be found in the Supplemental Appendix.

[14]We will call a measure μ 'absolutely continuous' if it is absolutely continuous with respect to the Lebesgue measure λ, i.e., if $\mu(A) = 0$ for every measurable set A for which $\lambda(A) = 0$. We need absolute continuity only when we prove inefficiency. All other results are obtained without that assumption.

Theorem 1 (Existence) *Consider an economy with conformity preferences and complete information.*[15]

1. *If the time horizon is finite ($T < \infty$), then the economy admits a unique symmetric MPE $g^* : \mathbf{X} \times \mathbf{\Theta} \times \{1, \cdots, T\} \mapsto X$ such that for all $t \in \{1, \ldots, T\}$, for all $(x_{t-1}, \theta_t) \in \mathbf{X} \times \mathbf{\Theta}$*

$$g^*_{T-(t-1)}(x_{t-1},\ \theta_t) = \sum_{a \in \mathbb{A}} c^a_{T-(t-1)}\, x^a_{t-1} + \sum_{a \in \mathbb{A}} d^a_{T-(t-1)}\, \theta^a_t + e_{T-(t-1)}\, \overline{\theta} \qquad \mathbb{P} - a.s.$$

where $c^a_\tau, d^a_\tau, e_\tau \geq 0$, $a \in \mathbb{A}$, and $e_\tau + \sum_{a \in \mathbb{A}}(c^a_\tau + d^a_\tau) = 1$, $0 \leq \tau \leq T$. Moreover, the equilibrium is also unique in the class of subgame perfect equilibria (SPE), meaning that there does not exist any non-Markovian SPE for our economy.

2. *If the time horizon is infinite ($T = \infty$), then the economy admits a (not necessarily unique) symmetric stationary MPE $g^* : \mathbf{X} \times \mathbf{\Theta} \mapsto X$ such that*

$$g^*(x_{t-1},\ \theta_t) = \sum_{a \in \mathbb{A}} c^a\, x^a_{t-1} + \sum_{a \in \mathbb{A}} d^a\, \theta^a_t + e\, \overline{\theta} \qquad \mathbb{P} - a.s.$$

where $c^a, d^a, e \geq 0$, for $a \in \mathbb{A}$, and $e + \sum_{a \in \mathbb{A}}(c^a + d^a) = 1$.[16]

The proof of the existence theorem requires some subtle arguments. While referring to the Appendix for details, a few comments here in this respect will be useful. Consider the (infinite dimensional) choice problem of each agent $a \in \mathbb{A}$. To be able to apply standard variational arguments to this problem it is necessary to bound the marginal effect of any infinitesimal change dx^a on the agent's value function. To this end, the Envelope theorem (as e.g., in Benveniste and Scheinkman (1979)) is not enough, as dx^a affects agent a's value function not only directly, but also indirectly, that is through its effects on all agents $b \in \mathbb{A}\backslash a$'s choices, which in turn affect agent a's value function. The marginal effect of any infinitesimal change dx^a is then an infinite sum, and each term of sum consists in turn of an infinite sum of endogenous marginal effects from all agents $b \in \mathbb{A}\backslash a$'s policy functions.[17]

In our economy, with quadratic utility, policy functions are necessarily linear and, provided we show that equilibria are interior, symmetric MPE's in pure strategies can be represented by a policy function which is obtained as a fixed point of a recursive map which can be directly studied.[18] Extending the existence proof to general preferences would require therefore sufficient

[15]The existence theorem in this section can be extended with straightforward modifications to the case of incomplete information. We refer the reader to Section 7 for the extension.

[16]Several assumptions can be relaxed while guaranteeing existence. In particular, the symmetry of the neighborhood structure can be substantially relaxed. See Section 7.1 for the discussion.

[17]The methodology used by Santos (1991) to prove the smoothness of the policy function in infinite dimensional recursive choice problems also does not apply.

[18]The class of economies we study are theoretically equivalent to a class of stochastic games, with an infinite number of agents, and uncountable state spaces. Ready-to-use results for the existence of "pure strategy" Markov-perfect equilibria for these environments do not exist. For the state of the art in that literature, see Mertens and Parthasarathy (1987) and Duffie et al. (1994). See also Mertens (2002) and Vieille (2002) for surveys.

conditions on the structural parameters to control the curvature of the policy function of each agent's decision problem. We conjecture that this can be done although sufficient conditions do not appear transparently from our proof.

3.2 The parameters of the policy function

By exploiting the linearity of the policy functions, our method of proof is constructive, producing a direct and useful recursive computational characterization for the parameters of the symmetric policy function at equilibrium. We repeatedly exploit this characterization in the next section e.g., when performing comparative dynamics exercises. Consider the choice problem of agent 0. For any T-period economy, agent 0's dynamic program yields a FOC that takes the following form (see Lemma 3 in the Supplemental Appendix).

$$x_1^0 = \left(\frac{1}{\Delta_T}\right)\left(\alpha_1\, x_0^0 + \alpha_2\, \theta_1^0 + \sum_{a\neq 0} \gamma_T^b\, x_1^b + \mu_T\, \bar{\theta}\right) \tag{5}$$

with $\Delta_T := \alpha_1 + \alpha_2 + \sum_{a\neq 0} \gamma_T^b + \mu_T$, where the coefficients Δ_T, γ_T^b, and μ_T are the effects on agent zero's discounted expected marginal utility of changes in agents 0 and b's first period actions and the change in the level of $\bar{\theta}$, respectively.

Let L_T be a map induced by (5) s.t. $(\hat{c}, \hat{d}, \hat{e}) = L_T(c, d, e)$, by matching coefficients of the policy on both sides of (5), i.e., for each $a \in \mathbb{A}$

$$\begin{aligned} \hat{c}^a &= \Delta_T^{-1}\left(\alpha_1 \mathbf{1}_{\{a=0\}} + \textstyle\sum_{b\neq 0} \gamma_T^b\, c^{a-b}\right) \\ \hat{d}^a &= \Delta_T^{-1}\left(\alpha_2 \mathbf{1}_{\{a=0\}} + \textstyle\sum_{b\neq 0} \gamma_T^b\, d^{a-b}\right) \\ \hat{e} &= \Delta_T^{-1}\left(\mu_T + e\textstyle\sum_{b\neq 0} \gamma_T^b\right) \end{aligned} \tag{6}$$

Let $L_{c,d,e} := \{(c, d, e) : e \geq 0, c^a \geq 0, d^a \geq 0, \forall a \text{ and } e + \sum_a (c^a + d^a) = 1\}$ be the space of nonnegative coefficient sequences summing to 1. The existence of an equilibrium policy for the first period of a T-period economy is then equivalent to the existence of a coefficient sequence (c_T^*, d_T^*, e_T^*) which is the fixed point of the map $L_T : L_{c,d,e} \to L_{c,d,e}$ induced by (5).

The parameters of the map L_T, namely $\Delta_T, \left(\gamma_T^b\right)_{a\neq 0}, \mu_T$, depend only on the continuation equilibrium coefficients $(c_s^*, d_s^*, e_s^*)_{s=1}^{T-1}$ in a linear fashion (see (A.11), (A.13), and (A.15) for their detailed expressions).

For $T = 1$, the parameters of L_1 are dictated directly by the underlying preferences, namely $\Delta_1 = \alpha_1 + \alpha_2 + 2\alpha_3$, $\gamma_1^1 = \gamma_1^{-1} = \alpha_3$, $\gamma_1^b = 0$, for all $b \neq -1, 0, 1$, and $\mu_1 = 0$. Thus, the map L_1 defined by the system in (6) becomes a contraction mapping whose unique fixed point is computed as the unique root to a second-order difference equation that satisfies transversality conditions toward both infinities. Consequently, the equilibrium policy coefficients are computed as in the next Theorem.

Theorem 2 (Recursive algorithm) *Consider a finite-horizon T-period economy with conformity preferences ($\alpha_i > 0$, $i = 1, 2, 3$) and complete information.*

(i) The map L_1 for a one-period economy, defined in (6), admits a unique fixed point. We compute the (exponential) coefficient sequence in closed-form. For any $a \in \mathbb{A}$,

$$c_1^{*a} = r_1^{|a|} \left(\frac{\alpha_1}{\alpha_1 + \alpha_2} \right) \left(\frac{1 - r_1}{1 + r_1} \right) \quad \text{and} \quad d_1^{*a} = r_1^{|a|} \left(\frac{\alpha_2}{\alpha_1 + \alpha_2} \right) \left(\frac{1 - r_1}{1 + r_1} \right) \tag{7}$$

where $r_1 = \left(\frac{\Delta_1}{2\,\alpha_3} \right) - \sqrt{\left(\frac{\Delta_1}{2\,\alpha_3} \right)^2 - 1}$ and $\Delta_1 = \alpha_1 + \alpha_2 + 2\,\alpha_3$.

(ii) The coefficients $(c_s^*, d_s^*, e_s^*)_{s=2}^T$ are computed recursively as the unique fixed points of the contraction maps $L_s : L_{c,d,e} \to L_{c,d,e}$, $s = 2, \ldots, T$, defined in (6), whose parameters $\Delta_s, (\gamma_s^a)_{a \neq 0}, \mu_s$ depend linearly only on the continuation coefficients $(c_\tau^*, d_\tau^*, e_\tau^*)_{\tau=1}^{s-1}$, as defined in (A.11), (A.13), and (A.15).

(iii) Moreover, $\lim_{T \to \infty} (c_T^*, d_T^*, e_T^*) = (c^*, d^*, e^*)$ exists and it is the coefficient sequence of a stationary Markovian equilibrium policy function for the infinite-horizon economy.

Fixed point calculations take less than a few seconds on an ordinary computer, for each period. Finally, the sequence of fixed point maps that we compute at each iteration converges to a policy sequence, which turns out to be an infinite-horizon stationary MPE. The convergence is very rapid, under a few minutes.

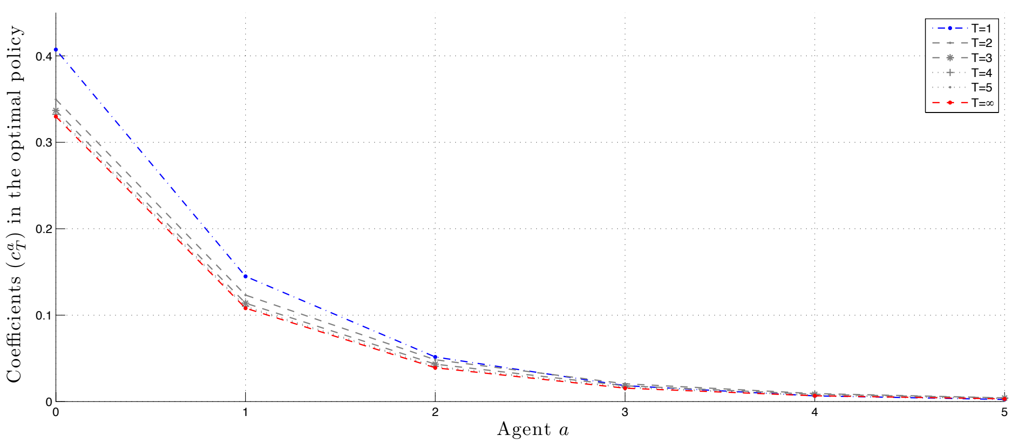

Figure 1: Non-stationary Optimal Policy.

Social interaction economies are usually plagued with multiple equilibria and all previous existence results in the literature require some form of '*Moderate Social Influence*' assumption, which roughly means that the effect on marginal utility of a change in individual's own choice ($\Delta_T = \alpha_1 + \alpha_2 + \sum_{a\neq 0} \gamma_T^b + \mu_T$ here) should dominate the sum of the effects on marginal utility of changes in peers' choices ($\sum_{a\neq 0} \gamma_T^b$ here) (see e.g. Glaeser and Scheinkman (2003)). In fact, as long as either one of the parameters α_1 or α_2 is positive (which is true for both by Assumption 1), this is the case ($\Delta_T > \sum_{a\neq 0} \gamma_T^b$) for the economies we study. This is why, in contrast to other models in the literature, no matter how large the interaction parameter α_3 is relative to the others, the equilibrium stays unique for finite-horizon economies.

3.3 Ergodicity

Given the characterization of the parameters of the policy function at hand, we are also able to determine the long-run behavior of the equilibrium process emerging from the class of dynamic models we study. To that end, let an infinite-horizon economy with conformity preferences be given and let g^* be a symmetric stationary MPE for that economy (recall that Theorem 1 does not guarantee that a unique such g^* exists). Let π_0 be an initial distribution on the space of action profiles $\mathbf{X}$. Given π_0, the stationary MPE g^* induces an equilibrium process $(x_t \in \mathbf{X})_{t=0}^{\infty}$ and an associated transition function Q_{g^*}. This latter generates iteratively a sequence of distributions $(\pi_t)_{t=1}^{\infty}$ on the configuration space $\mathbf{X}$, i.e., for $t = 0, 1, \ldots$

$$\pi_{t+1}(A) = \pi_t Q_{g^*}(A) = \int_{\mathbf{X}} Q_{g^*}(x_t, A)\ \pi_t(dx_{t+1})$$

We show first that, given the induced equilibrium process, the transition function Q_{g^*} admits an *invariant distribution* π, i.e., $\pi = \pi\, Q_{g^*}$ and that the equilibrium process starting from π is *ergodic*.[19]

Furthermore, we show that, for any initial distribution π_0 and a stationary Markovian policy function g^*, the equilibrium process $(x_t \in \mathbf{X})_{t=0}^{\infty}$ converges in distribution to the invariant distribution π, independently of π_0. This also implies that π is the *unique* invariant distribution of the equilibrium process $(x_t \in \mathbf{X})_{t=0}^{\infty}$. More specifically,

Theorem 3 (Ergodicity) *Suppose the process $\left((\theta_t^a)_{t=-\infty}^{\infty}\right)_{a\in\mathbb{A}}$ is i.i.d. with respect to a and t according to ν. The equilibrium process $(x_t \in \mathbf{X})_{t=0}^{\infty}$ induced by a symmetric stationary Markov perfect equilibrium of an economy with conformity via the policy function $g^*(x_{t-1},\ \theta_t)$ and the unique invariant measure π as the initial distribution is ergodic; π is the joint distribution of*

$$x_t = \left(\frac{e\,\overline{\theta}}{1-C} + \sum_{s=1}^{\infty} \sum_{b_1\in\mathbb{A}} \cdots \sum_{b_s\in\mathbb{A}} c^{b_1}\cdots c^{b_{s-1}}\, d^{b_s}\, \theta_{t+1-s}^{a+b_1+\cdots+b_s} \right)_{a\in\mathbb{A}} \tag{8}$$

[19] We call a Markov process (x_t) with state space $\mathbf{X}$ under a probability measure P ergodic if $\frac{1}{T}\sum_{t=1}^{T} f(x_t) \to \int f dP$ P-almost surely for every bounded measurable function $f : \mathbf{X} \to \mathbb{R}$. See, e.g., Duffie et al. (1994) for a similar usage.

where $C := \sum_{a\in\mathbb{A}} c^a$ is the sum of coefficients in the stationary policy function that multiply corresponding agents' last period choices. Moreover, the sequence $(\pi_t)_{t=1}^{\infty}$ of distributions generated by the equilibrium process $(x_t \in \mathbf{X})_{t=0}^{\infty}$ converges to π in the topology of weak convergence for probability measures, independently of any arbitrary initial distribution π_0.[20]

4 Characterization of equilibrium

Exploiting the linear structure of our economies we can study equilibria in some detail.

First of all, we study the *parameters of the policy function.* Furthermore, we study a fundamental statistical property of equilibrium, *cross-sectional auto-correlation of actions.* In fact, although any agent $a \in \mathbb{A}$ interacts directly only with a small subset of the population, at equilibrium, each agent's optimal choice is correlated with those of all the other agents.

4.1 Policy Function

The coefficients $c^b_{T-(t-1)}$ and $d^b_{T-(t-1)}$ (resp. c^b and d^b in the case of infinite-horizon economies) may be viewed as a measure for the total impact of the action x^{a+b}_{t-1} and of the preference shock θ^{a+b}_t of agent $a+b$, respectively, on the optimal current choice of agent a; where b abstractly and concisely represents the *social distance* between the two agents.

Consider first a finite-horizon economy. Since the policy function for this economy is well-defined, the coefficients $c^b_{T-(t-1)}$ and $d^b_{T-(t-1)}$ satisfy

$$\lim_{|b|\to\infty} c^{a+b}_{T-(t-1)} = \lim_{|b|\to\infty} d^{a+b}_{T-(t-1)} = 0$$

The impact of an agent $a+b$ on agent a tends to zero as $|b| \to \infty$. In this sense, linear conformity economies display weak social interactions. Furthermore, equilibrium policy functions are non-stationary in the finite economy, as rational forward-looking agents change their behavior optimally through time. In the final periods, for example, social interactions lose weight relative to individual characteristics; see Figure 1.[21]

Finally, as we have shown in Section 3.2,

$$\lim_{T\to\infty} c_T = c, \ \lim_{T\to\infty} d_T = d, \ \text{ and } \lim_{T\to\infty} e_T = e$$

The finite-horizon parameters converge (uniformly) to the infinite-horizon stationary policy parameters.

[20] A sequence of probability measures (λ_t) is said to converge weakly (or in the topology of weak convergence for probability measures) to λ if, for any bounded, measurable, continuous function $f : \mathbf{X} \to \mathbb{R}$, $\lim_{t\to\infty} \int f\, d\lambda_t = \int f d\lambda$ almost surely (see e.g. Kallenberg (2002), p.65).

[21] We plot in Figure 1 only one side of the policy coefficient sequence to get a close-up view of the change in equilibrium behavior. The left hand side is the mirror image of that due to symmetry. Parameter values for this figure are $\frac{\alpha_1}{\alpha_3} = 1$, $\frac{\alpha_3}{\alpha_2} = 10$, and $\beta = .95$

4.2 Cross-sectional Auto-correlations

Let $\rho_{a,T}$ denote the conditional correlation between the first-period equilibrium actions of agents a-step away from each other, in the T-period economy, given $x_0 \in \mathbf{X}$:[22]

$$\rho_{a,T} \quad = \quad \frac{Cov\left(x_1^0, x_1^a \,\middle|\, x_0\right)}{Var\left(x_1^0 \,\middle|\, x_0\right)}. \tag{9}$$

Exploiting the equilibrium characterization provided by Theorems 1 and 2, and the independence of preference shocks across agents, we can compute conditional covariances at equilibrium:

$$Cov\left(x_1^0, x_1^a \,\middle|\, x_0\right) = Var(\theta) \sum_{a_1 \in \mathbb{A}} d_T^{a_1}\, d_T^{a_1 - a}. \tag{10}$$

The expression $\sum_{a_1 \in \mathbb{A}} d_T^{a_1}\, d_T^{a_1-a}$ is the discrete self-convolution of the equilibrium policy sequence $d_T = \left(d_T^{a_1}\right)_{a_1 \in \mathbb{A}}$, where a acts as the shift parameter.[23] In Figure 2 we show how the convolution behaves with respect to the distance a, for the same set of parameters as in Figure 1. Substituting the form in (10) back in (9) for both terms, we obtain

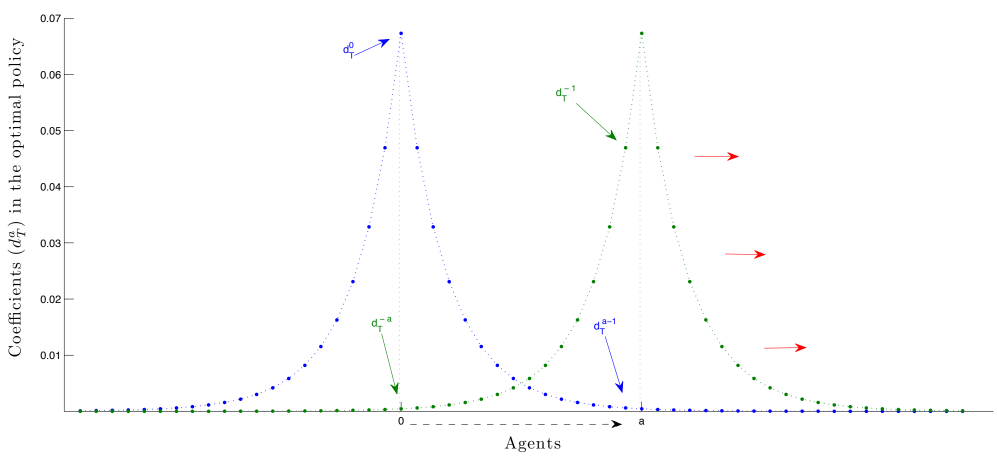

Figure 2: Convolution of the Policy Coefficient Sequence.

[22]The correlation between the first-period optimal choices of agents a and b, is $\frac{Cov\left(x_1^a, x_1^b \,\middle|\, x_0\right)}{\sqrt{Var\left(x_1^a \,\middle|\, x_0\right) Var\left(x_1^b \,\middle|\, x_0\right)}}$. Due to the symmetry imposed on our economy, such correlations are independent of agents' labels but depends only on $|b-a|$. Consequently, we can define the conditional correlation function with distances computed relative to any agent, in particular agent 0. $\rho_{a,T}$ also denotes the conditional correlation between any period $\tau+1$ equilibrium actions of agents a-step away from each other, in a $T+\tau$-period economy, given $x_\tau \in \mathbf{X}$.

[23]See (D.1) for the derivation.

$$\rho_{a,T} = \frac{\sum_{b\in\mathbb{A}} d_T^b \, d_T^{b-a}}{\sum_{b\in\mathbb{A}} d_T^b \, d_T^b} \tag{11}$$

Exploiting the recursive algorithm provided by Theorem 2, we can compute these autocorrelations easily for any finite economy.

We proceed with a characterization of the shape of the conditional correlation function $\rho_{a,T}$ over time T and across social space a. It represents a summary of the implications of equilibrium in our dynamic economy; and we shall exploit it when studying identification in Section 6.

First of all, $\rho_{a,T}$ is declining in a, for any T; see Figure 4 for an example with the same parametrization we used above for the policy weights in Figure 1. In an infinite-horizon economy, the limit unconditional correlation ρ_b between the actions of agents a and $a+b$ is independent of x_0.[24] In Figure 3, we report the correlation functions in both the mild and strong conformity parameterizations as a function of social distance, b.[25] Two effects are worth mentioning here. Firstly, both correlation functions converge to zero as the distance between two agents become arbitrarily large. Secondly, this convergence is much faster in the case of mild interactions than in the case of strong interactions. For example, the correlation between the equilibrium choices of agent a and agent $a+3$ (or $a-3$ due to symmetry) is about 7% in the case of mild interactions whereas it is about 75% in the case of strong interactions. The correlation between the equilibrium choices of agent a and agent $a+6$ are about 0% and 40% respectively. The strength of the desire to conform built in individuals' preferences determine endogenously, at equilibrium, the size of the effective neighborhood with which an individual interacts.

Importantly, while $\rho_{a,T}$ declining in a, for any T, the rate of decline cannot be ranked in T, given a; this also can be seen from Figure 4.

In particular, given a T-period economy, consider the *T-period rate of convergence of the spatial autocorrelations*, for $a \geq 0$,

$$r_{a,T} = \frac{\rho_{a+1,T}}{\rho_{a,T}}.$$

We show analytically that $r_{a,1}$ declines monotonically and becomes constant at the tail in a.[26] On the other hand, $r_{a,T}$ is typically non-monotonic in a, for longer horizons, including for $T = \infty$; see Figure 5. This is the case because each agent's policy function results from the composition of two distinct effects, the cross-sectional interaction and the dynamic effect of actions today in the future. The first effect vanishes exponentially with distance at equilibrium, while the rate at which the dynamic effect vanishes is not constant. In fact, the sum of the two effects becomes hyperbolic in distance, i.e., it decreases very sharply when you go beyond an agent's neighborhood

[24] Since the stationary MPE is ergodic, from Lemma 2 (i) and Theorem 3 imply that as t gets arbitrarily large, the conditional t-period covariance between agents 0 and a converges to its unconditional counterpart at the limit distribution.

[25] See Section 5.2 for the parameter values for the mild and strong interaction cases.

[26] See the proof of Proposition 1 for the argument.

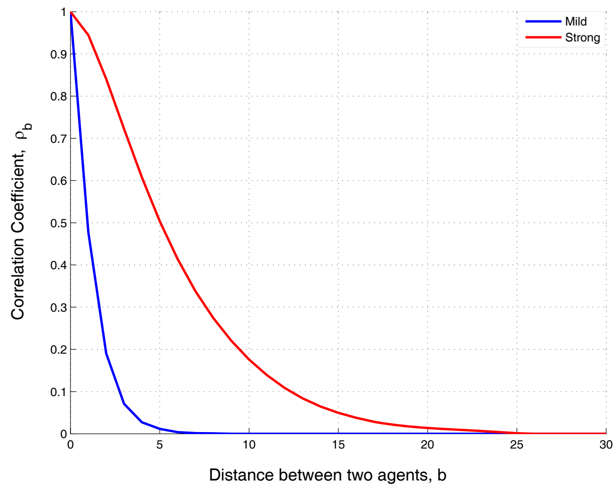

Figure 3: Correlation function at the ergodic distribution for Mild and Strong Interactions.

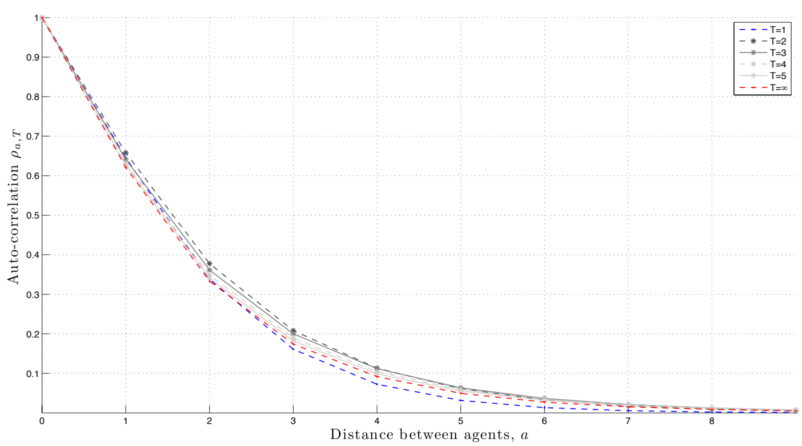

Figure 4: Cross-sectional Auto-correlations.

and then stabilizes with social distance. The non-monotonicity of the rate of convergence $r_{a,T}$ is the result of the self-convolution of an hyperbolic sequence of policy coefficients (see Equation (10) and Figure 2), which enters in the computation of the covariance. In the case of static economies (or equivalently, of economies in their last period), the dynamic effect is not present, the sequence of policy coefficients is therefore exponential (rather than hyperbolic), and the rate of convergence $r_{a,1}$ is monotonically decreasing.

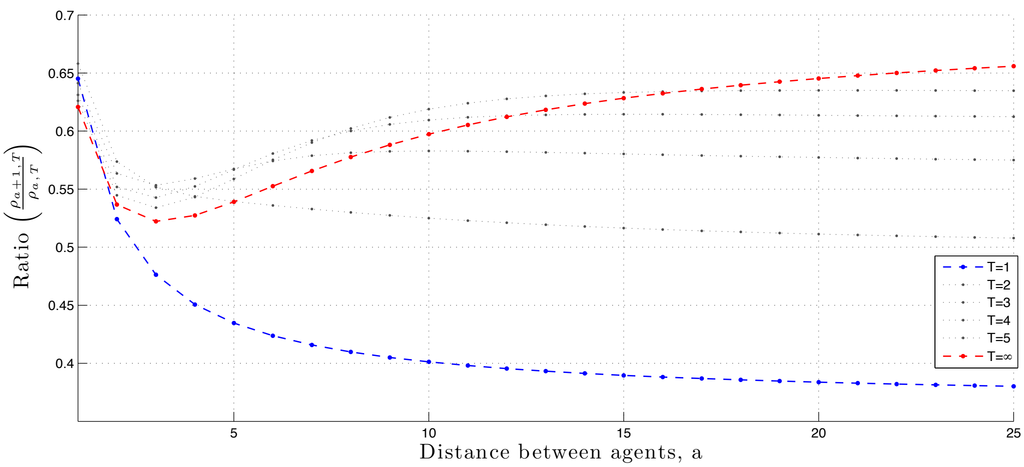

Figure 5: Rate of Convergence of the Auto-correlations.

Finally, consider the *T-period rate of tail convergence of the spatial autocorrelations*,

$$r_T := \lim_{a\to\infty} r_{a,T} = \lim_{a\to\infty}\left(\frac{\rho_{a+1,T}}{\rho_{a,T}}\right)$$

Similarly, let the same rate for the infinite-horizon economy ($T=\infty$) be represented by r.

Proposition 1 (Tail Convergence Monotonicity) *The*[27] *rate r_T is monotone increasing with respect to the length of the economy,*

$$r_{T+1} > r_T, \text{ for finite } T \geq 1.$$

Moreover, the sequence of tail convergence rate for finite-horizon economies converges to that of the infinite-horizon economy as the horizon length gets larger and the limit rate is strictly less than 1:

$$\lim_{T\to\infty} r_T = r < 1.$$

[27]The proof is in Supplemental Appendix D.

In other words, even though the autocorrelation functions might behave non-monotonically for shorter social distances, they eventually converge (as social distance $a \to \infty$) to an exponential rate in the tail. Moreover, rates of tail convergence are higher the farther is the final period of the economy (as $T \to \infty$). This is because rational agents choose to correlate their actions more with their neighbors in early periods and progressively less so as they approach the end of their social interactions. Finally, as the infinite-horizon limit is approached, the rate of tail convergence becomes stationary (as to be expected since finite-horizon equilibria approximate the stationary infinite-horizon equilibrium). We use this intuition to the fullest extent when discussing identification in Section 6.

5 Equilibrium Properties and Comparative dynamics

In this section we first study the welfare properties of equilibrium and then we use the characterization of equilibria we obtained to produce several simulations illustrating various comparative dynamics exercises of interest.

5.1 (In)efficiency

Social interactions are modelled in this paper as a preference externality, that is, by introducing a dependence of agent a's preferences on his/her peers' actions. Not surprisingly, therefore, equilibria will not be efficient in general. In this section we also characterize the form the inefficiency takes when social interactions are modelled as preferences for conformity.

A benevolent social planner, taking into account the preference externalities and at the same time treating all agents symmetrically, maximizes the expected discounted utility of a generic agent, say of agent $a \in \mathbb{A}$, by choosing a symmetric choice function h in $CB(\mathbf{X} \times \mathbf{\Theta}, X)$, the space of bounded, continuous, and X-valued measurable functions. The choice of h induces, in a recursive way, a sequence of choices for any agent $b \in \mathbb{A}$, given (x_0, θ_1), by

$$x_t^b\,(h)\left(x^{t-1}, \theta^t\right) = h_{T-(t-1)}(R^b\, x_{t-1}, R^b\, I_0\, \theta_t), \qquad \text{for } t = 1, \ldots, T. \tag{12}$$

Definition 2 (Recursive Planning Problem) *Let a T-period linear economy with social interactions and conformity preferences be given. Let π_0 be an absolutely continuous distribution on the initial choice profiles with a positive density. A symmetric Markovian choice function $g : \mathbf{X} \times \mathbf{\Theta}^{I(0)} \times \{1, \ldots, T\} \to X$ is said to be* ***efficient*** *if it is a solution, for all $a \in \mathbb{A}$, and for* *all* $t = 1, \ldots, T,$ *to*[28]

$$\max_{\{h \in CB(\mathbf{X}\times\mathbf{\Theta},X)\}} \int \sum_{t=1}^{T} \beta^{t-1} \Bigg(-\alpha_1 \left(x_{t-1}^{a}(h)\left(x^{t-1},\theta^{t}\right) - x_{t}^{a}(h)\left(x^{t-1},\theta^{t}\right)\right)^2$$
$$-\alpha_2 \left(\theta_t^a - x_t^a(h)\left(x^{t-1},\theta^t\right)\right)^2$$
$$-\alpha_3 \left(x_t^{a-1}(h)\left(x^{t-1},\theta^t\right) - x_t^a(h)\left(x^{t-1},\theta^t\right)\right)^2$$
$$-\alpha_3 \left(x_t^{a+1}(h)\left(x^{t-1},\theta^t\right) - x_t^a(h)\left(x^{t-1},\theta^t\right)\right)^2 \Bigg) \prod_{t=1}^{T} \mathbb{P}(d\theta_t)\,\pi_0(dx_0)$$

given π_0.[29]

As noted, preferences for conformity introduce an externality in each agent $a \in \mathbb{A}$'s decision problem, which depends directly on the actions of agents in neighbourhood $N(a)$ and, indirectly, on the actions of all agents in the economy. In equilibrium, agents do not internalize the impact of their choices on other agents today and in the future. More precisely,

Theorem 4 (Inefficiency of equilibrium) *A symmetric MPE of a conformity economy is inefficient.*

Furthermore, an efficient policy function will tend to weight less heavily the agent's own-effect and more heavily other agents' effects, relative to the equilibrium policy. This effect, hence the inefficiency, are neatly exhibited by comparing the equations determining policy weights in the planner (E.3) and equilibrium (A.2) scenarios. The (absolute value of the) weights the planner's equation associates on neighbors' choices is twice as large as the weights associated to neighbors in the equilibrium equation $\left(\frac{\alpha_3}{\alpha_1+\alpha_2+2\alpha_3}\right)$. As a consequence, the relative weights that the planner assigns to neighbors' choices are always higher than the ones that each agent uses in equilibrium.[30]

A graphic representation of the inefficiency is obtained in Figure 6, which presents the coefficient plot for the equilibrium policy of a one-period economy (equivalently the final period of any finite-horizon economy): c_{eqbm} (blue dots), and for the planner's solution, c_{planner} (red dots), respectively, for a given agent $a \in \mathbb{A}$, and for a given set of parameter values ($\frac{\alpha_1}{\alpha_2} = \frac{\alpha_2}{\alpha_3} = 1$, and $\beta = .95$).[31]

5.2 Comparative Dynamics: Peer Effects

The strength of the agents' preferences for conformity depends on the size of the preference parameter α_3 relatively to α_1 and α_2. A policy function is represented in Figure 7, which compares

[28] We adopt the convention that $h_{T-(t-2)}(R^a\, x_{t-2}, R^a\, I_0\, \theta_{t-1}) = x_0^a$ when $t = 1$.

[29] This problem can be written recursively; see Supplemental Appendix E.

[30] Normalizing the relative coefficients to form a probability measure (see the argument in the proof of Lemma 2 (*iv*)), we have that the measure obtained from the planner's policy is a *mean-preserving spread* of the measure obtained at equilibrium.

[31] We call this parametrization the *mild-interaction* case in Section 5.2.

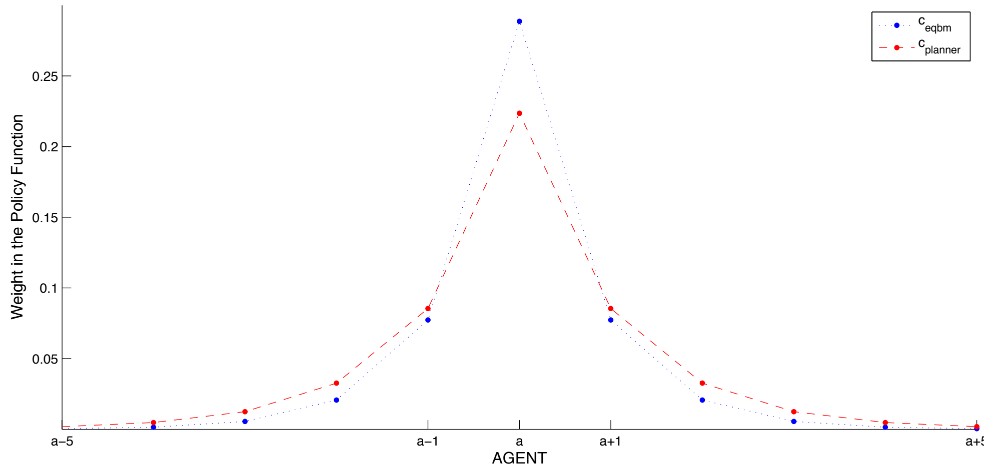

Figure 6: Inefficiency of equilibrium.

a case with *mild* preferences for conformity (with parametrization $\frac{\alpha_1}{\alpha_2} = \frac{\alpha_2}{\alpha_3} = 1$)[32] with one with *strong* preferences for conformity (with parametrization $\frac{\alpha_1}{\alpha_2} = 1$, $\frac{\alpha_2}{\alpha_3} = \frac{1}{20}$). On the x-axis, we plot agent a and his neighbors, while on the y-axis, we plot the weights $(c^b)_{b\in\mathbb{A}}$ that the symmetric policy function g associates with the last period actions of agents $(a+b)_{b\in\mathbb{A}}$. While each agent's interaction neighborhood is only composed of two agents, in effect local interactions involve indirectly larger groups. How large are the groups depends endogenously on the strength of the agents' preferences for conformity. Notice e.g., that in Figure 7, local interactions involve effectively a group of about ten neighbors when preferences for conformity are mild and involve a group of about thirty neighbors when preferences for conformity are strong. Furthermore, for the same cases of mild and strong conformity, we compare in Figure 8 the case in which neighborhoods are overlapping, $N(a) = \{a-1, a+1\}$, with the case of non-overlapping one-sided neighborhoods, $N(a) = \{a+1\}$.[33] Two effects are present here. Firstly, as in Figure 7, an increase in the strength of the interaction parameter spreads the interaction effects over a larger social geography. Secondly, this spread is observed most significantly in the case of non-overlapping neighborhoods due to the uni-directional nature of the interactions.

At the ergodic stationary distribution, when the dependence of the agents' actions in equilibrium are independent of the initial configuration of actions x_0, such correlations in endogenously

[32]The discount rate is fixed at $\beta = .95$ in all the simulations unless mentioned otherwise.

[33]In this case, the policy function is

$$x_t^a = g(R^a x_{t-1}, R^a \theta_t) = \sum_{b\geq 0} c^b\, x_{t-1}^{a+b} + \sum_{b\geq 0} d^b\, \theta_t^{a+b} + e\,\overline{\theta}.$$

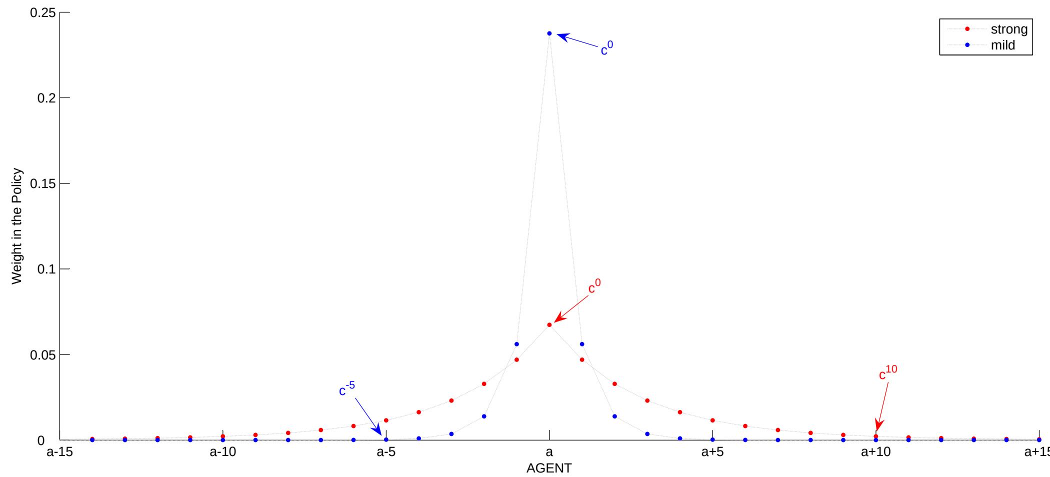

Figure 7: Weights on past history in the stationary policy function.

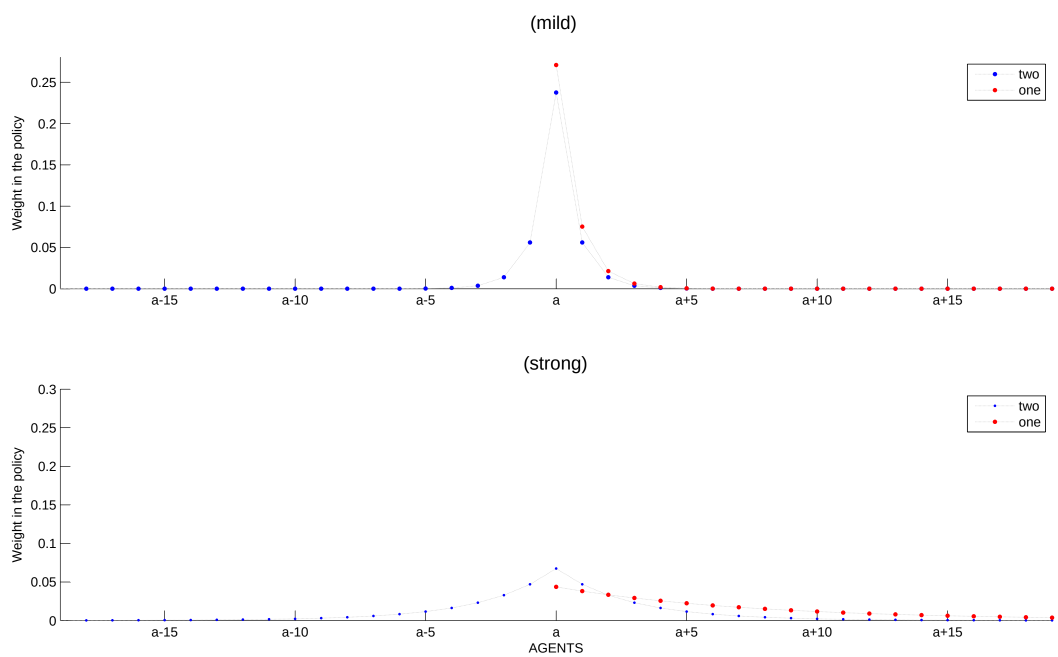

Figure 8: One-sided vs. two-sided interactions.

formed groups is manifested in a phenomenon which we refer to as *local norms of behavior* (see Figure 9).[34] In Figure 9, we plot 100 neighboring agents on the x-axis and their optimal choices drawn from the limit distribution at the same future date, on the y-axis. In the top panel, clearly the optimal actions are more spread and do not follow a significant pattern. In the bottom panel though, the optimal choices are more concentrated and follow a clear path. This is due to the fact that, in equilibrium agents conform to the actions of neighboring agents, leading the way to the creation of similar local behavior. In the bottom panel of Figure 9, we observe groups of agents (e.g., in the neighborhood of agent 20) choosing relatively low actions, and other groups (e.g., in the neighborhood of agent 70) choosing instead high actions. Two interesting aspects of this phenomenon are firstly that every individual uses the same symmetric policy function to make his choices and all heterogeneity is captured by random types and we still have high spatial correlation and high spatial variation. Secondly, the initial configuration of actions is irrelevant since the limit distribution of individual actions in this economy is ergodic.

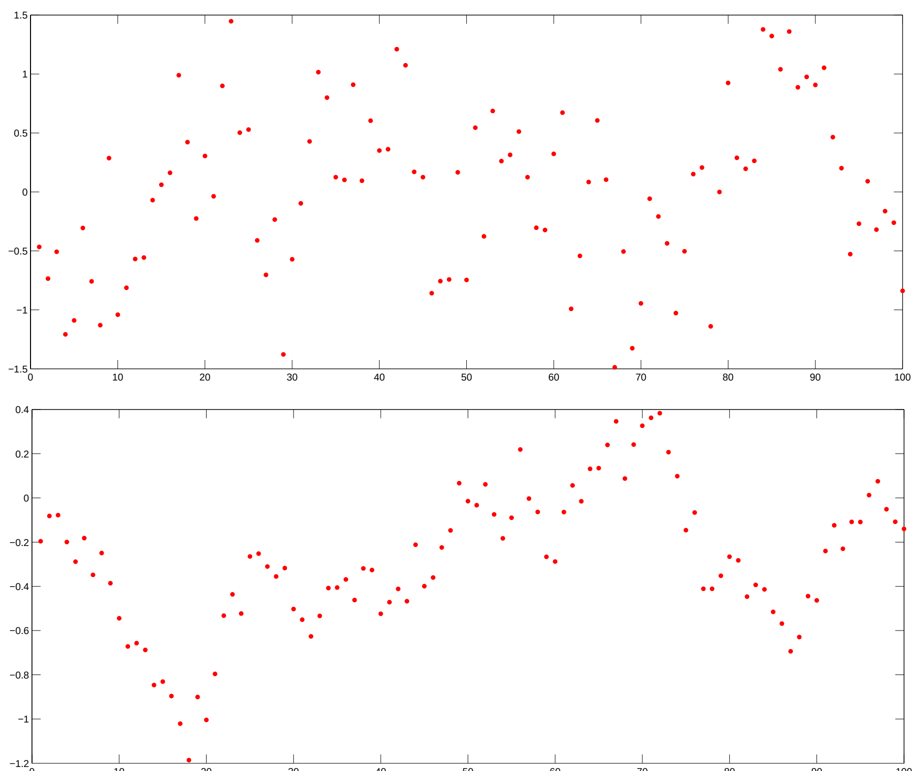

Figure 9: Ergodic Limit of *Mild* (top) and *Strong* (bottom) Interactions for 100 neghboring agents.

[34] See Appendix H for details about how we simulate the ergodic stationary distribution of actions of the economy.

5.3 Comparative Dynamics: Information

In this section we focus on comparing complete information ($I(a) = \mathbb{A}$) and incomplete information ($I(a) \neq \mathbb{A}$) economies. We show at the end of Section A in the Supplemental Appendix how our theoretical analysis can be extended to the incomplete information.[35] In this section we simply note that the policy function with incomplete information takes the following form:

$$x_t^a = g(R^a x_{t-1}, R^a I_0\,\theta_t) = \sum_{b\in\mathbb{A}} c^b\, x_{t-1}^{a+b} + \sum_{b\in I(0)} d^a\,\theta_t^{a+b} + e\,\bar{\theta},$$

where $R^a I_0 = I_a$ is the information set of agent a, which contains at any time t the preference shocks of all agents $b \in I(a)$. In Figure 10, we record the effect of an expansion of the information set (individuals whose types are observed by agent a) on best responses. We start with an information structure in which each agent observes his own type only. We then increase the number of types observed by each agent a (maintaining the symmetry of two-sided interactions) up to the complete information limit. The red dots represent the optimal weights in the policy

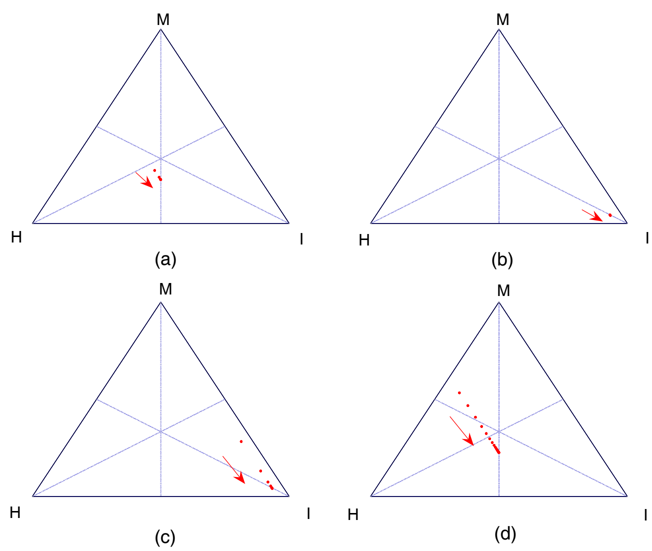

Figure 10: Effect of Information on Interactions.

function of an of agent a as a response to the informational structure. The lower left vertex represents (**H**)istory, the total sum of weights assigned to last period's choices. The lower right vertex represents (**I**)nformation, the sum of weights on current types observed. Finally, the upper vertex represents average information, (**M**)ean type, $\bar{\theta}$. In part (a), we have *mild preferences* for conformity once again. The dots are concentrated near the middle of the triangle (equal weights

[35] In fact, even our welfare results and those regarding identification can be easily extended to the case of incomplete information.

on history, information, and mean type) and they do not move much as a response to changes in the amount of current information. Part (d) is the counterpart with *strong interactions*. Hence the significant change from almost no weight on current information to almost equal weights. Individuals use the information in the best possible way by putting more weight on it in their policy functions. This is due to the fact that forming expectations more precisely how the neighbors will behave becomes more important for each agent, due to the increased strength of interactions. Part (c) is *mild interactions* but *strong own-type effect* ($\frac{\alpha_1}{\alpha_2} = \frac{1}{20}$, $\frac{\alpha_2}{\alpha_3} = 20$) and part (b) is *strong interactions* and *strong own-type effect* ($\frac{\alpha_1}{\alpha_2} = \frac{1}{20}$, $\frac{\alpha_2}{\alpha_3} = 1$). We do not see much change in (b), although most of the total weight is put on information. This is mainly due to the fact that any agent a cares so much about his current type that, he neglects the other effects. In (c), although the own-effect is still strong, due to the strength of interactions, each agent uses the average information to form the best expectations regarding the behavior of the other agents. As the amount of information increases, each agent forms better expectations by transferring the policy weight from average information to precise information on close neighbors.

6 Identification

We study here the identification of social interactions in the context of our linear dynamic economy with conformity. Identification obtains when the restrictions imposed on actions at equilibrium by preferences for conformity are distinct from those imposed by other relevant structural models.

The issue of identification in economies with social interactions has been posed by Manski (1993), who restricts the analysis to linear economies in which the social interactions operate through the mean action in a pre-specified group. More recently, the literature has derived conditions under which identification can be obtained in non-linear models (Brock and Durlauf (2001b)). But even in the context of linear in means models, conditions for identification can be obtained if the population of agents can be partitioned into a sequence of finitely-populated non-overlapping groups (e.g., Graham and Hahn (2005)); or into a sequence of overlapping but asymmetric groups (Bramoullé et al. (2009), Davezies et al. (2009)).[36]

In this paper we focus on dynamic economies and hence we ask in particular whether the restrictions imposed by dynamic equilibrium can help identifying social interactions. For this reason, we stack the desk against ourselves and study linear economies whose social interaction structure is characterized by symmetric overlapping groups.[37] In particular, we first study the

[36] Blume, Brock, Durlauf, and Ioannides (2011), Blume and Durlauf (2005), Brock and Durlauf (2007), Graham (2011), and Manski (1993, 2000, 2007) survey the main questions pertaining to identification in this social context. Other recent contributions include Blume, Brock, Durlauf, and Jayaraman (2011), Evans et al. (1992), Glaeser and Scheinkman (2001), Ioannides and Zabel (2008), and Zanella (2007).

[37] In a related context, De Paula (2009) and Brock and Durlauf (2010) also exploit the properties of dynamic equilibrium, the discontinuity in adoption curves in their continuous time model, to identify social interactions.

identification of the *dynamic structure*; that is, distinguishing the properties of dynamic social interaction economies from those of myopic (hence static) economies. The second series of results we derive regards instead the identification of *social interactions*, that is, distinguishing preferences for conformity from an alternative structural model characterized by (cross-sectionally) correlated preferences across agents. This specific alternative model is focal because correlated preferences could be generally due to some sort of assortative matching or positive selection into social groups, which induces agents with correlated preferences to interact socially.

6.1 Dynamic Rationality vs. Myopia

In this section we compare equilibrium configurations of dynamic economies with rational agents with those of economies with myopic agents. When agents are myopic, even economies with a dynamic structure, e.g., when agents' actions at time t depend on their previous actions, are effectively static. These economies have been extensively studied in the theoretical and empirical literature on social interactions, following the mathematical physics literature in statistical mechanics on interacting particle systems. Suppose that myopic agents, when called to make a choice, act as if they expect never to be called to act again.[38] Given initial history x_{t-1} and realization θ_t, each myopic agent $a \in \mathbb{A}$ chooses $x_t^a \in X$ to maximize

$$u(x_{t-1}^a, x_t^a, x_t^{a-1}, x_t^{a+1}, \theta_t^a) := -\alpha_1(x_{t-1}^a - x_t^a)^2 - \alpha_2(\theta_t^a - x_t^a)^2 - \alpha_3(x_t^{a-1} - x_t^a)^2 - \alpha_3(x_t^{a+1} - x_t^a)^2$$

There exists then a unique symmetric policy function for any agent a, $g^{a,m}$ (m for myopic) such that

$$g^{a,m}(x_{t-1}, \theta_t; \alpha) := \sum_{b \in \mathbb{A}} c^{b,m} x_{t-1}^{a+b} + \sum_{b \in \mathbb{A}} d^{b,m} \theta_t^{a+b}$$

where we make explicit the dependence of the policy function on the preference parameters $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.[39] The coefficients of the policy function $g^{a,m}$ are equal to the ones of the unique

Finally, Patacchini, Rainone, and Zenou (2011) exploits the intertemporal variation in individual choices relative to individual and peers' characteristics to identify social interactions in education from contextual and correlated effects. See also Cabral (1990) for an early discussion of these issues and Young (2009); see Blume, Brock, Durlauf, and Ioannides (2011) for an up to date survey.

[38] See e.g., Blume and Durlauf (2001), Brock and Durlauf (2001b) and Glaeser and Scheinkman (2003) for a comprehensive survey. Liggett (1985) is the standard reference for the mathematical literature.

[39] In some of the literature, myopic agents are modelled not only as assuming that all agents in the economy only interact once, but also that their neighbors are not changing their previous period actions. In this case an agent a's policy function is

$$x_t^a = \beta_1 x_{t-1}^a + \beta_2 \theta_t^0 + \beta_3 x_{t-1}^{-1} + \beta_3 x_{t-1}^1.$$

It can be shown (see e.g., Glaeser and Scheinkman (2003)), that the ergodic stationary distribution of actions in this economy coincides with that of *myopic* agents as defined in the text. As a consequence, our identification results extend to this economy as well.

MPE policy function of a one-period ($T = 1$) social interactions economy: $c^{b,m} = c_1^b$, $d^{b,m} = d_1^b$, for $b \in \mathbb{A}$. In this sense, myopic models are nested within the class of dynamic models we study.

In the following we ask whether the spatial correlations generated by the long-run stationary distribution of an infinite-horizon model can be distinguished from those obtained as the limit distribution of a myopic model. Let $g^a(x_{t-1},\ \theta_t^a; \alpha)$ denote agent a's policy function from the dynamic social interaction model, where we make once again explicit the dependence of the policy function on α. We say that $(x_t^a)_{t\geq 1}^{a\in\mathbb{A}}$ is a stochastic process induced by the dynamic economy with parameters α if it satisfies

$$x_t^a = g^a(x_{t-1},\ \theta_t^a; \alpha),\ \text{for any } a \in \mathbb{A} \text{ and any } t \geq 1$$

We instead say that $(x_t^a)_{t\geq 1}^{a\in\mathbb{A}}$ is a stochastic process induced by the myopic economy with parameters α if it satisfies

$$x_t^a = g^{a,m}(x_{t-1}, \theta_t; \alpha),\ \text{for any } a \in \mathbb{A} \text{ and any } t \geq 1$$

We are now ready to introduce our definition of identification of social interactions.

Definition 3 *Let $(x_t^a)_{t\geq 1}^{a\in\mathbb{A}}$ denote a stochastic process induced by the dynamic economy with parameters α. We say that the dynamic economy with parameters α is identified with respect to myopic economies if there does not exist an $\hat{\alpha}$, such that the process $(x_t^a)_{t\geq 1}^{a\in\mathbb{A}}$ is also induced by a myopic economy with parameter $\hat{\alpha}$.*

The characterization of the spatial correlation of actions at equilibrium for different time-horizons T, which we provided in Section 4.2, gives us a straightforward answer to the identification question. Recall in fact that the coefficients of the policy function $g^{a,m}$ are equal to the ones of the unique MPE policy function of a one-period ($T = 1$) social interactions economy. Recall also that the covariances between agents's choices obtained from data generated by a typical model of infinite-horizon stationary social interactions are fundamentally different from those generated by a static one-period model. In particular, we have shown in Section 4.2 that, for a typical choice of α,

$$r_{a,T} = \frac{\rho_{a+1,T}}{\rho_{a,T}}$$

is non-monotonic in a, for longer horizon economies; and so is r_a, the ratio of the limit economy with $T = \infty$); while $r_{a,1}$ declines monotonically in a, for any α; see Figure 5. Moreover, the limit unconditional covariances inherit the (non)-monotonicity features of their one-step conditional counterparts. Finally, by continuity, the non-monotonicity property necessarily holds for an open set of the parameter space, and is hence *robust*. Summarizing, then, we have the following.[40]

[40]We conjecture that the identification result obtains for the largest part of the parameter space, except for very low discount factors β. Our numerical simulations to that end agree. For example, we present in Appendix F.1 the

Proposition 2 (Rationality vs. Myopia) *A dynamic economy with parameter α is identified with respect to myopic economies, for a robust subset α.*

Finally, consider an econometrician fitting a static (myopic) model through data generated by the dynamic equilibrium of an economy with parameter α. From Proposition 1, $r_1(\alpha) < r(\alpha)$ for any possible α. As a consequence, the parameter $\hat{\alpha}$ estimated by the econometrician imposing the static (myopic) structure on the data, will satisfy $r_1(\hat{\alpha}) = r(\alpha) > r_1(\alpha)$. From (B.1), however, r_1 is monotonically decreasing in $\left(\frac{\Delta_1}{2\alpha_3}\right)$.[41] As a consequence,

$$\left(\frac{\hat{\alpha}_3}{\hat{\alpha}_1 + \hat{\alpha}_2 + 2\hat{\alpha}_3}\right) > \left(\frac{\alpha_3}{\alpha_1 + \alpha_2 + 2\alpha_3}\right),$$

and the econometrician **overestimates** the social interaction effects.

6.2 Social Interactions vs. Selection

In our dynamic economies, spatial correlation of individual actions at each time is induced by social interactions and preference for conformity. But spatial correlation of actions could be induced in principle also by spatial correlation of preference types, with no social interaction. Take two agents, e.g., agent a and agent b. A positive correlation between x_t^a and x_t^b could be due to a positive correlation between θ_t^a and θ_t^b. In this last case, preferences for conformity and social interactions would play no role in the correlation of actions at equilibrium. Rather, such correlation would be due to the fact that agents have correlated preferences. As we already noted, correlated preferences could be generally due to some sort of assortative matching or positive selection in social interaction, which induces agents with correlated preferences to interact socially.

In our economy, at a symmetric Markov perfect equilibrium, each agent $a \in \mathbb{A}$ acts according to the policy function $g^a_{T-(t-1)}(x_{t-1},\ \theta_t^a; \alpha)$, where we make once again explicit the dependence of the policy function on the preference parameters $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. If $T = \infty$, the policy function is stationary $g^a(x_{t-1},\ \theta_t^a; \alpha)$. Recall that the parameter α_3 represents the weight of conformity in each agent's preferences. It follows that $\alpha_3 = 0$ corresponds to an economy with no social interactions. We say that $(x_t^a)_{t\geq 1}^{a\in\mathbb{A}}$ is a stochastic process induced by α and $(\theta_t^a)_{t\geq 1}^{a\in\mathbb{A}}$ if it satisfies

$$x_t^a = g^a_{T-(t-1)}(x_{t-1},\ \theta_t^a; \alpha),\ \text{for any } a \in \mathbb{A} \text{ and any } t = 1, \ldots, T$$

We are now ready to construct our definition of identification of social interactions.

results of a simulation where we report r_a as a function of a, at the stationary distribution, for different levels of strength of interaction proxied by the ratio $\left(\frac{2\alpha_3}{\Delta_1}\right)$. More precisely, we set $\left(\frac{\alpha_1}{\alpha_2}\right) = 1$, $\left(\frac{2\alpha_3}{\Delta_1}\right) \in \{0.1, 0.2, 0.75, 0.9\}$ and $\beta = .95$ and obtain non-monotonicity throughout.

[41] Remember that $\Delta_1 := \alpha_1 + \alpha_2 + 2\,\alpha_3$.

Definition 4 *Let* $(x_t^a)_{t\geq 1}^{a\in\mathbb{A}}$ *denote a stochastic process induced by* α *and* $(\theta_t^a)_{t\geq 1}^{a\in\mathbb{A}}$*, where* $(\theta_t^a)_{t\geq 1}^{a\in\mathbb{A}}$ *is i.i.d. across agents and serially uncorrelated, that is, where* $Cov(\theta_t^a, \theta_t^b) = Cov(\theta_t^a, \theta_{t+1}^a) = 0$ *for any* $a \neq b \in \mathbb{A}$ *and any* $t \geq 1$*. We say that* α is identified *if there does not exist an* $\hat{\alpha}$*, with* $\hat{\alpha}_3 = 0$*, such that the process* $(x_t^a)_{t\geq 1}^{a\in\mathbb{A}}$ *is also induced by* $\hat{\alpha}$ *and some stochastic process* $\{\hat{\theta}_t^a\}_{t\geq 1}^{a\in\mathbb{A}}$ *. We say that* social interactions are identified *if some* α*, with* $\alpha_3 > 0$*, is identified.*

The conditions for identification of social interactions can be weakened by restricting the stochastic process $\{\hat{\theta}_t^a\}_{t\geq 1}^{a\in\mathbb{A}}$. We say that α is (resp. social interactions are) *identified relative to a set of preference shocks* if $\{\hat{\theta}_t^a\}_{t\geq 1}^{a\in\mathbb{A}}$ in Definition 4 is required to belong to a set of preference shocks which satisfies some specific restriction.

Finally, the conditions for idenfication of social interactions can be strengthened by limiting the observable properties of the process $(x_t^a)_{t\geq 1}^{a\in\mathbb{A}}$.

We first consider the case of an infinite horizon economy: policy functions are stationary and an ergodic distribution exists. In this context, we study first the possibility of obtaining identification by observing the properties of the stationary distribution of actions rather than the whole panel $(x_t^a)_{t\geq 1}^{a\in\mathbb{A}}$. We then pass on to identification tout court, that is exploiting the whole dynamic restrictions imposed by the model on $(x_t^a)_{t\geq 1}^{a\in\mathbb{A}}$, not just the restrictions on the stationary distribution. We shall see that results are negative in both cases, that is, identification is not obtained in general. Secondarily, we study identification relative to a series of relevant restrictions on the stochastic process for preference shocks $\{\hat{\theta}_t^a\}_{t\geq 1}^{a\in\mathbb{A}}$. These restrictions are meant to capture natural properties of the selection mechanism which induces agents to display spatially correlated preferences.

6.2.1 Infinite horizon (stationary) economies

Consider first the stationary distribution of actions as identified by its implied spatial correlation function ρ_b.

Proposition 3 *Social interactions are not identified by the properties of the spatial correlation function* ρ_b *of the stationary distribution of actions in infinite horizon economies.*

The proof is simple and instructive hence is given below.

Proof: We have shown in Section 3.3 that the stationary distribution of our dynamic economy with social interactions, that is, $\alpha_3 > 0$, and i.i.d. preference shock process $\{\theta_t^a\}_{t\geq 1}^{a\in\mathbb{A}}$, is given by the ergodic measure π in (8), i.e. π is the joint distribution of

$$x_t = \left(\frac{e(\alpha)\,\overline{\theta}}{1 - C(\alpha)} + \sum_{s=1}^{\infty} \sum_{b_1} \cdots \sum_{b_s} c(\alpha)^{b_1} \cdots c(\alpha)^{b_{s-1}} \left(d(\alpha)^{b_s}\; \theta_{t+1-s}^{a+b_1+\cdots+b_s} \right) \right)_{a\in\mathbb{A}}$$

Consider now an alternative specification of our economy with *no interactions* between agents ($\hat{\alpha}_3 = 0$) and *no habits* ($\hat{\alpha}_1 = 0$) but simply a preference shock process $\{\hat{\theta}_t^a\}_{t\geq 1}^{a\in\mathbb{A}}$ and *own type effects* with $\hat{\alpha}_2 > 0$. For this economy, equilibrium choice of agent a at time t is given by $x_t^a = \hat{\theta}_t^a$. As long as the process $\{\hat{\theta}_t^a\}_{t\geq 1}^{a\in\mathbb{A}}$ is the one where

$$\hat{\theta}_t^a := \frac{e\left(\alpha\right)\overline{\theta}}{1 - C\left(\alpha\right)} + \sum_{s=1}^{\infty}\sum_{b_1}\cdots\sum_{b_s} c\left(\alpha\right)^{b_1}\cdots c\left(\alpha\right)^{b_{s-1}}\left(d\left(\alpha\right)^{b_s}\ \theta_{t+1-s}^{a+b_1+\cdots+b_s}\right)$$

the probability distributions that the two specifications (with and without interactions)) generate on the observables of interest, $\{x_t^a\}_{t\geq 1}^{a\in\mathbb{A}}$, are identical. Hence, one cannot identify from the stationary distribution of choices which specification generates the data. ■

More generally, we investigate if the dynamic equilibrium restrictions of our model are sufficient to identify social interactions.

Proposition 4 *Social interactions are* not identified *in infinite horizon economies.*[42]

An intuition about this result can be obtained by loosely reducing the identification of social interactions in infinite horizon economies to the well known problem of distinguishing a VAR from an MA(∞) process. Stacking in a vector $\mathbf{x}_t$ (resp. θ_t) the actions x_t^a over the index $a \in \mathbb{A}$ (resp. the preference shocks θ_t^a), policy functions can be loosely written as a VAR:

$$\mathbf{x}_t = \Phi\mathbf{x}_{t-1} + \delta_t, \quad \text{with } \delta_t = \Gamma\theta_t + e\overline{\theta}$$

where $E\left(\delta_t\delta_{t-\tau}\right) = 0$ for all $\tau > 0$. Under standard stationarity assumptions, the VAR has an MA(∞) representation

$$\mathbf{x}_t = \left(I_A - \Phi\mathbf{L}\right)^{-1}\delta_t = \ \delta_t + \Psi_1\delta_{t-1} + \Psi_2\delta_{t-2} + \ldots$$

for a sequence Ψ_1,$\Psi_2\ldots$ such that $\left(I_A - \Phi\mathbf{L}\right)\left(I_A + \Psi_1\mathbf{L} + \Psi_2\mathbf{L}^2 + \ldots\right) = I_A$. The argument in the proof of Proposition 4 therefore amounts to picking

$$\mathbf{x}_t = \hat{\theta}_t = \delta_t + \Psi_1\delta_{t-1} + \Psi_2\delta_{t-2} + \ldots.$$

6.2.2 Finite-horizon (non-stationary) economies

Consider now the case of a finite horizon economy. In this case the unique policy function and the distribution of actions are *non-stationary*, as we have shown, and hence identification might

[42]The proof is similar to the proof of Proposition 3 with the extra requirement that observational-equivalence should also hold for processes with an initial condition and for their intertemporal transitions. We put the proof in the Supplemental Appendix F.2 for interested readers.

obtain in those environments where correlated effects satisfy a weakly stationary law through time. Consider then a restriction to the class of admissible preference shock processes $\{\hat{\theta}_t^a\}_{t\geq 1}^{a\in\mathbb{A}}$ which satisfy the following *conditional covariance stationarity* restriction:

Definition 5 (Conditional Covariance Stationarity) *A process $\{\hat{\theta}_t^a\}_{t\geq 1}^{a\in\mathbb{A}}$ is said to be conditional covariance stationary if $Cov\left(\hat{\theta}_t^a, \hat{\theta}_t^b \,\middle|\, \hat{\theta}_{t-1}, \cdots, \hat{\theta}_1\right) = Z(a, b, \hat{\theta}_{t-1}, \ldots, \hat{\theta}_{t-n}) \in \mathbb{R}$, for $a, b \in \mathbb{A}$, $t = n+1, \ldots, T$.*[43]

This condition defines a large class of stochastic processes for which the covariance between the preference shocks of any two agents a and b, depends on at most a finite memory (represented by n) of past realizations of the process, and possibly the relative positions of agents a and b in the social group. It is a relatively weak and natural condition in that it allows for the tailoring of the intertemporal dependence of agents' types to their relative positions in the network; what it excludes is events in the distant past from having a significant effect on the *joint* determination of agents' types today, when one has information on shock realizations in the recent past.[44]

Definition 6 (Linear Conditional Expectations) *Conditional expectations are linear in the n past realizations, i.e., for $a \in \mathbb{A}$, $t \leq T$.*

$$E\left[\hat{\theta}_t^a \mid \hat{\theta}_{t-1}, \cdots, \hat{\theta}_1\right] = L(R^a\,\hat{\theta}_{t-1}, \cdots, R^a\,\hat{\theta}_{t-n}) \tag{13}$$

where $L : (\Theta^{\mathbb{A}})^n \to X$ is a linear map.

Conditional covariance stationarity and linearity of expectations of preference shocks is in fact sufficient for identification of social interactions.

Proposition 5 *Social interactions are* identified *relative to processes satisfying the* conditional covariance stationarity *and* linear conditional expectations *restrictions.*

While the spatial autocorrelations between agents' choices have limited memory across periods in the absence of interaction effects, they vary in presence of social interactions.[45] Notice that we

[43] The original definition is due to Mandelbrot (1967) who provides the conditional spectral analysis of *sporadically varying* random functions in the mathematical theory of information transmission with noise. In his environment, he requires $E\left[\hat{\theta}_t\,\hat{\theta}_{t+n} \,\middle|\, 1 \leq t < t+n \leq T\right]$ to be independent of t. Ours is a slightly weaker condition since it uses fixed finite memory. For more recent usage of conditional covariance restrictions see the Time Series literature studying persistence of conditional variances, especially Bollerslev and Engle (1993), Bollerslev et al. (1994), and Engle and Bollerslev (1986).

[44] All existing social interaction models we can think of have stochastic structures that are special cases of this class. More specifically, they typically assume either time-independent or finite memory Markov structures to model exogenous effects; see e.g., Brock and Durlauf (2001b), Conley and Topa (2003), De Paula (2009), Glaeser and Scheinkman (2001), and Young (2009).

[45] More specifically, they can be ordered with respect to their spatial rate of tail convergence; see Proposition 1.

implicitly assume that the economy starts with the initial condition $(x_0, \theta_0, \ldots, \theta_{-(n-1)})$. Another way to proceed is to assume that $n \leq T-3$, which guarantees that the economy lasts longer than the memory of the spatial auto-correlation process, so that the non-stationarity of the equilibrium process can manifest itself in the dynamics of spatial auto-correlations. This is at the heart of the proof, whose details follow.

Proof: Consider a finite-horizon, T-period economy with $T \geq 2$ and $n \leq T-3$. In the absence of interactions ($\hat{\alpha}_3 = 0$), agent a's final period optimal choice is[46]

$$x_T^a \;=\; c_1(\hat{\alpha})\, x_{T-1}^a + d_1(\hat{\alpha})\, \hat{\theta}_T^a \tag{14}$$

Thanks to the linearity of the policy functions across periods with $\hat{\alpha}_3 = 0$, any path of shock realizations $(\hat{\theta}_1, \ldots, \hat{\theta}_{T-1})$, given x_0, generates a path of configurations $(x_0, x_1, \ldots, x_{T-1})$. Thus, conditioning on all imaginable choice paths spans all imaginable preference shock paths, given that the observables are generated by the above-mentioned policy functions. The a-step covariance between equilibrium choices of agent 0 and a in case of *interactions* is then given by

$$\begin{aligned} Cov\left(x_T^0\,,\, x_T^a \,\middle|\, x_{T-1}, \ldots, x_0\right) &= Cov\left(x_T^0\,,\, x_T^a \,\middle|\, x_{T-1}\right), \quad \forall\,(x_0, \ldots, x_{T-1}) \\ &= Cov\left(\sum_{b_1 \in \mathbb{A}} d_1^{b_1}\, \theta_T^{b_1}, \sum_{b_2 \in \mathbb{A}} d_1^{b_2}\, \theta_T^{a+b_2}\right) \\ &= Var(\theta) \sum_{b_1 \in \mathbb{A}} d_1^{b_1}\, d_1^{b_1 - a} \end{aligned} \tag{15}$$

thus the covariance term is independent of the conditioned upon path. So, in order the specification with no interactions to be observationally indistinguishable from the interactions case, the a-step conditional covariances, computed using (14) should satisfy, for all $(x_0, \hat{\theta}_1, \ldots, \hat{\theta}_{T-1})$

$$\begin{aligned} Cov\left(x_T^0\,,\, x_T^a \,\middle|\, x_{T-1}\right) &= d_1(\hat{\alpha})^2\, Cov\left(\hat{\theta}_T^0\,, \hat{\theta}_T^a \,|\, \hat{\theta}_{T-1}, \ldots, \hat{\theta}_1, x_0\right) \\ &= d_1(\hat{\alpha})^2\, Z(0, a, \hat{\theta}_{T-1}, \ldots, \hat{\theta}_{T-n}) \\ &= d_1(\hat{\alpha})^2\, \bar{Z}(0, a) \end{aligned} \tag{16}$$

The function $\bar{Z}$ is implicitly defined to capture the fact that to match the covariance term in (15), the covariance in (16) can depend only on the relative positions of agents and on nothing else. This is not an assumption but an observational-equivalence restriction. The same observational-equivalence idea should hold for all $a \in \mathbb{A}$, in period $T-1$. Similar calculations in period $T-1$

[46] In the absence of interactions, agents solve (dynamic) individual maximization problems. See Appendix F.3 for the derivations.

yield

$$
\begin{aligned}
Cov\left(x^0_{T-1}\,,\, x^a_{T-1}\,\middle|\, x_{T-2}\right) &= Var(\theta) \sum_{b_1 \in \mathbb{A}} d_2^{b_1}\, d_2^{b_1-a} \\
&= \sum_{b_1} \sum_{b_2} \hat{d}_2^{b_1}\, \hat{d}_2^{b_2}\, Cov(\hat{\theta}^{b_1}_{T-1}, \hat{\theta}^{a+b_2}_{T-1} \mid \hat{\theta}_{T-2}, \ldots, \hat{\theta}_{T-1-n}) \\
&= \sum_{b_1} \sum_{b_2} \hat{d}_2^{b_1}\, \hat{d}_2^{b_2}\, Z(0, a, \hat{\theta}_{T-2}, \ldots, \hat{\theta}_{T-1-n}), \quad \forall(\hat{\theta}_{T-2}, \ldots, \hat{\theta}_{T-1-n}) \\
&= \sum_{b_1} \sum_{b_2} \hat{d}_2^{b_1}\, \hat{d}_2^{b_2}\, \bar{Z}(0, a) \qquad (17)
\end{aligned}
$$

where the first equality is as in (15); second is the restriction imposed by observable indistinguishability and the policies for the no-interaction case derived in Appendix F.3; third is conditional covariance stationarity; finally fourth is by conditional covariance stationarity across periods using (16). We know that for arbitrarily large a, from (16)

$$
\lim_{a\to\infty} \frac{Cov\left(x^0_{T}\,,\, x^{a+1}_{T}\,\middle|\, x_{T-1}\right)}{Cov\left(x^0_{T}\,,\, x^{a}_{T}\,\middle|\, x_{T-1}\right)} = \lim_{a\to\infty} \frac{\bar{Z}(0, a+1)}{\bar{Z}(0, a)} = r_1 \qquad (18)
$$

and from (17)

$$
\lim_{a\to\infty} \frac{Cov\left(x^0_{T-1}\,,\, x^{a+1}_{T-1}\,\middle|\, x_{T-2}\right)}{Cov\left(x^0_{T-1}\,,\, x^{a}_{T-1}\,\middle|\, x_{T-2}\right)} = r_2 = \lim_{a\to\infty} \frac{\sum_{b_1}\sum_{b_2} \hat{d}_2^{b_1}\, \hat{d}_2^{b_2}\, \bar{Z}(0, a+1)}{\sum_{b_1}\sum_{b_2} \hat{d}_2^{b_1}\, \hat{d}_2^{b_2}\, \bar{Z}(0, a)} \qquad (19)
$$

This implies, using Lemma 7 in Appendix D that the policy weight sequence $\hat{d}_2$ should converge at the higher rate r_2. A final repetition of the same exercise should give for all $a \in \mathbb{A}$, in period $T-2$

$$
Cov\left(x^0_{T-2}\,,\, x^a_{T-2}\,\middle|\, x_{T-3}\right) = \sum_{b_1} \sum_{b_2} \hat{d}_3^{b_1}\, \hat{d}_3^{b_2}\, \bar{Z}(0, a) \qquad (20)
$$

From the definition of $\mu_{3,s}$ in (F.3) and Lemma 7, $\mu_{3,s}$ inherits the maximum of the rates of $\hat{c}_1, \hat{c}_2, \hat{d}_1$ and $\hat{d}_2$, which is r_2. So, from (F.5), $\hat{d}_3$ converges at the rate r_2. However, using the same idea of indistinguishability, as $a \to \infty$, it must be that

$$
r_3 = \lim_{a\to\infty} \frac{Cov\left(x^0_{T-2}\,,\, x^{a+1}_{T-2}\,\middle|\, x_{T-3}\right)}{Cov\left(x^0_{T-2}\,,\, x^{a}_{T-2}\,\middle|\, x_{T-3}\right)} = \lim_{a\to\infty} \frac{\sum_{b_1}\sum_{b_2} \hat{d}_3^{b_1}\, \hat{d}_3^{b_2}\, \bar{Z}(0, a+1)}{\sum_{b_1}\sum_{b_2} \hat{d}_3^{b_1}\, \hat{d}_3^{b_2}\, \bar{Z}(0, a)} = r_2 \qquad (21)
$$

which is a contradiction to Proposition 1. Therefore, there does not exist a conditional covariance stationary preference shock process $\{\hat{\theta}^a_t\}^{a\in\mathbb{A}}_{t\geq 1}$ that generates an equilibrium choice process $\{x^a_t\}^{a\in\mathbb{A}}_{t\geq 1}$ under the no interactions specification ($\hat{\alpha}_3 = 0$) that is observationally equivalent to the process generated by the local interactions ($\alpha_3 \neq 0$) process. This concludes the proof. ■

7 Extensions

The class of social interaction economies we studied in this paper has been restricted along several dimensions to better provide a stark theoretical analysis. Some of these restrictions, however, turn out to be important in applications and empirical work. In this section, therefore, we illustrate how our analysis can be extended to study more general neighborhood network structures for social interactions, more general stochastic processes for preference shocks, the addition of global interactions, that is, interactions at the population level, and the effects of stock variables which carry habit effects.

7.1 General Neighborhood Network Structures

Throughout the paper, we studied symmetric neighborhood structures. This is generalized easily. Consider an arbitrary neighborhood structure (not necessarily translation invariant), $N : \mathbb{A} \to 2^{\mathbb{A}}$. Suppose also that a generic agent a's preferences are represented by the utility function u^a defined as

$$\begin{aligned} u^a\left(x_{t-1}^a, x_t^a, \{x_t^b\}_{b\in N(a)}, \theta_t^a\right) \;:=\; & -\alpha_{a,1}\left(x_{t-1}^a - x_t^a\right)^2 - \alpha_{a,2}\left(\theta_t^a - x_t^a\right)^2 \\ & - \sum_{b\in N(a)} \alpha_{a,b}\left(x_t^b - x_t^a\right)^2 \end{aligned}$$

Notice that we allow for the preferences of any two agents a and b to be arbitrarily different in their parametrization, provided either $\alpha_{a,1} > 0$ or $\alpha_{a,2} > 0$ and $\sum_{b\in N(a)} \alpha_{a,b} < \infty$ so that peer effects are bounded. Under this specification, best-responses are well defined, linear, interior, and well-behaved.

Proposition 6 *An MPE exists (not necessarily unique) and the policy function of an arbitrary agent $a \in \mathbb{A}$ at equilibrium has the following form*

$$g_{T-(t-1)}^a\left(x_{t-1}, \theta_t\right) = \sum_{b\in\mathbb{A}} c_{T-(t-1)}^{a,b}\, x_t^b + \sum_{b\in\mathbb{A}} d_{T-(t-1)}^{a,b}\, \theta_t^b + e_{T-(t-1)}^a\, \bar{\theta}$$

for $t = 1, \ldots, T$, where, as before, all coefficients are non-negative and sum up to 1. Moreover, uniqueness of the MPS obtains if there exists a positive constant $K < 1$ such that for each individual $a \in \mathbb{A}$

$$\frac{\sum_{b\in N(a)} \alpha_{a,b}}{\alpha_{a,1} + \alpha_{a,2} + \sum_{b\in N(a)} \alpha_{a,b}} < K.$$

Thus, for uniqueness of equilibrium, it is sufficient that the relative composition of the peer effects within the determinants of individual choice be uniformly bounded. Under this condition,[47] best

[47] A related condition is referred to, in the literature, as the *Moderate Social Influence* condition; see e.g. Glaeser and Scheinkman (2003), Horst and Scheinkman (2006), and Ballester, Calvó-Armengol, and Zenou (2006) for restrictions in a similar spirit.

response profile induces a contraction operator and a unique equilibrium is obtained for any finite-horizon economy.

Ergodicity (relative to a given MPE) and welfare results extend straightforwardly, as do identification results. Notably, our positive identification result for non-stationary economies, Proposition 5, also extends: The tail convergence property of equilibrium is not peculiar to the line structure. For any given neighborhood structure, as long as the relative composition of the peer effects and the size of the neighborhoods are uniformly bounded (one can find weaker conitions), we get an economy with infinite range interactions where interactions '*decay at infinity*' (see e.g. Föllmer, 1974) as the social distance between agent a and any other agent becomes arbitrarily large and Proposition 1 extends (altough the convergence rate would be reference agent dependent, r^a_T, due to heterogeneity). Furthermore, since preference parameters of any agent a are stationary, in a finite-horizon economy, correlations of equilibrium actions between agents vary only due to interactions for preference processes that satisfy a Conditional Covariance Stationarity restriction.

7.2 General Stochastic Processes for Preference Shocks

The agents in our model make their decisions based on past behavior and current shocks. Our analysis however extends straightforwardly to economies where shocks are persistent across time as long as the economy is one of complete information.[48] We give here, as an illustration, an example of Markov dependence, where at any given period the probability of next period shocks depends on current realizations.

Consider any T-period economy with $T \leq \infty$. Recall from Section 2 that preference shocks $\theta_t := (\theta^a_t)_{a\in\mathbb{A}}$ are defined on the canonical probability space $(\mathbf{\Theta}, \mathcal{F}, \mathbb{P})$, where $\mathbf{\Theta} := \{(\theta^a)_{a\in\mathbb{A}} : \theta^a \in \Theta\}$. Let $Q : \mathbf{\Theta} \times \mathcal{F} \to \mathbb{R}_+$ be a *transition function* such that

(i) for any period t and any $\theta \in \mathbf{\Theta}$, $Q(\theta, A) = Pr\{\theta_{t+1} \in A \,|\, \theta_t = \theta\}$, for all $A \in \mathcal{F}$.

(ii) for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{F}$-measurable.

Any agent $a \in \mathbb{A}$ solves the problem in (1) with persistent shocks where the expectation operator acts on the distribution induced by Q and other agents' strategies. Naturally, the optimal choices depend on the conditioned upon variable θ_t as before, but the coefficients also depend on θ_t because the expectations change according to that latter. All this is summarized in

Proposition 7 *An MPE exists and is unique for $T < \infty$, and the policy function of an arbitrary agent $a \in \mathbb{A}$ at equilibrium has the following form*

$$g^a_{T-(t-1)}\left(x_{t-1}, \theta_t\right) = \sum_{b\in\mathbb{A}} c^{a,b}_{T-(t-1)}(\theta_t)\, x^b_{t-1} + \sum_{b\in\mathbb{A}} d^{a,b}_{T-(t-1)}(\theta_t)\, \theta^b_t + e^a_{T-(t-1)}\left(\theta_t\right)$$

[48]The mathematical issues arising in dynamic models with incomplete information are both well-known and outside the scope of the present paper. See Mailath and Samuelson (2006) for an extensive survey.

for $t = 1, \ldots, T$ all coefficients are non-negative and sum up to 1.

7.3 Global Interactions

Introducing global determinants of individual behavior into our framework is also relatively straightforward.[49] In particular, consider an economy in which the preferences of each agent $a \in \mathbb{A}$ depend also on the average action of the agents in the economy. Let the average action given a choice profile x be defined as

$$p(x) := \lim_{n\to\infty} \frac{1}{2n+1} \sum_{a=-n}^{n} x^a,$$

when the limit exists. Let $\mathbf{X_e}$ denote the set of all configurations such that the associated average action exists:

$$\mathbf{X_e} := \left\{ x \in \mathbf{X} : \exists\, p(x) := \lim_{n\to\infty} \frac{1}{2n+1} \sum_{a=-n}^{n} x^a \right\}$$

The preferences of the agent $a \in \mathbb{A}$ in period t are described by the instantaneous utility function $u : \mathbf{X_e} \times \Theta \to \mathbb{R}$ of the conformity class

$$\begin{aligned} & u^a\left(x_{t-1}^a, x_t^a, \{x_t^b\}_{b\in N(a)}, \theta_t^a, p(x_t)\right) := \\ & -\alpha_1\,(x_{t-1}^a - x_t^a)^2 - \alpha_2\,(\theta_t^a - x_t^a)^2 - \sum_{b\in N(a)} \alpha_{3,b}\,(x_t^b - x_t^a)^2 - \alpha_4\,(p(x_t) - x_t^a)^2 \end{aligned}$$

Given $x \in \mathbf{X_e}$, the initial configuration of actions, a symmetric stationary MPE of a dynamic economy with local and global interactions is a map $g : \mathbf{X} \times \boldsymbol{\Theta} \times X \to X$ and a map $F : X \to X$ such that, for each $a \in \mathbb{A}$:

$$\begin{aligned} g\left(R^a\, x_{t-1}, R^a\, \theta_t, p_t\right) \;=\; & \arg\max_{x_t^a \in X} E\Bigg[u\left(x_{t-1}^a, x_t^a, \left\{ g\left(R^b\, x_{t-1}, R^b\, \theta_t, p_t\right)\right\}_{b\in N(a)}, \theta_t^a, p_t\right) \\ & + \beta V_g\left(R^a \left(x_t^a, \left\{ g\left(R^b\, x_{t-1}, R^b\, \theta_t, p_t\right)\right\}_{b\neq a}\right), R^a\, \theta_{t+1}, p_{t+1}\right) \Bigg|\, (x_{t-1}, \theta_t)\Bigg] \end{aligned}$$

with

$$p_{t+1} = F\left(p_t\right), \qquad p_1 = p(x) \quad \text{ and } \quad p_t = p\left(x_t\right) \text{ almost surely.}$$ [50]

At a symmetric MPE, any agent rationally anticipates that all others play according to the policy function g and also anticipates the sequence of average actions $\{p(x_t)\}_{t\in\mathbb{N}}$ to be determined recursively via the map F.

[49] With respect to the analysis of MPE with local and global interactions in finite economies (as e.g., in Blume and Durlauf (2001) and in Glaeser and Scheinkman (2003)), a few technical subtleties arise in our economy due to the infinite number of agents. The techniques we use are extensions of the ones we used in a previous paper, Bisin, Horst, and Özgür (2006). We refer the reader to this paper for details. Some of the needed mathematical analysis is developed in Föllmer and Horst (2001) and Horst and Scheinkman (2006).

For this economy, we can show that the endogenous sequence of average actions $\{p(x_t)\}_{t\in\mathbb{N}}$ exists almost surely if the initial configuration x belongs to $\mathbf{X_e}$, and that it follows a deterministic recursive relation.[51]

Proposition 8 *A symmetric stationary MPE of our dynamic economy with local and global interactions exists and the policy function of an arbitrary agent* $a \in \mathbb{A}$ *at equilibrium has the form*

$$g^a(x_{t-1}, \theta_t) = \sum_{b\in\mathbb{A}} c_b\, x_{t-1}^{a+b} + \sum_{b\in\mathbb{A}} d_b\, \theta_t^{a+b} + e\,\bar{\theta} + B^*(p(x))$$

for some positive coefficients $(c^b)_{b\in\mathbb{A}}$*,* $(d^b)_{b\in\mathbb{A}}$*,* e*, and some constant* $B^*(p(x))$ *that depends only on the initial average action,* $p(x)$*.*

7.4 Social Accumulation of Habits

In this section, we generalize the class of the economies we have studied to encompass a richer structure of dynamic dependence of agents' actions at equilibrium. Consider an economy where preferences of agent $a \in \mathbb{A}$ are represented by a utility function

$$u\left(S_t^a, x_t^a, \{x_t^b\}_{b\in N(a)}, \theta_t^a\right) \;:=\; -\alpha_1 \left(S_t^a - x_t^a\right)^2 - \alpha_2 \left(\theta_t^a - x_t^a\right)^2 - \sum_{b\in N(a)} \alpha_{3,b} \left(x_t^b - x_t^a\right)^2$$

where S_t^a represents an accumulated stock variable,

$$S_{t+1}^a = (1-\delta)\, S_t^a + x_t^a$$

For instance, S_t^a captures what the addiction literature calls a "*reinforcement effect*" on agent a's substance consumption.

Proposition 9 *An MPE exists and is unique for* $T < \infty$*, and the policy function of an arbitrary agent* $a \in \mathbb{A}$ *at equilibrium has the following form*

$$g_{T-(t-1)}^a\left(S_t, \theta_t\right) = \sum_{b\in\mathbb{A}} c_{T-(t-1)}^b\, S_t^{a+b} + \sum_{b\in\mathbb{A}} d_{T-(t-1)}^b\, \theta_t^{a+b} + e_{T-(t-1)}\,\bar{\theta}$$

for $t = 1, \ldots, T$*, where, as before, all coefficients are non-negative and sum up to 1.*

Note that in equilibrium each agent's choice depends on the stock of his neighbors' actions, that is, on their long-term behavioral patterns rather than just their previous period actions. Also, as the final period approaches, agent a assigns *uniformly* higher weights to his own stock.

[51] Linearity is crucial for these results. Only in this case, in fact, can the dynamics of average actions $\{p(x_t)\}_{t\in\mathbb{N}}$ be described recursively. In models with more general local interactions, the average action typically is not a sufficient statistic for the aggregate behavior of the configuration x; hence a recursive relation typically fails to hold. In such more general cases, the analysis must be pursued in terms of empirical fields. Interested reader should consult Föllmer and Horst (2001).

8 Conclusion

Social interactions provide a rationale for several important phenomena at the intersection of economics and sociology. The theoretical and empirical study of economies with social interactions, however, has been hindered by several obstacles. Theoretically, the analysis of equilibria in these economies induces generally intractable mathematical problems: equilibria are represented formally by a fixed point in configuration of actions, typically an infinite dimensional object; and embedding equilibria in a full dynamic economy adds a second infinite dimensional element to the analysis. Computationally, these economies are also generally plagued by a curse of dimensionality associated to their large state space. Finally, in applications and empirical work, social interactions are typically identified, even with population data, only under heroic assumptions.

In this paper we have attempted to show how some of these obstacles to the study of economies with social interactions can be overcome. Admittedly, we have restricted our analysis to linear economies, but in this context we have been able i) to obtain several desirable theoretical properties, like existence, uniqueness, ergodicity; ii) to develop simple recursive methods to rapidly compute equilibria; and iii) to characterize several general properties of dynamic equilibria. Furthermore, while linearity in principle renders the identification problem in static economies with social interaction almost insurmountable, we have been able to exploit the properties of dynamic equilibria in non-stationary economies to produce a positive identification result.

In conclusion, we believe that the class of dynamic linear economies with social interactions we have studied in this paper can be fruitfully and easily employed in applied and empirical work.

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SUPPLEMENTAL APPENDIX

This supplement contains proofs of the results in the main text and the details about the simulations we used to generated the Figures. All notation is as defined in the main text unless explicitly noted otherwise.

A The existence proof

We prove here Theorem 1. The proof is constructive and works in three steps, by induction on the length of the economy.[52]

Step 1: Existence, uniqueness and the Markov property for $T = 1$. In this symmetric environment, it is enough to analyze the optimization problem of a single agent, say of agent $0 \in \mathbb{A}$. We will allow for arbitrary initial histories so that one can interpret the current step either as a one-period economy or the last period of a finite-horizon economy. We will show that, agents will use only the information contained in the previous period choices x_0 and current type realizations θ_1. Let any t-length history $(x^{t-1}, \theta^t) = (x_{-(t-1)}, \theta_{-(t-2)}, \ldots, x_{-1}, \theta_0, x_0, \theta_1)$ of previous choices and preference shock realizations be given. Agent 0 solves

$$\max_{x_1^0 \in X} \left\{ -\alpha_1 \left(x_0^0 - x_1^0\right)^2 - \alpha_2 \left(\theta_1^0 - x_1^0\right)^2 - \alpha_3 \left(x_1^{-1} - x_1^0\right)^2 - \alpha_3 \left(x_1^1 - x_1^0\right)^2 \right\} \tag{A.1}$$

The first order condition

$$2 \left[\alpha_1 \left(x_0^0 - x_1^0\right) + \alpha_2 \left(\theta_1^0 - x_1^0\right) + \alpha_3 \left(x_1^{-1} - x_1^0\right) + \alpha_3 \left(x_1^1 - x_1^0\right)\right] = 0$$

implies that

$$x_1^0 = \Delta_1^{-1} \left(\alpha_1 \, x_0^0 + \alpha_2 \, \theta_1^0 + \alpha_3 \, x_1^{-1} + \alpha_3 \, x_1^1\right) \tag{A.2}$$

where $\Delta_1 := \alpha_1 + \alpha_2 + 2\alpha_3 > 0$. This choice is feasible (in X) since it is a convex combination of elements of X, a convex set by assumption. The objective function (A.1) is strictly concave in x_1^0, thus x_1^0 in (A.2) is the unique optimizer. The form in (A.2) suggests that showing the existence of a symmetric equilibrium in the continuation given history (x^{t-1}, θ^t) is equivalent to finding the fixed point of an operator $L_1 : B\left((\mathbf{X} \times \mathbf{\Theta})^t, X\right) \to B\left((\mathbf{X} \times \mathbf{\Theta})^t, X\right)$ that acts on the class of bounded measurable functions $x_1 : (\mathbf{X} \times \mathbf{\Theta})^t \to X$ according to

$$(L_1 x_1)\left(x^{t-1}, \theta^t\right) = \Delta_1^{-1} \left(\alpha_1 \, x_0^0 + \alpha_2 \, \theta_1^0 + \alpha_3 \, x_1\left(R^{-1} \, x^{t-1}, R^{-1} \, \theta^t\right) + \alpha_3 \, x_1\left(R \, x^{t-1}, R \, \theta^t\right)\right)$$

[52] We laid out the problem in its recursive form for clarity in Section 2. Our method of proof attacks the sequence problem directly.

Clearly, L_1 is a self-map. We show next that it is a contraction. Endow $B\left((\mathbf{X}\times\mathbf{\Theta})^t, X\right)$ with the sup norm which makes $\left(B\left((\mathbf{X}\times\mathbf{\Theta})^t, X\right), ||\cdot||_\infty\right)$ a Banach space. Pick $x_1, \hat{x}_1 \in B\left((\mathbf{X}\times\mathbf{\Theta})^t, X\right)$. We have for all $\left(x^{t-1}, \theta^t\right)$

$$
\begin{aligned}
&\Big|\left(L_1 x_1\right)\left(x^{t-1},\theta^t\right) - \left(L_1 \hat{x}_1\right)\left(x^{t-1},\theta^t\right)\Big| \\
&\qquad = \Delta_1^{-1}\Big|\alpha_1\, x_0^0 + \alpha_2\,\theta_1^0 + \alpha_3\, x_1\left(R^{-1}\, x^{t-1}, R^{-1}\,\theta^t\right) + \alpha_3\, x_1\left(R\, x^{t-1}, R\,\theta^t\right) \\
&\qquad - \alpha_1\, x_0^0 - \alpha_2\,\theta_1^0 - \alpha_3\,\hat{x}_1\left(R^{-1}x^{t-1}, R^{-1}\theta^t\right) - \alpha_3\,\hat{x}_1\left(R\, x^{t-1}, R\,\theta^t\right)\Big| \\
&\qquad = \Delta_1^{-1}\Big|\alpha_3\,\left(x_1\left(R^{-1}\, x^{t-1}, R^{-1}\,\theta^t\right) - \hat{x}_1\left(R^{-1}\, x^{t-1}, R^{-1}\,\theta^t\right)\right) \\
&\qquad + \alpha_3\,\left(x_1\left(R\, x^{t-1}, R\,\theta^t\right) - \hat{x}_1\left(R\, x^{t-1}, R\,\theta^t\right)\right)\Big| \\
&\qquad \le \left(\frac{\alpha_3}{\Delta_1}\right)\Big|x_1\left(R^{-1}\, x^{t-1}, R^{-1}\,\theta^t\right) - \hat{x}_1\left(R^{-1}\, x^{t-1}, R^{-1}\,\theta^t\right)\Big| \\
&\qquad + \left(\frac{\alpha_3}{\Delta_1}\right)\Big|x_1\left(R\, x^{t-1}, R\,\theta^t\right) - \hat{x}_1\left(R\, x^{t-1}, R\,\theta^t\right)\Big| \\
&\qquad \le \left(\frac{2\,\alpha_3}{\Delta_1}\right)\|x_1 - \hat{x}_1\|_\infty
\end{aligned}
$$

The coefficient $2\alpha_3\,\Delta_1^{-1} < 1$ since $\alpha_i > 0$, for $i = 1, 2, 3$. Hence L_1 is a contraction on $B\left((\mathbf{X}\times\mathbf{\Theta})^t, X\right)$. Thus, by Banach Fixed Point Theorem (see e.g., Aliprantis and Border (2006), p.95) L_1 has a unique fixed point x_1^* in $B\left((\mathbf{X}\times\mathbf{\Theta})^t, X\right)$. Next, we argue that this equilibrium strategy must be Markovian and should take the convex combination form as in the statement of Theorem 1.

Lemma 1 (Markov Property and the Convex Combination Form) *Unique symmetric equilibrium strategy x_1^* is* Markovian*: For any t-length history (x^{t-1}, θ^t), it depends solely on last period equilibrium choices and current preference shock realizations, i.e. $x_1^*(x^{t-1}, \theta^t) = g_1(x_0, \theta_1)$, for some $g_1 : \mathbf{X}\times\mathbf{\Theta} \to X$. Moreover, the Markovian policy function g_1 has the convex combination form as in the statement of the theorem.*

Proof: Let

$$
G := \left\{
\begin{array}{l}
g : \mathbf{X}\times\mathbf{\Theta} \to X \text{ s.t. } g(x,\ \theta) = \sum_{a\in\mathbb{A}} c^a\, x^a + \sum_{a\in\mathbb{A}} d^a\,\theta^a + e\,\overline{\theta} \\
\text{with} \\
\text{ (i) } c^a, d^a, e \ge 0 \text{ and } e + \sum_{a\in\mathbb{A}}(c^a + d^a) = 1 \\
\text{ (ii) } (\tfrac{1}{2})c^{a+1} + (\tfrac{1}{2})c^{a-1} \ge c^a, \forall a \neq 0 \\
\text{ (iii) } c^b \le c^a, \forall a, b \in \mathbb{A} \text{ with } |b| > |a|. \\
\text{ (iv) } c^a = c^{-a},\ \forall a \in \mathbb{A} \\
\text{and properties (ii), (iii), and (iv) also holding for the } d = (d^a)_{a\in\mathbb{A}} \text{ sequence.}
\end{array}
\right\} \tag{A.3}
$$

be the class of functions that are convex combinations (i) of one-period before history, current types and average type, having the (ii) 'convexity', (iii) 'monotonicity', and (iv) 'symmetry' properties. Property (ii) states that the rate of 'spatial' (cross-sectional) convergence of the policy weights is non-increasing in both directions, relative to the origin. Monotonicity property, (iii), has a very natural interpretation: agent b's effect on agent 0's marginal utility is smaller than agent a's effect on it, if a is closer to 0 than b is. Finally, (iv) says that the policy weights are symmetric around 0. Let $g \in G$ be such that after any history $(x^{t-1}, \theta^t) = (x_{-(t-1)}, \theta_{-(t-2)}, \ldots, x_{-1}, \theta_0, x_0, \theta_1)$

$$x_1(x^{t-1}, \theta^t) = g(x_0, \theta_1)$$

and let (c, d, e) be the coefficient sequence associated with g. Applying L_1 to x_1 (hence to g), we get

$$\begin{aligned}(L_1 x_1)\left(x^{t-1}, \theta^t\right) &= \Delta_1^{-1}\left(\alpha_1\, x_0^0 + \alpha_2\, \theta_1^0 + \alpha_3\, g\left(R^{-1}x_0, R^{-1}\theta_1\right) + \alpha_3\, g\left(R\, x_0, R\, \theta_1\right)\right) \\ &= \Delta_1^{-1}\left[\alpha_1\, x_0^0 + \alpha_2\, \theta_1^0 + \alpha_3\left(\sum_{a\in\mathbb{A}} c^a\, x_0^{a-1} + \sum_{a\in\mathbb{A}} d^a\, \theta_1^{a-1} + e\,\bar{\theta}\right)\right. \\ &+ \left.\alpha_3\left(\sum_{a\in\mathbb{A}} c^a\, x_0^{a+1} + \sum_{a\in\mathbb{A}} d^a\, \theta_1^{a+1} + e\,\bar{\theta}\right)\right] \end{aligned} \tag{A.4}$$

Rearranging terms gives

$$\begin{aligned} &= \Delta_1^{-1}\Bigg(x_0^0(\underbrace{\alpha_1 + \alpha_3\, c^{-1} + \alpha_3\, c^1}_{\Delta_1 \hat{c}^0}) + \theta_1^0(\underbrace{\alpha_2 + \alpha_3\, d^{-1} + \alpha_3\, d^1}_{\Delta_1 \hat{d}^0}) + 2\alpha_3\, e\,\bar{\theta} \\ &+ \sum_{a\neq 0}(\underbrace{\alpha_3\, c^{a-1} + \alpha_3\, c^{a+1}}_{\Delta_1 \hat{c}^a})\, x_0^a + \sum_{a\neq 0}(\underbrace{\alpha_3\, d^{a-1} + \alpha_3\, d^{a+1}}_{\Delta_1 \hat{d}^a}\, \theta_1^a \Bigg) \end{aligned} \tag{A.5}$$

The function after the last equality sign is linear in x_0, θ_1 and $\bar{\theta}$. So, $L_1 x_1$ preserves the same linear form. By definition of the new coefficient sequence $(\hat{c}, \hat{d}, \hat{e})$ in (A.5), each element of the sequence is nonnegative since each element of the original one was so. New coefficients sum up to 1 since convex combination form of g makes the sum of the coefficients inside the two parentheses on the right hand side of (A.4) equal to 1. Thus, the total sum of coefficients on the right hand side of (A.4) is $\Delta_1^{-1}(\alpha_1 + \alpha_2 + 2\alpha_3) = 1$, which proves property (i). The final form in (A.5) is just a regrouping of elements in (A.4). Let $(\hat{c}^a)_{a\in\mathbb{A}}$ be the new coefficient sequence associated with

$L_1 x_T$ as defined in equation (A.5). Pick $a \neq 0$ in $\mathbb{A}$,

$$\begin{aligned}
\hat{c}^{a+1} + \hat{c}^{a-1} &\geq \left(\frac{\alpha_3}{\Delta_1}\right)\left(c^a + c^{a+2}\right) + \left(\frac{\alpha_3}{\Delta_1}\right)\left(c^{a-2} + c^a\right) \\
&\geq \left(\frac{\alpha_3}{\Delta_1}\right)\left(2c^{a+1} + 2c^{a-1}\right) \\
&= 2\left(\frac{\alpha_3}{\Delta_1}\right)\left(c^{a+1} + c^{a-1}\right) \\
&= 2\hat{c}^a
\end{aligned}$$

By definition of $\hat{c}$ in (A.5), first inequality is strict if $|a| = 1$, is an equality otherwise; second inequality is by property (ii) on c; last equality is once again by definition of $\hat{c}$ in (A.5). Therefore, for any $a \neq 0$ in $\mathbb{A}$, $\hat{c}^{a+1} + \hat{c}^{a-1} \geq 2\hat{c}^a$, which is property (ii). Pick any $a, b \in \mathbb{A}$ with $|a| < |b|$.

$$\begin{aligned}
\hat{c}^a &= \left(\frac{\alpha_3}{\Delta_1}\right) c^{a-1} + \left(\frac{\alpha_3}{\Delta_1}\right) c^{a+1} = \left(\frac{\alpha_3}{\Delta_1}\right) c^{|a|-1} + \left(\frac{\alpha_3}{\Delta_1}\right) c^{|a|+1} \\
&\geq \left(\frac{\alpha_3}{\Delta_1}\right) c^{|b|-1} + \left(\frac{\alpha_3}{\Delta_1}\right) c^{|b|+1} = \left(\frac{\alpha_3}{\Delta_1}\right) c^{b-1} + \left(\frac{\alpha_3}{\Delta_1}\right) c^{b+1} \\
&= \hat{c}^b
\end{aligned}$$

First equality is from (A.5); second by property (iv) of G in (A.3); the inequality is property (iii) of G in (A.3); next equality is due to property (iv) of G again; and finally the last equality is by (A.5). Hence, property (iii) in (A.3) holds for the new sequence. We next show that $\hat{c}$ satisfies (iv) in (A.3).

$$\begin{aligned}
\hat{c}^a &= \left(\frac{\alpha_3}{\Delta_1}\right) c^{a-1} + \left(\frac{\alpha_3}{\Delta_1}\right) c^{a+1} \\
&= \left(\frac{\alpha_3}{\Delta_1}\right) c^{-a-1} + \left(\frac{\alpha_3}{\Delta_1}\right) c^{-a+1} \\
&= \hat{c}^{-a}
\end{aligned}$$

where first equality is by (A.5); the second is due to (iv) of G in (A.3); finally the last is again by (A.5).

Thus, the restriction of L_1 to the subspace (call it B_G) of bounded measurable functions that agree with an element of G after any history, maps elements of B_G into itself. Moreover, endowed with the sup norm, B_G is a closed subset of $B\left((\mathbf{X} \times \mathbf{\Theta})^t, X\right)$ since it is defined by equality and inequality constraints, hence a complete metric space in its own right. Since L_1 is a contraction on this latter as we just showed, it is so on B_G too and the unique fixed point x_1^* in $B\left((\mathbf{X} \times \mathbf{\Theta})^t, X\right)$ must lie in B_G. Since the choice of t was arbitrary, the unique symmetric equilibrium in a one-period (continuation) economy, after any length history must be Markovian and should assume the convex combination form stated in the theorem. This concludes the proof of Lemma 1. ■

This proves Step 1, namely that the statement of the Theorem is true for 1-period economies. Next, we prove that this result generalizes to any finite-horizon economy.

Step 2: Induction, T-1 implies T. Let $T \geq 2$. Assume that the statement of Theorem 1 is true up to $T-1$-period. The T-period economy can be *separated* into a first period and a $T-1$-period continuation economy. By hypothesis, there exists a unique symmetric MPE, $g : \mathbf{X} \times \mathbf{\Theta} \times \{1, \cdots, T-1\} \mapsto X$, for the $T-1$-period continuation economy. Agent 0 believes that all other agents, including his own reincarnations, will use that unique symmetric equilibrium map from period 2 on, i.e., for any agent $b \in \mathbb{A}$,

$$x_t^b \left(x^{t-1}, \theta^t\right) = g_{T-(t-1)}(R^b\, x_{t-1}, R^b\, I_0\, \theta_t), \qquad \text{for all } t = 2, \ldots, T$$

Given any t-length history (x^{t-1}, θ^t), the current strategies of all other agents $(x_1^b)_{b \neq 0}$, and the fact that $(x_t^b)_{t \geq 2}^{b \in \mathbb{A}}$ are induced by g, agent 0 solves

$$\max_{x_1^0 \in X} \Bigg\{ -\alpha_1 \left(x_0^0 - x_1^0\right)^2 - \alpha_2 \left(\theta_1^0 - x_1^0\right)^2 - \alpha_3 \left(x_1^{-1} - x_1^0\right)^2 - \alpha_3 \left(x_1^1 - x_1^0\right)^2 \tag{A.6}$$
$$+E\Bigg[\sum_{\tau=2}^{T} \beta^{\tau-1} \Big(-\alpha_1 \left(x_{\tau-1}^0 - x_\tau^0\right)^2 - \alpha_2 \left(\theta_\tau^0 - x_\tau^0\right)^2$$
$$-\alpha_3 \left(x_\tau^{-1} - x_\tau^0\right)^2 - \alpha_3 \left(x_\tau^1 - x_\tau^0\right)^2\Big) \Bigg| \left(x^{t-1}, \theta^t\right) \Bigg] \Bigg\}$$

The form of the optimal choices on the equilibrium path can be characterized as in the following Lemma.

Lemma 2 (Convexity and Monotonicity) *Given a T-period economy, equilibrium choices satisfy the following properties:*

(i) For any period $t \geq 2$, x_t^a can be written as [53]

$$x_t^a = \sum_{b_1 \in \mathbb{A}} \cdots \sum_{b_{t-1} \in \mathbb{A}} c_{T-(t-1)}^{b_1} \cdots c_{T-1}^{b_{t-1}}\, x_1^{a+b_1+\cdots+b_{t-1}} \tag{A.7}$$
$$+ \sum_{s=1}^{t-1} \sum_{b_1 \in \mathbb{A}} \cdots \sum_{b_{s-1} \in \mathbb{A}} c_{T-(t-1)}^{b_1} \cdots c_{T-(t-(s-1))}^{b_{s-1}} \left(\sum_{b_s \in \mathbb{A}} d_{T-(t-s)}^{b_s} \theta_{t-(s-1)}^{a+b_1+\cdots+b_s} + e_{T-(t-s)}\, \overline{\theta} \right)$$

[53] We use in expression (A.7) the convention that in the sum after the plus sign, for $s = 1$, the summand becomes $\sum_{b_s \in \mathbb{A}} d_{T-(t-s)}^{b_s} \theta_{t-(s-1)}^{a+b_1+\cdots+b_s} + e_{T-(t-s)}\, \overline{\theta}$.

(ii) For any $a, b \in \mathbb{A}$, any $t \geq 2$,

$$|a| \leq |b| \implies \frac{\partial x_t^0}{\partial x_1^a} \leq \frac{\partial x_t^0}{\partial x_1^b}$$

(iii) $\frac{\partial}{\partial x_1^a}\left(2x_t^0 - x_t^1 - x_t^{-1}\right) \leq 0$

(iv) $\frac{\partial}{\partial x_1^0}\, x_t^0 \leq \left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right) \frac{\partial}{\partial x_1^0} x_{t-1}^0$

Proof: *(i)* This part is simply by iterated application of the policy maps, i.e.,

$$\begin{aligned}
x_t^a &= g_{T-(t-1)}(R^a\, x_{t-1}, R^a\, \theta_t) \\
&= \sum_{b_1\in\mathbb{A}} c_{T-(t-1)}^{b_1}\, x_{t-1}^{a+b_1} + \sum_{b_1\in\mathbb{A}} d_{T-(t-1)}^{b_1}\, \theta_t^{a+b_1} + e_{T-(t-1)}\, \overline{\theta} \\
&= \sum_{b_1\in\mathbb{A}} c_{T-(t-1)}^{b_1}\, \underbrace{g_{T-t}(R^{a+b_1}\, x_{t-2}, R^{a+b_1}\, \theta_{t-1})}_{x_{t-1}^{a+b_1}} + \sum_{b_1\in\mathbb{A}} d_{T-(t-1)}^{b_1}\, \theta_t^{a+b_1} + e_{T-(t-1)}\, \overline{\theta}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{b_1\in\mathbb{A}} c_{T-(t-1)}^{b_1} \left(\sum_{b_2\in\mathbb{A}} c_{T-(t-2)}^{b_2}\, x_{t-2}^{a+b_1+b_2} + \sum_{b_2\in\mathbb{A}} d_{T-(t-2)}^{b_2}\, \theta_{t-1}^{a+b_1+b_2} + e_{T-(t-2)}\, \overline{\theta} \right) \\
&\quad + \sum_{b_1\in\mathbb{A}} d_{T-(t-1)}^{b_1}\, \theta_t^{a+b_1} + e_{T-(t-1)}\, \overline{\theta} \\
&\vdots \\
&= \sum_{b_1\in\mathbb{A}} \cdots \sum_{b_{t-1}\in\mathbb{A}} c_{T-(t-1)}^{b_1} \cdots c_{T-1}^{b_{t-1}}\, x_1^{a+b_1+\cdots+b_{t-1}} \\
&+ \sum_{s=1}^{t-1} \sum_{b_1\in\mathbb{A}} \cdots \sum_{b_{s-1}\in\mathbb{A}} c_{T-(t-1)}^{b_1} \cdots c_{T-(t-s)+1}^{b_{s-1}} \left(\sum_{b_s\in\mathbb{A}} d_{T-(t-s)}^{b_{\tau+1}} \theta_{t-(s-1)}^{a+b_1+\cdots+b_\tau} + e_{T-(t-s)}\, \overline{\theta} \right)
\end{aligned}$$

(ii) For $t = 2$, $\frac{\partial}{\partial x_1^a} x_2^0 = c_1^a \geq c_1^b = \frac{\partial}{\partial x_1^b} x_2^0$ by (A.3) (iii). Suppose the claim is true for $t \leq k$ and let $t = k+1$. Assume w.l.o.g that $a < b$. Let $\underline{s} := \max\{s \in \mathbb{A} :\ s \leq \frac{a+b}{2}\}$ and $\bar{s} : \min = \{s \in \mathbb{A} : s \geq \frac{a+b}{2}\}$. This implies that $\frac{\partial}{\partial x_1^a}\, x_k^s - \frac{\partial}{\partial x_1^b}\, x_k^s \geq 0$ (≤ 0) for $s \leq \underline{s}$ ($s \geq \bar{s}$). Due to the assumed symmetry, $\left[\frac{\partial}{\partial x_1^a}\, x_k^{\underline{s}-s} - \frac{\partial}{\partial x_1^b}\, x_k^{\underline{s}-s}\right] = \left[\frac{\partial}{\partial x_1^{a-\underline{s}+s}}\, x_k^0 - \frac{\partial}{\partial x_1^{b-\underline{s}+s}}\, x_k^0\right]$ and $\left[\frac{\partial}{\partial x_1^{a-\bar{s}-s}}\, x_k^0 - \frac{\partial}{\partial x_1^{b-\bar{s}-s}}\, x_k^0\right] = \left[\frac{\partial}{\partial x_1^a}\, x_k^{\bar{s}} - \frac{\partial}{\partial x_1^b}\, x_k^{\bar{s}}\right]$. This implies that for any $s > 0$

$$\left[\frac{\partial}{\partial x_1^{a-\underline{s}+\tau}}\, x_k^0 - \frac{\partial}{\partial x_1^{b-\underline{s}+\tau}}\, x_k^0\right] \;=\; -\left[\frac{\partial}{\partial x_1^{a-\bar{s}-\tau}}\, x_k^0 - \frac{\partial}{\partial x_1^{b-\bar{s}-\tau}}\, x_k^0\right]$$

Thus, we can use this to separate $\mathbb{A}$ into $\{s \in \mathbb{A} : s \leq \underline{s}\}$ $\{s \in \mathbb{A} : s \geq \bar{s}\}$ and rearrange the sum

$$\begin{aligned}
\frac{\partial}{\partial x_1^a} x_t^0 - \frac{\partial}{\partial x_1^b} x_t^0 &= \sum_{s \in \mathbb{A}} c_{T-k}^s \left[\frac{\partial}{\partial x_1^a} x_k^s - \frac{\partial}{\partial x_1^b} x_k^s \right] \\
&= \sum_{s \in \mathbb{A}} c_{T-k}^s \left[\frac{\partial}{\partial x_1^{a-s}} x_k^0 - \frac{\partial}{\partial x_1^{b-s}} x_k^0 \right] \\
&= \sum_{\tau \geq 0} \left(c_{T-k}^{\underline{s}-s} - c_{T-k}^{\bar{s}+s} \right) \left[\frac{\partial x_k^0}{\partial x_1^{a-\underline{s}+s}} - \frac{\partial x_k^0}{\partial x_1^{b-\underline{s}+s}} \right] \\
&\geq 0
\end{aligned}$$

The term in the brackets is nonnegative by hypothesis. Since $a < b$, $\underline{s} \geq 0$ which implies that $c_{T-k}^{\underline{s}} \geq c_{T-k}^{\bar{s}}$. But this implies that $c_{T-k}^{\underline{s}-s} \geq c_{T-k}^{\bar{s}+s}$ for any $s \geq 0$ which means that the argument in the parenthesis is nonnegative too. So, the claim is true. The analysis for the case $a > b$ is a straightforward modification of the same argument.

(iii) Using the t-th period equilibrium policy

$$\begin{aligned}
\frac{\partial}{\partial x_1^a} \left(2x_t^0 - x_t^1 - x_t^{-1} \right) &= \frac{\partial}{\partial x_1^a} \Bigg[2 \sum_{b \in \mathbb{A}} c_{T-(t-1)}^b \, x_{t-1}^b + \sum_{b \in \mathbb{A}} d_{T-(t-1)}^b \, \theta_t^b + e_{T-(t-1)} \, \overline{\theta} \\
&- \sum_{b \in \mathbb{A}} c_{T-(t-1)}^{b-1} \, x_{t-1}^b + \sum_{b \in \mathbb{A}} d_{T-(t-1)}^{b-1} \, \theta_t^b + e_{T-(t-1)} \, \overline{\theta} \\
&- \sum_{b \in \mathbb{A}} c_{T-(t-1)}^{b+1} \, x_{t-1}^b + \sum_{b \in \mathbb{A}} d_{T-(t-1)}^{b+1} \, \theta_t^b + e_{T-(t-1)} \, \overline{\theta} \Bigg] \\
&= \sum_{b \in \mathbb{A}} \left(2c_{T-(t-1)}^b - c_{T-(t-1)}^{b-1} - c_{T-(t-1)}^{b+1} \right) \frac{\partial}{\partial x_1^a} \, x_{t-1}^b \leq 0 \qquad \text{(A.8)}
\end{aligned}$$

The weights in the last parenthesis are negative by property (ii) in (A.3). By iteratively applying the policy functions from period t backwards, at each iteration the weights on one-period before choices would all be positive and one preserves the convex combination form. This process ends after $t-1$ iteration, the end result being a convex combination of $(x_1^b)_{b \in \mathbb{A}}$, θ^t and $\bar{\theta}$. Thus, the weight on x_1^a is positive, which makes the last term in the last line positive. Therefore the claim is true.

(iv) Let $t \geq 2$.

$$\begin{aligned}
\frac{\partial}{\partial x_1^0} x_t^0 &= \sum_{a \in \mathbb{A}} c_{T-(t-1)}^a \frac{\partial x_{t-1}^a}{\partial x_1^0} = \sum_{a \in \mathbb{A}} c_{T-(t-1)}^a \frac{\partial x_{t-1}^0}{\partial x_1^a} \\
&\leq \sum_{a \in \mathbb{A}} c_{T-(t-1)}^a \frac{\partial x_{t-1}^0}{\partial x_1^0} = C_{T-(t-1)} \frac{\partial x_{t-1}^0}{\partial x_1^0}
\end{aligned}$$

First and second equalities and the first inequality are by the definition of the policy mapping and (i) of Lemma 2; $C_{T-(t-1)}$ is the sum of coefficients on the past history in the period t policy.

Since $g_{T-(t-1)}$ satisfies (A.12), coefficients should match and we should have

$$0 = c^a_{T-(t-1)}\,\Delta_{T-(t-1)} - \alpha_1\, I_{\{a=0\}} - \sum_{b\neq 0} \gamma^b_{T-(t-1)}\, c^{a-b}_{T-(t-1)}$$

$$\text{summing over a,} \qquad 0 = C_{T-(t-1)}\Delta_{T-(t-1)} - \alpha_1 - \sum_{b\neq 0} \gamma^b_{T-(t-1)} C_{T-(t-1)}$$

But $\Delta_{T-(t-1)} = \alpha_1 + \alpha_2 + \sum_{b\neq 0}\gamma^b_{T-(t-1)} + \mu_{T-(t-1)}$ by definition. So,

$$C_{T-(t-1)} = \frac{\alpha_1}{\Delta_{T-(t-1)} - \sum_{b\neq 0}\gamma^b_{T-(t-1)}} = \frac{\alpha_1}{\alpha_1+\alpha_2+\mu_{T-(t-1)}} \leq \frac{\alpha_1}{\alpha_1+\alpha_2} \tag{A.9}$$

Thus,

$$\frac{\partial}{\partial x^0_1} x^0_t \;\leq\; C_{T-(t-1)} \frac{\partial}{\partial x^0_1} x^0_{t-1} \leq \left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right) \frac{\partial}{\partial x^0_1} x^0_{t-1}$$

which concludes the proof of Lemma 2. ■

Thanks to the linearity of the optimal future choices as shown in Lemma 2, agent 0's problem (A.6) is differentiable with respect to x^0_1 and the unconstrained ($x^0_1 \in \mathbb{R}$) first order condition is

$$\begin{aligned} 0 &= \alpha_1\left(x^0_0 - x^0_1\right) + \alpha_2\left(\theta^0_1 - x^0_1\right) + \alpha_3\left(x^{-1}_1 - x^0_1\right) + \alpha_3\left(x^1_1 - x^0_1\right) \\ &+ E\Bigg[\sum_{\tau=2}^{T}\beta^{\tau-1}\Bigg(-\alpha_1\left(x^0_{\tau-1} - x^0_\tau\right)\frac{\partial}{\partial x^0_1}\left(x^0_{\tau-1} - x^0_\tau\right) + \alpha_2\left(\theta^0_\tau - x^0_\tau\right)\frac{\partial}{\partial x^0_1} x^0_\tau \\ &- \alpha_3\left(x^{-1}_\tau - x^0_\tau\right)\frac{\partial}{\partial x^0_1}\left(x^{-1}_\tau - x^0_\tau\right) - \alpha_3\left(x^1_\tau - x^0_\tau\right)\frac{\partial}{\partial x^0_1}\left(x^1_\tau - x^0_\tau\right)\Bigg)\,\Bigg|\,\left(x^{t-1}, \theta^t\right)\Bigg] \end{aligned} \tag{A.10}$$

Agent 0's problem (A.6) is strictly concave in his choice x^0_1 since the second partial of the objective function with respect to x^0_1, $-\Delta_T$ by definition, is negative, or

$$\begin{aligned} \Delta_T \;:=\; &\alpha_1 + \alpha_2 + 2\alpha_3 + \sum_{\tau=2}^{T}\beta^{\tau-t}\Bigg(\alpha_1\left(\frac{\partial}{\partial x^0_1}\left(x^0_{\tau-1} - x^0_\tau\right)\right)^2 + \alpha_2\left(\frac{\partial}{\partial x^0_1} x^0_\tau\right)^2 \\ &+\alpha_3\left(\frac{\partial}{\partial x^0_1}\left(x^{-1}_\tau - x^0_\tau\right)\right)^2 + \alpha_3\left(\frac{\partial}{\partial x^0_1}\left(x^1_\tau - x^0_\tau\right)\right)^2\Bigg) > 0 \end{aligned} \tag{A.11}$$

Consequently, the FOC characterizes the *unique maximizer* of the unconstrained problem ($x^0_1 \in \mathbb{R}$). The following Lemma shows that equation (A.10) has a much simpler representation.

Lemma 3 (Interiority) *Equation (A.10) can be written alternatively as*

$$0 = -x^0_1\,\Delta_T + \alpha_1\, x^0_0 + \alpha_2\,\theta^0_1 + \sum_{a\neq 0}\gamma^a_T\, x^a_1 + \mu_T\,\bar\theta \tag{A.12}$$

where $\Delta_T := \alpha_1 + \alpha_2 + \sum_{a\neq 0}\gamma^a_T + \mu_T$ and the coefficients α_1, α_2, $(\gamma^a_T)_{a\neq 0}$, and μ_T are non-negative.

Proof: The coefficient of x_1^a in (A.12), γ_T^a, is the total effect of a change in x_1^a ($a \neq 0$) on the expected discounted marginal utility of agent 0 (the right hand side of (A.10)), i.e.,

$$\begin{aligned}
\gamma_T^a \;:=\; & \alpha_3\, I_{\{a\in\{-1,1\}\}} \\
& -\sum_{\tau=2}^{T}\beta^{\tau-1}\Big(\alpha_1\,\frac{\partial}{\partial x_1^a}\left(x_{\tau-1}^0-x_\tau^0\right)\frac{\partial}{\partial x_1^0}\left(x_{\tau-1}^0-x_\tau^0\right)+\alpha_2\,\frac{\partial}{\partial x_1^a}\,x_\tau^0\,\frac{\partial}{\partial x_1^0}\,x_\tau^0 \\
& +\alpha_3\,\frac{\partial}{\partial x_1^a}\left(x_\tau^{-1}-x_\tau^0\right)\frac{\partial}{\partial x_1^0}\left(x_\tau^{-1}-x_\tau^0\right)+\alpha_3\,\frac{\partial}{\partial x_1^a}\left(x_\tau^1-x_\tau^0\right)\frac{\partial}{\partial x_1^0}\left(x_\tau^1-x_\tau^0\right)\Big)
\end{aligned} \tag{A.13}$$

For any $\tau \geq 2$, the last two term in the summand for each period in equation (A.13) can be written as

$$\begin{aligned}
& \frac{\partial}{\partial x_1^0}\left(x_\tau^1-x_\tau^0\right)\left[\alpha_3\,\frac{\partial}{\partial x_1^a}\left(x_\tau^{-1}-x_\tau^0\right)+\alpha_3\,\frac{\partial}{\partial x_1^a}\left(x_\tau^1-x_\tau^0\right)\right] \\
= \;& \frac{\partial}{\partial x_1^0}\left(x_\tau^1-x_\tau^0\right)\left[\alpha_3\,\frac{\partial x_\tau^0}{\partial x_1^{a+1}}+\alpha_3\,\frac{\partial x_\tau^0}{\partial x_1^{a-1}}-2\alpha_3\frac{\partial x_\tau^0}{\partial x_1^a}\right] \\
\leq \;& 0
\end{aligned} \tag{A.14}$$

The equality is due to the symmetry of the policy function across agents; Lemma 2 *(ii)* and *(iii)* imply that the terms in the parentheses are non-positive and the terms in the brackets are non-negative, respectively. Similarly, the first terms in the summand in (A.13) can be written as

$$\begin{aligned}
& \alpha_1\,\frac{\partial}{\partial x_1^a}\left(x_\tau^0-x_{\tau-1}^0\right)\frac{\partial}{\partial x_1^0}\left(x_\tau^0-x_{\tau-1}^0\right)+\alpha_2\,\frac{\partial}{\partial x_1^a}\,x_\tau^0\frac{\partial}{\partial x_1^0}\,x_\tau^0 \\
\leq \;& \alpha_1\,\frac{\partial}{\partial x_1^a}x_\tau^0\frac{\partial}{\partial x_1^0}\left(x_\tau^0-x_{\tau-1}^0\right)+\alpha_2\,\frac{\partial}{\partial x_1^a}x_\tau^0\frac{\partial}{\partial x_1^0}\,x_\tau^0 \\
= \;& \frac{\partial x_\tau^0}{\partial x_1^a}\left[\left(\alpha_1+\alpha_2\right)\frac{\partial}{\partial x_1^0}x_\tau^0-\alpha_1\frac{\partial}{\partial x_1^0}x_{\tau-1}^0\right] \\
\leq \;& 0
\end{aligned}$$

which is nonpositive since for any $\tau \geq 2$

$$\frac{\partial}{\partial x_1^0}\,x_\tau^0 \leq \frac{\alpha_1}{\left(\alpha_1+\alpha_2\right)}\frac{\partial}{\partial x_1^0}x_{\tau-1}^0$$

due to Lemma 2 *(iv)*. Thus, we established the non-positiveness of each term of the summand for any period $\tau \geq 2$ in (A.13). Since, the latter is basically a finite weighted some of such terms with a negative sign in front, for any $a \in \mathbb{A}$, $\gamma_T^a \geq 0$. Finally we account for the coefficients multiplying $\bar{\theta}$ in equation (A.10) and show that

$$\begin{aligned}
\mu_T \;=\; & \frac{\partial}{\partial\bar{\theta}}\,E\Bigg[\sum_{\tau=2}^{T}\beta^{\tau-1}\Big(-\alpha_1\,\left(x_{\tau-1}^0-x_\tau^0\right)\frac{\partial}{\partial x_1^0}\left(x_{\tau-1}^0-x_\tau^0\right)+\alpha_2\,\left(\theta_\tau^0-x_\tau^0\right)\frac{\partial}{\partial x_1^0}\,x_\tau^0 \\
& -\alpha_3\,\left(x_\tau^{-1}-x_\tau^0\right)\frac{\partial}{\partial x_1^0}\left(x_\tau^{-1}-x_\tau^0\right)-\alpha_3\,\left(x_\tau^1-x_\tau^0\right)\frac{\partial}{\partial x_1^0}\left(x_\tau^1-x_\tau^0\right)\Big)\;\Bigg|\;\left(x^{t-1},\theta^t\right)\Bigg] \\
\geq \;& 0
\end{aligned} \tag{A.15}$$

Expectation washes out all individual θ_{τ}^{a}'s and we have only $\bar{\theta}$ apart from $(x_1^a)_{a\in\mathbb{A}}$ in each period's expression in (A.15). By symmetry of the form in Lemma (2) *(i)* across agents, the weight on $\bar{\theta}$ in x_t^0, is equal to the weight on $\bar{\theta}$ in x_t^1 and on x_t^{-1}. Thus, $\frac{\partial}{\partial\theta} E[(x_t^0 - x_t^1)] = \frac{\partial}{\partial\theta} E[(x_t^0 - x_t^{-1})] = 0$. This makes the second line of (A.15) equal to zero. By Lemma (2)-*(i)*, the weight on $\bar{\theta}$ in $E[x_t^0]$, $1 - \Pi_{s=2}^{t} C_{T-(t-1)}$ (residual of the sum of the effects of $\{x_1^b\}$) is bigger than that in $E[x_{t-1}^0]$, $1 - \Pi_{s=2}^{t-1} C_{T-(t-1)}$; hence the term $\frac{\partial}{\partial\theta} E[(x_{t-1}^0 - x_t^0)] \leq 0$. By Lemma (2)-*(i)*, $\frac{\partial}{\partial\theta} E[x_t^0] \geq 0$. By Lemma (2)-*(iv)*, $\frac{\partial}{\partial x_1^0}\left(x_t^0 - x_{t-1}^0\right) \leq 0$. All these together imply that the expression in (A.15) is non-negative. Each $E[x_\tau^b]$ in (A.10) can be written as a convex combination of $(x_1^a)_{a\in\mathbb{A}}, \bar{\theta}, x_0^0, \theta_1^0$, with the help of Lemma 2-*(i)*. Since at each iteration, convex combination structure is preserved, it is so at the end too. Then, the sum of coefficients in each of the differences involving those variables in the parentheses is zero. This in turn implies that the total sum of coefficients in (A.10) is zero. Thus, the alternative formulation in (A.12) is true. This concludes the proof of Lemma 3. ■

By isolating the choice x_1^0 on the left hand side, we can write the maximizer of the unconstrained problem as a convex combination of x_0^0, θ_1^0, $(x_1^a)_{a\neq 0}$ and $\bar{\theta}$

$$x_1^0 \quad = \quad \Delta_T^{-1}\left(\alpha_1\, x_0^0 + \alpha_2\, \theta_1^0 + \sum_{a\neq 0} \gamma_T^a\, x_1^a + \mu_T\, \bar{\theta}\right) \tag{A.16}$$

Each of these are elements of X, a convex set. Thus, the optimal choice of the unconstrained problem is in the feasible set of the constrained problem, hence it is its unique maximizer. The form in (A.16) implies that showing the existence of a symmetric equilibrium policy for the first period of a T-period economy is equivalent to finding the fixed point of an operator $L_T : B\left((\mathbf{X}\times\mathbf{\Theta})^t, X\right) \to B\left((\mathbf{X}\times\mathbf{\Theta})^t, X\right)$ that acts on the class of bounded measurable functions $x_1 : (\mathbf{X}\times\mathbf{\Theta})^t \to X$ according to

$$\left(L_T\; x_1\right)\left(x^{t-1}, \theta^t\right) = \Delta_T^{-1}\left(\alpha_1\, x_0^0 + \alpha_2\, \theta_1^0 + \sum_{a\neq 0} \gamma_T^a\, x_1\left(R^a\, x^{t-1}, R^a\, \theta^t\right) + \mu_T \bar{\theta}\right) \tag{A.17}$$

Clearly L_T is a self-map. Using straightforward modifications of the arguments in the proof of **Step 1**, one obtains for $x_1, \hat{x}_1 \in B\left((\mathbf{X}\times\mathbf{\Theta})^t, X\right)$ that

$$\Big|\left(L_T\, x_1\right)\left(x^{t-1}, \theta^t\right) - \left(L_T\, \hat{x}_1\right)\left(x^{t-1}, \theta^t\right)\Big| \leq \sum_{a\neq 0}\left(\frac{\gamma_T^a}{\Delta_T}\right) \|x_1 - \hat{x}_1\|_\infty$$

The coefficient $\sum_{a\neq 0}\left(\frac{\gamma_T^a}{\Delta_T}\right) < 1$ since $\alpha_i > 0$, $i = 1, 2, 3$. Thus, L_T is a contraction on the Banach space of bounded measurable functions $\left(B((\mathbf{X}\times\mathbf{\Theta})^t, X), ||\cdot||_\infty\right)$, hence has a unique fixed point

x_1^*. Once again, by the same token as in Lemma 1, perfect equilibria are necessarily Markovian thus we can focus attention on Markovian strategies. As in the proof of Lemma 1, it suffices to show that $L_T(B_G) \subset B_G$. To that end, let $x_1 \in B_G$ be such that there exists a $g \in G$ for which after any history (x^{t-1}, θ^t), one has $x_1(x^{t-1}, \theta^t) = g(x_0, \theta_1)$; let (c, d, e) be the coefficient sequence associated with g. Applying L_T to x_1, we get

$$(L_T\, x_1)\left(x^{t-1}, \theta^t\right) = \Delta_T^{-1}\Bigg(\underbrace{[\alpha_1 + \sum_{a \neq 0} \gamma_T^a\, c^{-a}]}_{\Delta_T \hat{c}^0}\, x_0^0 + \underbrace{[\alpha_2 + \sum_{a_1 \neq 0} \gamma_T^a\, d^{-a}]}_{\Delta_T \hat{d}^0}\, \theta_1^0$$
$$+ \sum_{b \neq 0}\left\{\underbrace{[\sum_{a \neq 0} \gamma_T^a\, c^{b-a}]}_{\Delta_T \hat{c}^b}\, x_0^b + \underbrace{[\sum_{a \neq 0} \gamma_T^a\, d^{b-a}]}_{\Delta_T \hat{d}^b}\, \theta_1^b\right\} + \underbrace{[\mu_T + e \sum_{a \neq 0} \gamma_T^a]}_{\Delta_T \hat{e}}\, \bar{\theta}\Bigg) \quad \text{(A.18)}$$

The expression above is linear in x_0, θ_1 and $\bar{\theta}$. So, $L_T\, x_1$ is linear. By definition of the new coefficient sequence $(\hat{c}, \hat{d}, \hat{e})$, each element of the new sequence is nonnegative since each element of the original one was so and the new elements are positive weighted sums of the original ones. The total sum of the coefficients on the right hand side of (A.18) is $\Delta_T^{-1}(\alpha_1 + \alpha_2 + \sum_{a \neq 0} \gamma_T^a + \mu_T) = 1$ since (A.17) (which is equivalent to (A.18)) is a convex combination of elements and of functions that are convex combinations of elements of the convex set X. This proves property (i). The proof of the properties (ii), (iii), and (iv) follows identical arguments as in Lemma 1. Thus, the unique fixed point x_1^* should lie in the set B_G with an associated equilibrium Markovian policy function g_T^{**}.

Therefore, when the symmetric continuation equilibrium policies are Markovian, i.e., $g : \mathbf{X} \times\ \boldsymbol{\Theta} \times \{1, \cdots, T-1\} \mapsto X$, after any history (x^{t-1}, θ^t), the unique symmetric equilibrium policy in the first period, g_T^{**} is Markovian too. Since the choice of t was arbitrary, this must be true for any length history. Now, construct the policy function g^* as $g_T^*(x_0, \theta_1) = g_T^{**}(x_0, \theta_1)$ for any initial (x_0, θ_1); and $g^*_{T-(t-1)}(x_{t-1}, \theta_t) = g_{T-(t-1)}(x_{t-1}, \theta_t)$, for all $t \in \{2, \cdots, T\}$ and all (x_{t-1}, θ_t). But then, the function g^* is by construction the unique MPE of the T-period economy. This completes the induction step for any given $T \geq 2$. Therefore, the claim in Theorem 1 is true for any finite horizon economy.

Step 3: Convergence and stationarity. This step proves that the sequence of finite horizon symmetric Markovian equilibria tends to a stationary symmetric Markov Perfect equilibrium. To do that, we treat finite-horizon economies as finite truncations of the infinite-horizon economy. Let $G^\infty := \prod_{t=1}^\infty G$ be the infinite-horizon Markovian strategy set. For a fixed discount factor $\beta \in (0,1)$, let $L_\beta := \{\beta_T \in [0,1]^\infty \mid \beta_{T,t} = \beta^{t-1},\ \text{for } 1 \leq t \leq T, \text{and } \beta_{T,t} = 0,\ \text{for } t > T,\ \text{where } T \in \{1, 2, \ldots\} \cup \{\infty\}\}$ be the space of exponentially declining sequences (at the rate β) that are equal to zero after the T-th element. Endow L_β with the sup norm.

Lemma 4 (Compactness) *L_β and G endowed with the supnorm are compact metric spaces.*

Proof: Let $(\beta_{T_n})_n$ be a sequence lying in L_β that converges to $x = (x_t) \in [0,1]^\infty$. This implies that $\beta_{T_n,t} \to x_t$, for all $t \geq 1$, which in turn means that $x_t \in \{0, \beta^t\}$ by the construction of L_β. Moreover, if $x_t = 0$ for some t, $x_{t+\tau} = 0$ for all $\tau \geq 1$ since the terms β_{T_n} are geometric (finite or infinite) sequences. There are two possibilities: either $x = (1, \beta, \ldots, \beta^T, 0, 0, \ldots)$ or $x = \beta^t$ for all $t \geq 1$. Both lie in L_β which means that the limit of any convergent sequence in L_β lies in L_β. This establishes that L_β is closed. Given any $\epsilon > 0$, choose $N \geq 1$, a natural number, s.t. $\beta^N < \epsilon$. It is easy to see that any element in L_β lies in the ϵ-neighborhood (with respect to the sup metric) of one of the elements in the finite set $\{\beta_1, \beta_2, \ldots, \beta_N\} \subset L_\beta$. This establishes that L_β is totally bounded. Therefore, L_β is compact. We next show that G endowed with the sup norm is compact.

Let $H := \left\{x = (x^a)_{a \in \mathbb{A}} \mid x^a \leq \left(\frac{1}{2a}\right), \text{ for all } a \in \mathbb{A}\right\}$. Defined by inequality constraints, this set is closed under the sup norm. We will show that it is also totally bounded. For a given $\epsilon > 0$, one can find an $N \geq 1$ s.t. $\frac{1}{2N} < \epsilon$. Pick a sequence $\bar{x} \in H$. For any $a \in \mathbb{A}$ s.t. $|a| \geq N$, $[0, (2N)^{-1}] \subset B_\infty(x^a, \epsilon)$, the ϵ-ball around x^a with respect to the sup norm. For $|a| \leq N$, let $Y(a) := \{0, \epsilon, 2\epsilon, \ldots, k_a\epsilon, (2a)^{-1}\}$, where k_a is the greatest integer s.t. $k_a\epsilon \leq (2a)^{-1}$. The set

$$\left\{x \in H \mid x^a = \bar{x}^a, \text{ for } |a| \geq N, \text{ and } (x^{-(N-1)}, \ldots, x^0, \ldots, x^{N-1}) \in \prod_{|a| \leq N} Y(a), \text{ for } |a| \leq N\right\}$$

is a finite set of elements of H. Moreover, it is dense in H by construction. This establishes that H is totally bounded. Thus, H is compact under the sup norm.

Each $g \in G$ is associated with coefficients $((c^a, d^a)_a, e)$. Clearly, for any sequence of policies in G, $g_n \to g$ in sup norm if and only if the associated coefficients $((c^a_n, d^a_n)_a, e_n) \to ((c^a, d^a)_a, e)$ in sup norm. We know from (A.3) that c satisfies properties $(i), (ii)$ and (iii). Thus, for any $a \in \mathbb{A}$, $c^0 > c^1 > \ldots > c^{|a|}$, $c^a = c^{-a}$ and $\sum_{|b| \leq |a|} c^b < 1$. Combining all these, we have $2|a|c^a < \sum_{|b| \leq |a|} c^b < 1$ which in turn implies that $c^a < \frac{1}{2|a|}$, for all $a \in \mathbb{A}$. Same bounds hold for the d sequence. But then, the space of associated coefficient sequences, call it L_G, can be seen as a closed subset of H, a compact metric. Consequently, L_G is compact, thus sequentially compact. Pick a sequence $(g_n) \in G$ and let (c_n, d_n, e_n) be the associated coefficient sequence lying in L_G. Since L_G is sequentially compact, there exists a subsequence $(c_{m_n}, d_{m_n}, e_{m_n}) \to (c, d, e) \in L_G$. The latter, being an admissible coefficient sequence, is associated with the policy $g(x, \theta) := \sum_a c^a x^a + \sum_a d^a \theta^a + e\bar{\theta}$. Thus, the respective policy subsequence $g_{m_n} \to g \in G$. This establishes that G is sequentially compact hence compact. This concludes the proof of Lemma 4.

■

Given $g \in G^\infty$, let $x^a(g)$ be agent a's *strategy induced by* g, i.e., $x^a(g)(x^{t-1}, \theta^t) = g_t(R^a\, x_{t-1}, R^a\, \theta_t)$, for all $a \in \mathbb{A}$ and all (x^{t-1}, θ^t). Define the objective function U for agent 0 in the class of truncated economies as $U : G^\infty \times L_\beta \times G^\infty$ as

$$U(g^0\ ;\ \beta_T, g) := E\left[\sum_{t=1}^{\infty} \beta_{T,t}\, u\left(x_{t-1}^0(g^0), x_t^0(g^0), \{x_t^b(g)\}_{b\in N(0)}, \theta_t^0\right) \,\Big|\, (x_0, \theta_1)\right]$$

where u represents the conformity preferences and $N(0) = \{-1, 1\}$ as in Assumption 1. Let the feasibility correspondence $\Gamma : L_\beta \times G^\infty \to G^\infty$ be defined for $T < \infty$ as $\Gamma(\beta_T, g) = \{g^0 \in G^\infty \mid g_t^0(x,\theta) = \bar{\theta}, \forall t > T, \forall(x,\theta) \in \mathbf{X} \times \mathbf{\Theta}\}$, and for $T = \infty$ as $\Gamma(\beta_\infty, g) = G^\infty$. It is easy to see, thanks to the monotonicity of Γ in T (through β_T) and the compactness of G that Γ is a compact-valued and continuous correspondence. Moreover, as the next Lemma shows, the parameterized objective function U is continuous in g^0, the choice variable.

Lemma 5 (Continuity) *For any given $(\beta_T, g) \in L_\beta \times G^\infty$, $U(\cdot;\ \beta_T, g)$ is continuous on $\Gamma(\beta_T, g)$ with respect to the product topology.*

Proof: Since G endowed with the sup norm is a compact metric space due to Lemma 4, the metric $d(g, g') := \sum_{t=1}^{\infty} 2^{-t} ||g_t - g_t'||_\infty$ induces the product topology on G^∞ (see e.g., Aliprantis and Border (2006, p. 90)), where $|| \cdot ||_\infty$ is the supnorm as before. Let $(\beta_T, g) \in L_\beta \times G^\infty$ and $\epsilon > 0$ be given. Set $\epsilon' := (\frac{1-\beta}{1-\beta^{T+1}})\,\epsilon$. The period utility u is uniformly continuous since X is compact. Thus, one can choose a $\delta' > 0$ such that for any t, $|x_t^0 - y_t^0| < \delta'$ implies

$$\left| u\left(x_{t-1}^0, x_t^0, \{x_t^b(g)\}_{b\in N(0)}, \theta_t^0\right) - u\left(y_{t-1}^0, y_t^0, \{x_t^b(g)\}_{b\in N(0)}, \theta_t^0\right)\right| < \epsilon'.$$

Set $\delta = 2^{-T}\delta'$. Pick $g^0, g'^0 \in \Gamma(\beta^T, g)$ such that $d(g^0, g'^0) < \delta$. This implies that for all $t \le T$, $||g_t^0 - g_t'^0||_\infty < 2^T\delta = \delta'$ hence $|x_t^0(g^0) - x_t^0(g'^0)| < \delta$. Uniform continuity of u then implies that the period utility levels are uniformly bounded above by ϵ' for all periods $t \le T$. The claim therefore follows from

$$|U(g^0\ ;\ \beta_T, g) - U(g'^0\ ;\ \beta_T, g)| \quad < \quad \frac{1-\beta^{T+1}}{1-\beta}\epsilon' = \epsilon$$

■

For every T-period symmetric Markovian equilibrium policy sequence g^{*T}, define $g^{**T} \in G^\infty$ as

$$\forall t, \forall(x,\theta) \in \mathbf{X} \times \mathbf{\Theta}, g_t^{**T}(x,\theta) := \begin{cases} g_{T-(t-1)}^{*T}(x,\theta), & \text{if } t \le T \\ \bar{\theta}, & \text{if } t > T \end{cases}$$

G^∞ endowed with the product topology is compact since each G endowed with the supnorm is compact from Lemma 4. Since product topology is metrizable, say with metric d,[54] (G^∞, d) is a compact metric space hence the sequence $(g^{**T})_T$ has a convergent subsequence $(g^{**T_n})_{T_n}$in G^∞ that converges say to $g^* \in G^\infty$. Let $M : L_\beta \times G^\infty \to G^\infty$ be the correspondence of maximizers of

[54]See Footnote 55 for an example of metrization of product topology.

U given the value of the parameters. Also, let $\mathcal{E} : L_\beta \to G^\infty$ be the symmetric equilibrium correspondence for the sequence of finite economies. Since g^{*T_n} is a symmetric Markovian equilibrium for any T_n, for all $g^{T_n} \in G^\infty$ we have

$$\begin{aligned}
U(g^*_{T_n}\ ;\ \beta_{T_n}, g^*_{T_n}) &= E\left[\sum_{t=1}^{\infty} \beta_{T_n,t} u\left(x^0_{t-1}(g^{*T_n}), x^0_t(g^{*T_n}), \{x^b_t(g^{*T_n})\}_{b\in N(0)}, \theta^0_t\right) \Big| (x_0,\theta_1)\right] \\
&= E\left[\sum_{t=1}^{T_n+1} \beta^{t-1} u\left(x^0_{t-1}(g^{*T_n}), x^0_t(g^{*T_n}), \{x^b_t(g^{*T_n})\}_{b\in N(0)}, \theta^0_t\right) \Big| (x_0,\theta_1)\right] \\
&\geq E\left[\sum_{t=1}^{T_n+1} \beta^{t-1} u\left(x^0_{t-1}(g^{T_n}), x^0_t(g^{T_n}), \{x^b_t(g^{*T_n})\}_{b\in N(0)}, \theta^0_t\right) \Big| (x_0,\theta_1)\right] \\
&= E\left[\sum_{t=1}^{\infty} \beta_{T_n,t} u\left(x^0_{t-1}(g^{T_n}), x^0_t(g^{T_n}), \{x^b_t(g^{*T_n})\}_{b\in N(0)}, \theta^0_t\right) \Big| (x_0,\theta_1)\right] \\
&= U(g_{T_n}\ ;\ \beta_{T_n}, g^*_{T_n})
\end{aligned}$$

Thus, $g^*_{T_n} \in M(\beta_{T_n}, g^*_{T_n})$ for all T_n. Since U is continuous in the choice dimension due to Lemma 5 and that the feasibility correspondence Γ is continuous, by the Maximum Theorem (see Berge (1963), p. 115), the correspondence of maximizers, M, is upper hemi-continuous. This implies that if $(\beta_{T_n}, g^*_{T_n}) \to (\beta_\infty, g^*)$, then $g^* \in M(\beta_\infty, g^*)$ hence g^* is a symmetric MPE of the infinite-horizon economy. This implies immediately that the equilibrium correspondence $\mathcal{E}$ is upper hemi-continuous too.

Uniqueness of finite-horizon symmetric MPEs imply that $\mathcal{E}$ is single-valued hence continuous for $T < \infty$. Define $\mathcal{F}(\beta_T) := \mathcal{E}(\beta_T)$, for $T < \infty$ and let $\mathcal{F}(\beta_\infty) = g^*$. This way, $\mathcal{F}$ is continuous on the space L_β, which is compact under the supnorm by Lemma 4. Consequently, $\mathcal{F}$ is uniformly continuous. This means, for a given $\epsilon > 0$, we can pick $\delta > 0$ small enough so that $||\beta_T - \beta_{T'}||_\infty < \delta$ implies $d\left(\mathcal{F}(\beta_T), \mathcal{F}(\beta_{T'})\right) < \frac{\epsilon}{2}$. We know from the previous approximation that for $\beta_T \to \beta_\infty$ there is a subsequence $g^{*T_n} \to g^*$. Since $(\beta_T)_T$ is convergent, it is Cauchy. So, choose $T(\delta)$ large enough such that $\forall T, T' \geq T(\delta)$, $||\beta_T - \beta_{T'}|| < \delta$ and $\forall T_n \geq T(\delta)$, $||g^{*T_n} - g^*||_\infty < \frac{\epsilon}{2}$. Pick, then, any element T_n of the subsequence and any other element, T' such that $T_n, T' \geq T(\delta)$. We have

$$\begin{aligned}
d\left(g^{*T'}, g^*\right) &= d\left(\mathcal{F}(\beta_{T'}), \mathcal{F}(\beta_\infty)\right) \\
&\leq d\left(\mathcal{F}(\beta_{T'}), \mathcal{F}(\beta_{T_n})\right) + d\left(\mathcal{F}(\beta_{T_n}), \mathcal{F}(\beta_\infty)\right) \\
&< \frac{\epsilon}{2} + d\left(g^{*T_n}, g^*\right) \\
&< \epsilon
\end{aligned}$$

The first inequality is the triangle inequality; the second is due to the uniform continuity of $\mathcal{F}$ and the third is by the fact that $g^{*T_n} \to g^*$ uniformly. This proves that the whole sequence $g^{*T} \to g^*$

uniformly. The implication of this latter is that, as the finite-horizon economies approach the infinite-horizon economy, every two consecutive period, we make choices approximately with respect to the same MPE policy, hence g^* is stationary. This concludes **Step 3** which in turn establishes the proof of the statement of Theorem 1. ■

It is straightforward to extend our analysis to incomplete information, i.e. $I(a) \neq \mathbb{A}$. In this case the equilibrium concept becomes *Markov Sequential Equilibrium (MSE)*). We present here without proof the existence result since the proof is obtained through straightforward modifications of the arguments in the complete information case.

Theorem 5 (Existence - Incomplete Information) *Consider an economy with conformity preferences and with incomplete information.*

1. *For $T < \infty$, the economy admits a unique symmetric MSE $g^* : \mathbf{X} \times \mathbf{\Theta}^{I(0)} \times \{1, \cdots, T\} \mapsto X$ such that for all $t \in \{1, \ldots, T\}$,*

$$g^*_{T-(t-1)}(x_{t-1},\ I_0\,\theta_t) = \sum_{a\in\mathbb{A}} c^a_{T-(t-1)}\, x^a_{t-1} + \sum_{a\in I(0)} d^a_{T-(t-1)}\, \theta^a_t + e_{T-(t-1)}\,\overline{\theta} \qquad \mathbb{P}-a.s.$$

where $c^a_\tau, d^a_\tau, e_\tau \geq 0$ and $e_\tau + \sum_{a\in\mathbb{A}} c^a_\tau + \sum_{a\in I(0)} d^a_\tau = 1,\ 0 \leq \tau \leq T$.

2. *For $T = \infty$, the economy admits a symmetric MSE $g^* : \mathbf{X} \times \mathbf{\Theta}^{I(0)} \mapsto X$ such that*

$$g^*(x_{t-1},\ I_0\,\theta_t) = \sum_{a\in\mathbb{A}} c^a\, x^a_{t-1} + \sum_{a\in I(0)} d^a\, \theta^a_t + e\,\overline{\theta} \qquad \mathbb{P}-a.s.$$

where $c^a, d^a, e \geq 0$ and $e + \sum_{a\in\ \mathbb{A}} c^a + \sum_{a\in I(0)} d^a = 1$.

B Proof of the Recursive Computation Theorem

Here we prove Theorem 2. Consider a finite-horizon T-period economy with conformity preferences ($\alpha_i > 0,\ i = 1, 2, 3$) and complete information. For part (i), we simply assume that $T = 1$ and show that one can fit an exponentially declining sequence into equation A.2. Since that equation has a unique solution as argued in the existence proof, that solution must have exponentially declining coefficients. Matching the coefficients of the policy function using equation A.2, one gets for $a \neq 0$

$$d^{a+1}_1 = \left(\frac{\alpha_3}{\Delta_1}\right) d^{a+2}_1 + \left(\frac{\alpha_3}{\Delta_1}\right) d^a_1$$

Dividing both sides by d^a_1 and multiplying them by $\left(\frac{\Delta_1}{\alpha_3}\right)$, one gets

$$\left(\frac{\Delta_1}{\alpha_3}\right) \underbrace{\left(\frac{d^{a+1}_T}{d^a_1}\right)}_{r_1} = \underbrace{\left(\frac{d^{a+2}_1}{d^{a+1}_1}\right)}_{r_1} \underbrace{\left(\frac{d^{a+1}_1}{d^a_1}\right)}_{r_1} + 1$$

which induces a quadratic equation

$$r_1^2 - \left(\frac{\Delta_1}{\alpha_3}\right) r_1 + 1 = 0$$

whose determinant $\left(\frac{\Delta_1}{\alpha_3}\right)^2 - 4 > 0$ since $\Delta_1 = \alpha_1 + \alpha_2 + 2\alpha_3 > 2\alpha_3$ (remember that $\alpha_i > 0$ for $i = 1, 2, 3$). The equation has two positive roots, one bigger and one smaller than 1. The bigger root cannot work since it is explosive as $|a| \to \infty$. We pick the smaller root

$$0 < r_1 = \left(\frac{\Delta_1}{2\,\alpha_3}\right) - \sqrt{\left(\frac{\Delta_1}{2\,\alpha_3}\right)^2 - 1} < 1 \tag{B.1}$$

which is decreasing in $\left(\frac{\Delta_1}{2\,\alpha_3}\right)$ spanning the interval $(0, 1)$ for different values of the former in the interval $(1, \infty)$. Finally, the sum of coefficients can be written

$$\sum_{a \in \mathbb{A}} d_1^a = \sum_{a \in \mathbb{A}} d_1^0\, r_1^{|a|} = d_1^0 + 2\, d_1^0 \frac{r_1}{1 - r_1} = \frac{\alpha_2}{\alpha_1 + \alpha_2} \tag{B.2}$$

The first equality is due to the exponentiality of the sequence; the third uses the same argument as in (A.9) with $\mu_1 = 0$, for the coefficient sequence $(d_1^a)_{a \in \mathbb{A}}$. Solving for d_1^0 from above, we obtain

$$d_1^0 = \left(\frac{\alpha_2}{\alpha_1 + \alpha_2}\right)\left(\frac{1 - r_1}{1 + r_1}\right)$$

and finally thanks to exponentiality

$$d_1^a = r_1^{|a|}\left(\frac{\alpha_2}{\alpha_1 + \alpha_2}\right)\left(\frac{1 - r_1}{1 + r_1}\right), \qquad \text{for } a \in \mathbb{A}$$

The argument for the sequence $(c_1^a)_{a \in \mathbb{A}}$ is identical with one change: The sum of coefficients $\sum_a c_1^a = \left(\frac{\alpha_1}{\alpha_1 + \alpha_2}\right)$. This proves part (i) of the theorem.

For part (ii), observe that the parameters of the maps L_s, namely $\Delta_s, (\gamma_s^a), \mu_s$ are functions only of the continuation policy coefficients $(c_\tau^*, d_\tau^*, e_\tau^*)_{\tau=1}^{s-1}$ as defined in (A.11), (A.13), and (A.15), simply because these are "forward-looking" expressions. We saw in the induction step (**Step 2**) of the existence proof that L_s defined in this fashion becomes a contraction and has a unique fixed point, which is the coefficient sequence of the first-period policy of an s-period continuation. This establishes part (ii).

For part (iii), observe that each $g \in G$ is associated with coefficients $((c^a, d^a)_a, e)$. Clearly, for any sequence of policies in G, $g_n \to g$ in sup norm if and only if the associated coefficients $((c_n^a, d_n^a)_a, e_n) \to ((c^a, d^a)_a, e)$ in sup norm. In **Step 3** of the existence proof, we establish the convergence of the finite-horizon equilibrium policies to the stationary infinite-horizon MPE policy as the horizon expands. But this implies that the associated unique coefficient sequence also should converge, then, to the coefficient sequence of the infinite-horizon stationary MPE policy. This establishes part (iii) of Theorem 2. ∎

C Proof of Ergodicity

This section proves Theorem 3. Suppose that the process $\left((\theta_t^a)_{t=-\infty}^{\infty}\right)_{a\in\mathbb{A}}$ is i.i.d. with respect to a and t according to ν. Let π be the initial measure on the configuration space $\mathbf{X}$ which is the distribution of

$$x_0 = \left(\frac{e\,\overline{\theta}}{1-C} + \sum_{s=1}^{\infty}\sum_{b_1}\cdots\sum_{b_s} c^{b_1}\cdots c^{b_{s-1}}\left(d^{b_s}\,\theta_{1-s}^{a+b_1+\cdots+b_s}\right)\right)_{a\in\mathbb{A}} \tag{C.1}$$

Given that $(x_t \in \mathbf{X})_{t=0}^{\infty}$ is an equilibrium process generated by the stationary MPE g^* in Theorem 1, given x_0, one obtains on the equilibrium path

$$\begin{aligned}
x_1^a &= \sum_{b_1\in\mathbb{A}} c^{b_1}\,x_0^{a+b_1} + \sum_{b_1\in\mathbb{A}} d^{b_1}\,\theta_1^{a+b_1} + e\,\overline{\theta} \\
&= \sum_{b_1\in\mathbb{A}} c^{b_1}\left(\frac{e\,\overline{\theta}}{1-C} + \sum_{s=1}^{\infty}\sum_{b_1}\cdots\sum_{b_s} c^{b_1}\cdots c^{b_{s-1}}\left(d^{b_s}\,\theta_{1-s}^{a+b_1+\cdots+b_s}\right)\right) \\
&\quad + \sum_{b_1\in\mathbb{A}} d^{b_1}\,\theta_1^{a+b_1} + e\,\overline{\theta} \\
&= \frac{e\,\overline{\theta}}{1-C} + \sum_{s=1}^{\infty}\sum_{b_1}\cdots\sum_{b_s} c^{b_1}\cdots c^{b_{s-1}}\left(d^{b_s}\,\theta_{2-s}^{a+b_1+\cdots+b_s}\right)
\end{aligned}$$

which has the same form as in (C.1). Since the process $\left((\theta_t^a)_{t=-\infty}^{\infty}\right)_{a\in\mathbb{A}}$ is i.i.d., x_0^a and x_1^a are distributed identically when the initial measure is π. Since the choice of a was arbitrary, π is a stationary distribution of the Markov process $(x_t)_{t=0}^{\infty}$. Moreover, from Lemma 2 for a stationary policy function, on any path $(\theta_1, \theta_2, \ldots)$ of the stochastic process

$$\begin{aligned}
x_t^a &= \sum_{b_1}\cdots\sum_{b_t} c^{b_1}\cdots c^{b_t}\,x_0^{a+b_1+\cdots+b_t} \\
&\quad + \sum_{s=1}^{t}\sum_{b_1}\cdots\sum_{b_s} c^{b_1}\cdots c^{b_{s-1}}\left(d^{b_s}\,\theta_{t-(s-1)}^{a+b_1+\cdots+b_s} + e\,\overline{\theta}\right) \\
&= C^t\sum_{b_1}\cdots\sum_{b_t}\left(\frac{c^{b_1}\cdots c^{b_t}}{C^t}\right)x_0^{a+b_1+\cdots+b_t} \\
&\quad + \sum_{s=1}^{t}\sum_{b_1}\cdots\sum_{b_s} c^{b_1}\cdots c^{b_{s-1}}\left(d^{b_s}\,\theta_{t+1-s}^{a+b_1+\cdots+b_s} + e\,\overline{\theta}\right)
\end{aligned} \tag{C.2}$$

Thus, independent of the initial conditions, x_t^a converges pointwise to $x^a \in X$ where

$$x^a := \lim_{t\to\infty} x_t^a \;=\; \lim_{t\to\infty} \Big[C^t \sum_{b_1} \cdots \sum_{b_t} \left(\frac{c^{b_1} \cdots c^{b_t}}{C^t} \right) x_0^{a+b_1+\cdots+b_t} + \sum_{s=1}^{t} \sum_{b_1} \cdots \sum_{b_s} c^{b_1} \cdots c^{b_{s-1}} \left(d^{b_s}\, \theta_{t+1-s}^{a+b_1+\cdots+b_s} + e\,\overline{\theta} \right) \Big] \tag{C.3}$$

The first term of the previous expression $C^t \to 0$ since $C < 1$ due to $\alpha_i > 0$, for all i. The first term in the parentheses in the summand is a convex combination of uniformly bounded terms. Hence, the first part of the above expression goes to 0 as $t \to \infty$. Moreover, since the equilibrium is symmetric, the convergence is uniform across agents: $x_t \to x = (x^a)$ uniformly. Since the exogenous shock process is i.i.d, the law for the part after the plus sign in (C.3) is identical to the law for

$$\sum_{s=1}^{t} \sum_{b_1} \cdots \sum_{b_s} c^{b_1} \cdots c^{b_{s-1}} \left(d^{b_s}\, \theta_{1-s}^{a+b_1+\cdots+b_s} + e\,\overline{\theta} \right)$$

which is the 't-translated-into-the-past' version of the former. Thus, for any given initial value x_0, and a path $(\ldots, \theta_{-1}, \theta_0)$, the pointwise limit of x_t^a can be written as

$$x^a = \frac{e\,\overline{\theta}}{1-C} + \sum_{s=1}^{\infty} \sum_{b_1} \cdots \sum_{b_s} c^{b_1} \cdots c^{b_{s-1}} \left(d^{b_s}\, \theta_{1-s}^{a+b_1+\cdots+b_s} \right) \tag{C.4}$$

Let $\mathbb{P}_\infty(\cdot) := \prod_{t=0}^{\infty} \mathbb{P}(\cdot)$ and $\theta := (\ldots, \theta_{-1}, \theta_0,)$. Pick any $f \in C(\mathbf{X}, \mathbb{R})$, the set of bounded, continuous, and measurable, real-valued functions from $\mathbf{X}$ into $\mathbb{R}$. Let π_0 be an arbitrary initial distribution for x_0. We have

$$\begin{aligned}
\lim_{t\to\infty} \int f(x_t)\pi_t(dx_t) \;&=\; \lim_{t\to\infty} \int f\Bigg(\Bigg(C^t \sum_{b_1} \cdots \sum_{b_t} \left(\frac{c^{b_1} \cdots c^{b_t}}{C^t} \right) x_0^{a+b_1+\cdots+b_t} \\
&\quad + \sum_{s=1}^{t} \sum_{b_1} \cdots \sum_{b_s} c^{b_1} \cdots c^{b_{s-1}} \left(d^{b_s}\, \theta_{1-s}^{a+b_1+\cdots+b_s} + e\,\overline{\theta} \right) \Bigg)_{a\in\mathbb{A}} \Bigg) \mathbb{P}_\infty(d\theta)\pi_0(dx_0) \\
&= \int f\Bigg(\Bigg(\frac{e\,\overline{\theta}}{1-C} + \sum_{s=1}^{\infty} \sum_{b_1} \cdots \sum_{b_s} c^{b_1} \cdots c^{b_{s-1}} \left(d^{b_s}\, \theta_{1-s}^{a+b_1+\cdots+b_s} \right) \Bigg)_{a\in\mathbb{A}} \Bigg) \mathbb{P}_\infty(d\theta)\pi_0(dx_0) \\
&= \int f(x)\,\pi(dx)
\end{aligned}$$

The first equality is from (C.2); the second is due to Lebesgue Dominated Convergence theorem (see e.g. Aliprantis and Border (2006), p. 415); third is due to the continuity of f and the pointwise limit of x_t in (C.4). Thus, for any $f \in C(\mathbf{X}, \mathbb{R})$, $\lim_{t\to\infty} \int f d\pi_t = \int f d\pi$, meaning that the sequence of equilibrium distributions π_t generated by the exogenous law $\mathbb{P}$ and the stationary

MPE policy g^* *converges weakly* to the invariant distribution π. The choice of π_0 was arbitrary. Hence, for any initial distribution, the induced equilibrium process converges weakly to the same invariant distribution π. Therefore, π is the unique invariant distribution of the equilibrium process. Here is why: Suppose that $\hat{\pi}$ is another invariant distribution. This implies that the induced process starting with $\pi_0 = \hat{\pi}$ should satisfy $\pi_t = \hat{\pi}$, for all $t = 1, 2, \ldots$. From the above convergence argument $\pi_t \to \pi$ weakly. Hence $\hat{\pi} = \pi$.

Finally, to show ergodicity, pick an $f \in B(\mathbf{X}, \mathbb{R})$, the set of bounded, measurable, real-valued functions from $\mathbf{X}$ into $\mathbb{R}$. The process starting with π is stationary, hence $\pi_t = \pi$ for all $t = 0, 1, \ldots$. Since the process x_t is stationary, so is the process $(f(x_t))$. We can then use Birkhoff's Ergodic Theorem (see e.g. Aliprantis and Border (2006), p. 659) on the process $(f(x_t))$ to obtain

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} f(x_t) = \int f(x_t)\pi(dx_t)$$

almost surely. Since the choice of f was arbitrary, the last expression holds for all $f \in B(\mathbf{X}, \mathbb{R})$. Thus the equilibrium process $(x_t \in \mathbf{X})_{t=0}^{\infty}$ starting from initial distribution π is ergodic. This concludes the proof of Theorem 3. ■

It is straightforward to show that ergodicity extends to the case of incomplete information.

D Tail Convergence Monotonicity

We prove here Proposition 1. The proof is by induction on T. For $T = 1$, we know from Theorem 2 (i) that the policy coefficient sequence $(d_1^a)_{a\in\mathbb{A}}$ is exponentially declining on both sides of the origin, at the rate r_1. From the form of the policy function in Theorem 1, the conditional covariance between agents 0 and $a+1$, with $a \geq 0$ w.l.o.g., given x_0 is

$$\begin{aligned} Cov\left(x_1^0, x_1^{a+1} \middle| x_0\right) &= Cov\left(\sum_{b_1\in\mathbb{A}} d_1^{b_1}\,\theta_1^{b_1}, \sum_{b_2\in\mathbb{A}} d_1^{b_2}\,\theta_1^{a+1+b_2}\right) \\ &= \sum_{b_1\in\mathbb{A}} d_1^{b_1}\, Cov\left(\theta_1^{b_1}, \sum_{b_2\in\mathbb{A}} d_1^{b_2}\,\theta_1^{a+1+b_2}\right) \\ &= Var(\theta) \sum_{b_1\in\mathbb{A}} d_1^{b_1}\, d_1^{b_1-(a+1)} \end{aligned} \tag{D.1}$$

We will focus on the summation term in the last expression in (D.1). Write it as

$$\sum_{b_1 \in \mathbb{A}} d_1^{b_1} d_1^{b_1-(a+1)} = \sum_{b_1<0} d_1^{b_1} d_1^{b_1-(a+1)} + d_1^0 d_1^{-a-1} + \sum_{b_1>0} d_1^{b_1} d_1^{b_1-(a+1)}$$
$$= \sum_{b_1<0} d_1^{b_1} d_1^{b_1-(a+1)} + d_1^0 d_1^{-a-1} + \sum_{b_1 \geq 0} d_1^{b_1+1} d_1^{b_1-(a)}$$

$$= \sum_{b_1<0} d_1^{b_1} \left(r_1 d_1^{b_1-a}\right) + (d_1^0)^2 (r_1)^{a+1} + \sum_{b_1 \geq 0} \left(r_1 d_1^{b_1}\right) d_1^{b_1-a}$$
$$= r_1 \sum_{b_1 \in \mathbb{A}} d_1^{b_1} d_1^{b_1-a} + (d_1^0)^2 (r_1)^{a+1}$$
$$= r_1 \, Var(\theta)^{-1} Cov\left(x_1^0, x_1^a \,\middle|\, x_0\right) + (d_1^0)^2 \, r_1^{a+1}$$

The first equality is a partitioning, the second a simple change of variable, and the third is due to the symmetry and the exponentiality of the d_1^a sequence. Substituting the final expression back in (D.1) yields, for all $a \geq 0$

$$Cov\left(x_1^0, x_1^{a+1} \,\middle|\, x_0\right) = r_1 \, Cov\left(x_1^0, x_1^a \,\middle|\, x_0\right) + r_1^{a+1} \, Var(\theta) \, (d_1^0)^2 \tag{D.2}$$

which implies that the rate of decay of the covariances is greater than r_1, for any $a \geq 0$. Since the second term on the right hand side of (D.2) decays at the rate r_1, this implies that the ratio

$$\frac{r_1^{a+1} \, Var(\theta) \, (d_1^0)^2}{Cov\left(x_1^0, x_1^a \,\middle|\, x_0\right)} \tag{D.3}$$

decreases monotonically, and being non-negative, it converges. Actually it converges to zero. Here is why: Since the ratio is less than 1, suppose that it converges to $k \in (0, 1)$. This means from (D.2) that the limit rate of decay of the covariances is $r_1 + k$, greater than the rate for the term in the numerator in (D.3). Thus, the ratio in (D.3) should converge to zero at the limit, a contradiction to $k \in (0, 1)$. So, the limit of (D.3) is zero, which in turn implies from (D.2), after dividing both sides by $Cov\left(x_1^0, x_1^a \,\middle|\, x_0\right)$, that

$$\lim_{a \to \infty} \frac{Cov\left(x_1^0, x_1^{a+1} \,\middle|\, x_0\right)}{Cov\left(x_1^0, x_1^a \,\middle|\, x_0\right)} = r_1 \tag{D.4}$$

The argument is symmetric for $a \leq 0$; hence the sequence $\left\{Cov\left(x_1^0, x_1^a \,\middle|\, x_0\right)\right\}_{a \in \mathbb{A}}$ declines exponentially on both tails at the same rate r_1 and the statement is true for $T = 1$.

Now assume that the statement in Proposition 1 is true for economies up to $T - 1$ period. We will show that it should also hold for T-period economies. We will base the main induction arguments on the following Lemma.

Lemma 6 *The sequence* $\{\gamma_T^b\}_{b\in\mathbb{A}}$ *in Lemma 3 and the equilibrium coefficient sequence* (c_T, d_T) *for the first-period policy of a* T*-period economy have the following properties: The rate at which they decline at the tail satisfies, for* $T \geq 2$

$$\lim_{a\to\infty}\left(\frac{\gamma_T^{a+1}}{\gamma_T^a}\right) = \lim_{a\to-\infty}\left(\frac{\gamma_T^{a-1}}{\gamma_T^a}\right) = r_{T-1},$$

and

$$\lim_{a\to\infty}\left(\frac{d_T^{a+1}}{d_T^a}\right) = \lim_{a\to-\infty}\left(\frac{d_T^{a-1}}{d_T^a}\right) = r_T > r_{T-1}.$$

Proof: We prove this by induction on T. To do that, we first prove an intermediate Lemma on the convergence rates of convolutions of converging absolutely summable sequences that share the same characteristics with the γ_T and d_T sequences.

Lemma 7 *Let* (x_b) *and* (y_b) *be two symmetric absolutely summable* ($\sum_b |x_b| < \infty$ *and* $\sum_b |y_b| < \infty$) *sequences that satisfy*

$$\lim_{b\to\infty}\frac{x_{b+1}}{x_b} = r_x < r_y = \lim_{b\to\infty}\frac{y_{b+1}}{y_b} < 1$$

Let $S(a) := \sum_b x_b y_{a-b}$*. The function* $S : \mathbb{A} \to \mathbb{R}$ *is well defined and*

$$\lim_{a\to\infty}\frac{S(a+1)}{S(a)} = r_y.$$

Proof: For any $a \in \mathbb{A}$,

$$S(a) = \underbrace{\sum_{b<0} x_b\, y_{a-b}}_{A(a)} + \underbrace{\sum_{b>a} x_b\, y_{a-b}}_{B(a)} + \underbrace{\sum_{b=0}^{a} x_b\, y_{a-b}}_{C(a)} \tag{D.5}$$

which is well-defined since both sequences are bounded and exponentially converging at the tails. For large enough b, $|x_{b+1} - r_x x_b|$ and $|y_{b+1} - r_y y_b|$ can be made arbitrarily small by hypothesis. This means that

$$\begin{aligned}|A(a+1) - r_y A(a)| &= \left|\sum_{b<0} x_b\, y_{a+1-b} - r_y \sum_{b<0} x_b\, y_{a-b}\right| \\ &= \left|\sum_{b<0} x_b\, y_{a-b}\left(\frac{y_{a+1-b}}{y_{a-b}}\right) - r_y \sum_{b<0} x_b\, y_{a-b}\right|\end{aligned}$$

can be made arbitrarily small for large enough a. Similarly for

$$\begin{aligned}|B(a+1) - r_x B(a)| &= \left|\sum_{b>a+1} x_b\, y_{a+1-b} - r_x \sum_{b>a} x_b\, y_{a-b}\right| \\ &= \left|\sum_{b>a} x_b\, y_{a-b}\left(\frac{x_{b+1}}{x_b}\right) - r_x \sum_{b>a} x_b\, y_{a-b}\right|\end{aligned}$$

which becomes arbitrarily small as a gets arbitrarily large. The last part, $C(a)$, is trickier. Write it as

$$C(a) = y_a \sum_{b=0}^{a} x_b \left(\frac{y_{a-b}}{y_a} \right)$$

Since $r_y > r_x$ (or $\frac{r_x}{r_y} < 1$), $\frac{1}{r_y}$ is within the radius of convergence of $\sum_{b=0}^{\infty} x_b \lambda^b$. But then, result (1.2) on page 452 of Bojanic and Lee (1974) yields that

$$\lim_{a\to\infty} \sum_{b=0}^{a} x_b \left(\frac{y_{a-b}}{y_a} \right) = \underbrace{\sum_{b=0}^{\infty} x_b \left(\frac{1}{r_y} \right)^b}_{K} < \infty.$$

This implies that $|C(a) - y_a K|$ can be made arbitrarily small for large a. Now

$$|C(a+1) - r_y C(a)| \quad \leq \quad |C(a+1) - y_{a+1} K| + |y_{a+1} K - r_y C(a)| \to 0$$

as $a \to \infty$ since $|y_{a+1} - r_y y_a| \to 0$ by hypothesis. Now, we collect all pieces and put them together. First of all, $A(a)$ and $C(a)$ converges at the rate r_y which is greater than r_x the rate at which $B(a)$ converges. This means that

$$\frac{A(a) + C(a)}{S(a)} \to 1 \text{ and } \frac{B(a)}{S(a)} \to 0 \text{ as } a \to \infty \tag{D.6}$$

Thus, the rate at which $S(a)$ converges can be written for large a values

$$\begin{aligned} \frac{S(a+1)}{S(a)} &\approx \frac{r_y A(a) + r_x B(a) + r_y C(a)}{S(a)} \\ &= r_y + (r_x - r_y) \frac{B(a)}{S(a)} \end{aligned}$$

As $a \to \infty$, the ratio in the last expression goes to zero, thanks to (D.6) and this ends the proof.

■

Proof of the First part of Lemma 6: Let $u(t) := u\left(x_{t-1}^0, x_t^0, \{x_t^b\}_{b\in\{-1,1\}}, \theta_t^0\right)$ where u represents the conformity preferences in Assumption 1. Let $u_0(t) := \frac{\partial}{\partial x_1^0} u(t)$. From equation (A.13), γ_T^a can be written as

$$\begin{aligned} \gamma_T^a \quad &:= \quad \alpha_3 \, I_{\{a\in\{-1,1\}\}} \\ &+ \quad \sum_{\tau=2}^{T} \beta^{\tau-1} \left[\left(\frac{\partial x_{\tau-1}^0}{\partial x_1^a} \right) \frac{\partial}{\partial x_{\tau-1}^0} u_0(\tau) + \left(\frac{\partial x_\tau^0}{\partial x_1^a} \right) \frac{\partial}{\partial x_\tau^0} u_0(\tau) \right. \\ &+ \quad \left. \left(\frac{\partial x_\tau^{-1}}{\partial x_1^a} \right) \frac{\partial}{\partial x_\tau^{-1}} u_0(\tau) + \left(\frac{\partial x_\tau^1}{\partial x_1^a} \right) \frac{\partial}{\partial x_\tau^1} u_0(\tau) \right] \end{aligned} \tag{D.7}$$

We will present the argument for the second term inside the summand and the method of proof will apply to the remaining terms straightforwardly. Assume w.l.o.g. that $a \geq 0$.

$$\left(\frac{\partial x_\tau^0}{\partial x_1^a}\right)\frac{\partial}{\partial x_\tau^0}u_0(\tau) = \sum_{s\in\mathbb{A}}\left(\frac{\partial x_2^s}{\partial x_1^a}\right)\left(\frac{\partial x_\tau^0}{\partial x_2^s}\right)\frac{\partial}{\partial x_\tau^0}u_0(\tau)$$
$$= \sum_{s\in\mathbb{A}} c_{T-1}^{a-s}\left(\frac{\partial x_\tau^0}{\partial x_2^s}\right)\frac{\partial}{\partial x_\tau^0}u_0(\tau)$$

and the corresponding term for γ_T^{a+1} is

$$\sum_{s\in\mathbb{A}} c_{T-1}^{a+1-s}\left(\frac{\partial x_\tau^0}{\partial x_2^s}\right)\frac{\partial}{\partial x_\tau^0}u_0(\tau)$$

Thanks to *(i)* of Lemma 2, $\left(\frac{\partial x_\tau^0}{\partial x_2^s}\right)_{s\in\mathbb{A}}$ is an iterated convolution of the sequences $c_{T-2},\ldots,c_{T-\tau+1}$, which share the same convergence rates with $d_{T-2},\ldots,d_{T-\tau+1}$. So, iterated application of Lemma 7 yields that the rate of convergence of the sequence $\left(\frac{\partial x_\tau^0}{\partial x_2^s}\right)_{s\in\mathbb{A}}$ is r_{T-2}. But then a final application of Lemma 7 to the expression below yields

$$\lim_{a\to\infty}\sum_{s\in\mathbb{A}} c_{T-1}^{a+1-s}\left(\frac{\partial x_\tau^0}{\partial x_2^s}\right)\frac{\partial}{\partial x_\tau^0}u(\tau) = r_{T-1}\lim_{a\to\infty}\sum_{s\in\mathbb{A}} c_{T-1}^{a-s}\left(\frac{\partial x_\tau^0}{\partial x_2^s}\right)\frac{\partial}{\partial x_\tau^0}u(\tau)$$

Hence at the tail, the second term of the sum inside the brackets of (D.7) decays at the rate r_{T-1}. The same one-step transition argument applies to each term of the sum, in equation (D.7). Moreover, since (D.7) is a discounted sum, the entire expression is summable and it implies that

$$\lim_{a\to\infty}\left(\frac{\gamma_T^{a+1}}{\gamma_T^a}\right) = r_{T-1}$$

ranking the rates of convergences, as is argued in the Lemma. The method of proof for $a \leq 0$ is identical thanks to the symmetry of the environment.

Proof of the Second part of Lemma 6: For the second part of Lemma 6, let $D(r_{T-1})$ be the space of sequences that satisfies the properties in (A.3) and that converges at the tail at a rate $r_T \geq r_{T-1}$. This is a closed subset of the space of sequences that satisfy only the properties in (A.3), hence a complete metric space itself. Consequently, the unique coefficient sequence d_T that is the fixed point of the map in (A.16) should lie in $D(r_{T-1})$. Let $D'(r_{T-1}) \subset D(r_{T-1})$ be the space of sequences in $D(r_{T-1})$ whose convergence at the tail is *strictly greater* than r_{T-1}. We will show below that the unique solution of the map (*A*.16) should lie in the set $D'(r_{T-1})$ and hence should converge at a rate $r_T > r_{T-1}$ at the tail.

Pick agent $2a+1$ and assume w.l.o.g. that $a \geq 0$. Let $d_T \in D(r_{T-1})$. From (A.16) by matching coefficients

$$\begin{aligned}
\hat{d}_T^{2a+1} &= \Delta_T^{-1}\left[\sum_{b\neq 0}\gamma_T^b\, d_T^{2a+1-b}\right] \\
&= \Delta_T^{-1}\left[\sum_{b>a}\gamma_T^b\, d_T^{2a+1-b} + \gamma_T^a\, d_T^{a+1} + \sum_{b<a, b\neq 0}\gamma_T^b\, d_T^{2a+1-b}\right] \\
&= \Delta_T^{-1}\left[\sum_{b\geq a}\gamma_T^{b+1}\, d_T^{2a-b} + \gamma_T^a\, d_T^{a+1} + \sum_{b<a, b\neq 0}\gamma_T^b\, d_T^{2a+1-b}\right] \\
&= \Delta_T^{-1}\left[\sum_{b\geq a}\gamma_T^b\, d_T^{2a-b}\left(\frac{\gamma_T^{b+1}}{\gamma_T^b}\right) + \gamma_T^a\, d_T^{a+1} + \sum_{b<a, b\neq 0}\gamma_T^b\, d_T^{2a-b}\left(\frac{d_T^{2a+1-b}}{d_T^{2a-b}}\right)\right] \qquad \text{(D.8)}
\end{aligned}$$

The second equality is a partitioning of the sum taking agent a as the 'middle'; the first sum after the third equality is a simple shift and change of the dummy variable b; the first term after the first equality sign is by multiplying and dividing each term in the summand by γ_T^b; finally the last term after the fourth equality sign is by multiplying and dividing each term in the summand by d_T^{2a-b}. Since all elements involved are non-zero, the algebraic manipulation above is feasible. We can add to and substract from equation (D.8) the term $\Delta_T^{-1}\sum_{b<a,b\neq 0}\gamma_T^b\, d_T^{2a-b}\, r_{T-1}$ and rearrange the order of the terms to obtain

$$\begin{aligned}
\hat{d}_T^{2a+1} = \Delta_T^{-1}\Bigg[&\underbrace{\sum_{b\geq a}\gamma_T^b\, d_T^{2a-b}\left(\frac{\gamma_T^{b+1}}{\gamma_T^b}\right) + \sum_{b<a, b\neq 0}\gamma_T^b\, d_T^{2a-b}\, r_{T-1}}_{A} + \underbrace{\gamma_T^a\, d_T^{a+1}}_{B} \\
&+ \underbrace{\sum_{b<a, b\neq 0}\gamma_T^b\, d_T^{2a-b}\left\{\left(\frac{d_T^{2a+1-b}}{d_T^{2a-b}}\right) - r_{T-1}\right\}}_{C}\Bigg] \qquad \text{(D.9)}
\end{aligned}$$

The analogous expression for $\hat{d}_T^{2a}$ is given, after a similar partitioning with agent a as the middle agent, by

$$\begin{aligned}
\hat{d}_T^{2a} &= \Delta_T^{-1}\left[\sum_{b\neq 0}\gamma_T^b\, d_T^{2a-b}\right] \\
&= \Delta_T^{-1}\left[\underbrace{\sum_{b\geq a}\gamma_T^b\, d_T^{2a-1-b}\left(\frac{\gamma_T^{b+1}}{\gamma_T^b}\right)+\sum_{b<a,b\neq 0}\gamma_T^b\, d_T^{2a-1-b}\, r_{T-1}}_{A'}+\underbrace{\gamma_T^a\, d_T^a}_{B'}\right. \\
&\qquad \left.+\underbrace{\sum_{b<a,b\neq 0}\gamma_T^b\, d_T^{2a-1-b}\left\{\left(\frac{d_T^{2a-b}}{d_T^{2a-1-b}}\right)-r_{T-1}\right\}}_{C'}\right]
\end{aligned} \tag{D.10}$$

We showed above in the first part of the proof of Lemma 6 that as $b\to\infty$, the ratio $\left(\gamma_T^{b+1}/\gamma_T^b\right)\to r_{T-1}$. Consequently, the following limits hold

$$\lim_{a\to\infty}\Delta_T^{-1}\, A = r_{T-1}\lim_{a\to\infty}\hat{d}_T^{2a} \tag{D.11}$$

$$\lim_{a\to\infty}\Delta_T^{-1}\, A' = r_{T-1}\lim_{a\to\infty}\hat{d}_T^{2a-1} \tag{D.12}$$

$$\lim_{a\to\infty}\Delta_T^{-1}\, C = (r_T - r_{T-1})\lim_{a\to\infty}\hat{d}_T^{2a} \tag{D.13}$$

$$\lim_{a\to\infty}\Delta_T^{-1}\, C' = (r_T - r_{T-1})\lim_{a\to\infty}\hat{d}_T^{2a-1} \tag{D.14}$$

The expressions in (D.11) and (D.13) put together imply that as a gets large the ratio

$$\left(\frac{\hat{d}_T^{2a+1}}{\hat{d}_T^{2a}}\right)\approx r_T+\frac{\gamma_T^a\, d_T^{a+1}}{\hat{d}_T^{2a}}\geq r_T \tag{D.15}$$

and the expressions in (D.12) and (D.14) put together imply that

$$\left(\frac{\hat{d}_T^{2a}}{\hat{d}_T^{2a-1}}\right)\approx r_T+\frac{\gamma_T^a\, d_T^{a}}{\hat{d}_T^{2a-1}}\geq r_T \tag{D.16}$$

The last two expressions imply that the ratios $\frac{\gamma_T^a\, d_T^{a+1}}{\hat{d}_T^{2a}}$ and $\frac{\gamma_T^a\, d_T^{a}}{\hat{d}_T^{2a-1}}$ converge. This is because as a gets arbitrarily large, the numerator converges at the rate $r_{T-1}\, r_T$ and the denominator at a rate greater than equal to r_T^2.

At the unique fixed point, one has $\hat{d}_T = d_T$. Since both ratios after the plus signs are strictly less than one, one of the following two possibilities must hold: either (i) they converge to a positive constant less than one (the case where $r_T = r_{T-1}$) or (ii) they converge to zero (the case where $r_T > r_{T-1}$). We show now that the first case is not possible. Suppose it is. Then, $r_T = r_{T-1}$. This implies from (D.9) along with (D.11) and (D.13) that

$$r_T = \lim_{a\to\infty}\left(\frac{\hat{d}_T^{2a+1}}{\hat{d}_T^{2a}}\right) \quad = \quad r_T + \lim_{a\to\infty}\left(\frac{\gamma_T^a\, d_T^{a+1}}{\hat{d}_T^{2a}}\right) > r_T \tag{D.17}$$

a contradiction. Therefore the second case (ii) must be true. The argument for $a \leq 0$ is symmetric. Consequently, the unique coefficient sequence d_T that is the fixed point of the map in (A.16) should lie in $D'(r_{T-1})$. So our claim in the beginning is true and the unique sequence d_T that satisfies properties in (A.3) and the equation (A.16) converges at the tail at a rate $r_T > r_{T-1}$. This concludes the proof of Lemma 6. ■

For the rest of the proof of Proposition 1, assume that the statement in Proposition 1 is true for economies up to $T-1$ period. We will show that it should also hold for T-period economies. Consider first the covariance between agents 0 and $2a+1$, with $a \geq 0$ w.l.o.g.

$$\begin{aligned} Cov\left(x_1^0, x_1^{2a+1}\,\middle|\, x_0\right) &= Var(\theta)\sum_{b\in\mathbb{A}} d_1^b\, d_1^{b-(2a+1)} \\ &= Var(\theta)\left[\sum_{b\leq a} d_T^b\, d_T^{b-(2a+1)} + d_T^{a+1}\, d_T^{-a} + \sum_{b\geq a+2} d_T^b\, d_T^{b-(2a+1)}\right] \end{aligned}$$

$$\begin{aligned} &= Var(\theta)\left[\sum_{b\leq a} d_T^b\, d_T^{b-(2a+1)} + d_T^{a+1}\, d_T^{-a} + \sum_{b\geq a+1} d_T^{b+1}\, d_T^{b+1-(2a+1)}\right] \\ &= Var(\theta)\left[\sum_{b\leq a} d_T^b\, d_T^{b-(2a)}\left(\frac{d_T^{b-(2a+1)}}{d_T^{b-(2a)}}\right) + \sum_{b\geq a+1} d_T^b\, d_T^{b-(2a)}\left(\frac{d_T^{b+1}}{d_T^b}\right) + d_T^{a+1}\, d_T^{-a}\right] \end{aligned}$$

where the algebraic manipulation is the same as in the proof of Lemma 6. The analogous expression for agent $2a$, taking agent a as the agent in the middle, is

$$Cov\left(x_1^0, x_1^{2a}\,\middle|\, x_0\right) = Var(\theta)\left[\sum_{b\leq a} d_T^b\, d_T^{b-(2a-1)}\left(\frac{d_T^{b-2a}}{d_T^{b-(2a-1)}}\right) + \sum_{b\geq a+1} d_T^b\, d_T^{b-(2a-1)}\left(\frac{d_T^{b+1}}{d_T^b}\right) + d_T^a\, d_T^{-a}\right]$$

We know from Lemma 6 that as $a \to \infty$, the ratio $(d_T^{a+1}/d_T^a) \to r_T > r_{T-1}$. This implies, from the expressions above for the covariance terms, that for large a,

$$\frac{Cov\left(x_1^0, x_1^{2a+1}\,\middle|\, x_0\right)}{Cov\left(x_1^0, x_1^{2a}\,\middle|\, x_0\right)} \approx r_T + \frac{d_T^{a+1}\, d_T^{-a}}{Cov\left(x_1^0, x_1^{2a}\,\middle|\, x_0\right)} \geq r_T > r_{T-1} \tag{D.18}$$

and

$$\frac{Cov\left(x_1^0, x_1^{2a}\,\middle|\, x_0\right)}{Cov\left(x_1^0, x_1^{2a-1}\,\middle|\, x_0\right)} \approx r_T + \frac{d_T^a\, d_T^{-a}}{Cov\left(x_1^0, x_1^{2a-1}\,\middle|\, x_0\right)} \geq r_T > r_{T-1} \tag{D.19}$$

and straightforward modifications of the argument used in the proof of Lemma 6 implies that the ratios $d_T^{a+1}\, d_T^{-a}\, Cov\left(x_1^0, x_1^{2a}\,\middle|\, x_0\right)^{-1}$ and $d_T^a\, d_T^{-a}\, Cov\left(x_1^0, x_1^{2a-1}\,\middle|\, x_0\right)^{-1}$ both converge to zero and one obtains

$$\lim_{a\to\infty} \frac{Cov\left(x_1^0, x_1^{2a+1}\,\middle|\, x_0\right)}{Cov\left(x_1^0, x_1^{2a}\,\middle|\, x_0\right)} = r_T > r_{T-1} \tag{D.20}$$

$$\lim_{a\to\infty} \frac{Cov\left(x_1^0, x_1^{2a}\,\middle|\, x_0\right)}{Cov\left(x_1^0, x_1^{2a-1}\,\middle|\, x_0\right)} = r_T > r_{T-1} \tag{D.21}$$

thus the statement of Proposition 1 is true for any finite T-period economy. Clearly, $r_T \leq 1$ for any $T \geq 1$ since the non-negative d sequences sum up to less than 1. Hence, what we have is a monotone increasing sequence bounded from above by 1. Hence, the limit $r_\infty = \lim_{T\to\infty} r_T$ exists and is less than or equal to 1. Moreover, we know from Theorem 2 that the sequence of finite-horizon MPE coefficients converges to that of the infinite-horizon MPE coefficient sequence d, thus r_∞ is the tail convergence rate of the infinite-horizon MPE coefficient sequence d. Therefore $r_\infty < 1$ since otherwise that would contradict the summability of the sequence d. This establishes the proof of Proposition 1. ■

E Proof of Inefficiency

We give here the proof of Theorem 4 for economies with complete information. The extension of the line of proof to the incomplete information economies is straightforward.

We first write the planning problem recursively. For any agent $a \in \mathbb{A}$, for all $t = 1, \ldots, T$, and all $(x_{T-1}, \theta_T) \in \mathbf{X} \times \mathbf{\Theta}^{I(0)}$, let the value of using the choice rule h in the continuation be defined as

$$\begin{aligned}
V^{h,T-(t-1)}\left(R^a\, x_{T-1}, R^a\, \theta_T\right) &= -\alpha_1 \left(x_{t-1}^a - h_{T-(t-1)}(R^a\, x_{t-1}, R^a\, I_0\, \theta_t)\right)^2 \\
&-\alpha_2 \left(\theta_t^a - h_{T-(t-1)}(R^a\, x_{t-1}, R^a\, I_0\, \theta_t)\right)^2 \\
&-\alpha_3 \left(h_{T-(t-1)}(R^{a-1}\, x_{t-1}, R^{a-1}\, I_0\, \theta_t) - h_{T-(t-1)}(R^a\, x_{t-1}, R^a\, I_0\, \theta_t)\right)^2 \\
&-\alpha_3 \left(h_{T-(t-1)}(R^{a+1}\, x_{t-1}, R^{a+1}\, I_0\, \theta_t) - h_{T-(t-1)}(R^a\, x_{t-1}, R^a\, I_0\, \theta_t)\right)^2 \\
&+\beta \int V^{h,T-t}\left(R^a\, \left\{h_t(R^b\, x_{t-1}, R^b\, I_0\, \theta_t)\right\}_{b\in\mathbb{A}}, R^a\, I_0\, \theta_{t+1}\right) \mathbb{P}\left(d\theta_{t+1}\right)
\end{aligned}$$

which leads us to the following definition

Definition 7 (Recursive Planning Problem) *Let a T-period linear economy with social interactions and conformity preferences be given. Let π_0 be an absolutely continuous distribution on the initial choice profiles with a positive density. A symmetric Markovian choice function $g : \mathbf{X} \times \mathbf{\Theta}^{I(0)} \times \{1, \ldots, T\} \to X$ is said to be **efficient** if it is a solution, for all $a \in \mathbb{A}$, and for all $t = 1, \ldots, T$, to*

$$\begin{aligned}
\underset{\{h \in CB(\mathbf{X}\times\mathbf{\Theta}, X)\}}{\arg\max} \int \Bigg[& -\alpha_1 \left(x^0_{t-1} - h_{T-(t-1)}(R^a\, x_{t-1}, R^a\, I_0\, \theta_t)\right)^2 \\
& -\alpha_2 \left(\theta^a_t - h_{T-(t-1)}(R^a\, x_{t-1}, R^a\, I_0\, \theta_t)\right)^2 \\
& -\alpha_3 \left(h_{T-(t-1)}(R^{a-1}\, x_{t-1}, R^{a-1}\, I_0\, \theta_t) - h_{T-(t-1)}(R^a\, x_{t-1}, R^a\, I_0\, \theta_t)\right)^2 \\
& -\alpha_3 \left(h_{T-(t-1)}(R^{a+1}\, x_{t-1}, R^{a+1}\, I_0\, \theta_t) - h_{T-(t-1)}(R^a\, x_{t-1}, R^a\, I_0\, \theta_t)\right)^2 \\
& +\beta\, V^{h,T-t} \left(R^a \left\{h(R^b\, x_{t-1}, R^b\, I_0\, \theta_t)\right\}_{b\in\mathbb{A}}, R^a\, I_0\, \theta_{t+1}\right) \Bigg] \mathbb{P}\,(d\theta_t)\, \mathbb{P}\,(d\theta_{t+1})\, \pi_t\,(dx_{t-1})
\end{aligned}$$

where π_t is the distribution on t-th period choice profiles induced by π_0 and the planner's choice rule h.

We can now proceed with the proof.

Finite-Horizon: Take any finite horizon economy ($T < \infty$). We will use continuity arguments so endow the underlying space $\mathbf{X} \times \mathbf{\Theta}$ with the product topology. Product topology is metrizable, say by metric d[55]. In the final period of this finite horizon economy, with absolutely continuous distribution π_{T-1} on the space of choice profiles x_{T-1}[56] with a positive density, the planner maximizes ex-ante (before the realization of θ_T) the expected utility of a given agent, say of agent $0 \in \mathbb{A}$, by choosing a symmetric policy function $h \in CB(\mathbf{X} \times \mathbf{\Theta}, X)$, the space of bounded, continuous, and X-valued measurable functions. [57]

The space $\mathbf{X} \times \mathbf{\Theta}$ is compact with respect to the product topology since X and Θ are compact. Since the utility function is continuous and strictly concave in all arguments, the maximizer exists and it is unique. The necessary condition for optimality is summarized in the following lemma.

[55]Let $|\cdot|$ be the usual Euclidean norm. For any $(x, \theta), (x', \theta') \in \mathbf{X} \times \mathbf{\Theta}$, let

$$d\left((x, \theta), (x', \theta')\right) := \sum_{a\in\mathbb{A}} 2^{-a} \left(|x_a - x'_a| + |\theta_a - \theta'_a|\right)$$

Since $X = \Theta = [\underline{x}, \bar{x}]$ is a compact interval, this is a well-defined metric that metrizes the product topology on $\mathbf{X} \times \mathbf{\Theta}$. See also Aliprantis and Border (2006), p. 90.

[56]Starting with an initial π_0 which is absolutely continuous, the MPE policy function and the absolutely continuous preference shocks induce a sequence (π_t) of absolutely continuous distributions on t-period equilibrium choice profiles.

[57]Since the planner's choice rule is symmetric, the choice of agent 0 rather than another agent is inconsequential.

Lemma 8 *For any* $(x_{T-1}, \theta_T) \in \mathbf{X} \times \mathbf{\Theta}$,

$$\begin{aligned} 0 &= \alpha_1 \left(x^0_{T-1} - h(x_{T-1}, \theta_T)\right) + \alpha_2 \left(\theta^0_T - h(x_{T-1}, \theta_T)\right) \\ &+ \alpha_3 \left(h(R^{-1}\, x_{T-1}, R^{-1}\, \theta_T) - h(x_{T-1}, \theta_T)\right) + \alpha_3 \left(h(R\, x_{T-1}, R\, \theta_T) - h(x_{T-1}, \theta_T)\right) \\ &- \alpha_3 \left(h(x_{T-1}, \theta_T) - h(R\, x_{T-1}, R\, \theta_T)\right) - \alpha_3 \left(h(x_{T-1}, \theta_T) - h(R^{-1}\, x_{T-1}, R^{-1}\, \theta_T)\right) \end{aligned}$$

Proof: The proof uses an extension of the usual calculus of variation techniques to our symmetric strategic environment. We prove it for the class of bounded, continuous, and measurable, real-valued functions on $\mathbf{X} \times \mathbf{\Theta}$. Then, we use the restriction of the result to a subset of it, the space of bounded, continuous, and measurable, X-valued functions. Suppose that the function h provides the maximum for the planner's problem. For any other admissible function h', define $k = h' - h$. Consider now the expected utility from a one-parameter deviation from the optimal policy h, i.e.,

$$\begin{aligned} J(a) &:= \int u\left(x^0_{T-1}, (h + ak)(x_{T-1}, \theta_T), (h + ak)(R^{-1}\, x_{T-1}, R^{-1}\, \theta_T),\right. \\ &\qquad \left.(h + ak)(R\, x_{T-1}, R\, \theta_T), \theta^0_T\right) \mathbb{P}\,(d\theta_T)\, \pi_{T-1}\,(dx_{T-1}) \end{aligned}$$

where a is an arbitrary real number and u represents the conformity preferences in Assumption 1.. Since h maximizes the planner's problem, the function J must assume its maximum at $a = 0$. Leibnitz's rule for differentiation under an integral along with the chain rule for differentiation gives us

$$J'(a) := \int \left(u_2\, k + u_3\, k \circ R^{-1} + u_4\, k \circ R\right) d\mathbb{P}\, d\pi_{T-1}$$

where u_i is the partial derivative of u with respect to the i-th argument. For J to assume its maximum at $a = 0$, it must satisfy

$$\begin{aligned} J'(0) &:= \int \Big[u_2\left(x^0_{T-1}, h(x_{T-1}, \theta_T), h(R^{-1}\, x_{T-1}, R^{-1}\, \theta_T), h(R\, x_{T-1}, R\, \theta_T), \theta^0_T\right)\, k(x_{T-1}, \theta_T) \\ &+ u_3\left(x^0_{T-1}, h(x_{T-1}, \theta_T), h(R^{-1}\, x_{T-1}, R^{-1}\, \theta_T), h(R\, x_{T-1}, R\, \theta_T), \theta^0_T\right)\, k(R^{-1} x_{T-1}, R^{-1} \theta_T) \\ &+ u_4\left(x^0_{T-1}, h(x_{T-1}, \theta_T), h(R^{-1}\, x_{T-1}, R^{-1}\, \theta_T), h(R\, x_{T-1}, R\, \theta_T), \theta^0_T\right)\, k(R x_{T-1}, R \theta_T) \Big] \\ &\quad \times \mathbb{P}\,(d\theta_T)\, \pi_{T-1}\,(dx_{T-1}) \;=\; 0 \end{aligned}$$

for any arbitrary admissible deviation k. Suppose that the statement of the lemma is not true. This implies that there is an element $(\bar{x}, \bar{\theta}) \in \mathbf{X} \times \mathbf{\Theta}$ such that

$$\begin{aligned} 0 &\neq u_2\left(\bar{x}^0, h(\bar{x}, \bar{\theta}), h(R^{-1}\, \bar{x}, R^{-1}\, \bar{\theta}), h(R\, \bar{x}, R\, \bar{\theta}), \bar{\theta}^0\right) \\ &+ u_3\left(\bar{x}^1, h(R\, \bar{x}, R\, \bar{\theta}), h(\bar{x}, \bar{\theta}), h(R^2\, \bar{x}, R^2\, \bar{\theta}), \bar{\theta}^1\right) \\ &+ u_4\left(\bar{x}^{-1}, h(R^{-1}\, \bar{x}, R^{-1}\, \bar{\theta}), h(R^{-2}\, \bar{x}, R^{-2}\, \bar{\theta}), h(\bar{x}, \bar{\theta}), \bar{\theta}^{-1}\right) \end{aligned} \tag{E.1}$$

Assume w.l.o.g. that the above expression takes a positive value (the proof for the case with a negative value is identical). Since the utility function, its partials, and the deviation functions are all continuous with respect to the product topology, and that the measures π and $\mathbb{P}$ have positive densities, there exists a $(\pi \times \mathbb{P})$-positive measure neigborhood $A \subset \mathbf{X} \times \mathbf{\Theta}$ around $(\bar{x}, \bar{\theta})$ such that the above expression stays positive for all $(x_{T-1}, \theta_T) \in A$.[58] Assume that $a_1 = (\bar{x}, \bar{\theta})$, $a_2 = (R\,\bar{x}, R\,\bar{\theta})$, and $a_3 = (R^{-1}\,\bar{x}, R^{-1}\,\bar{\theta})$ are distinct points. Otherwise, since the underlying space X is a real interval and the maps R and R^{-1} are right and left shift maps, one can always pick a point in A that has that property.

Now choose $\epsilon > 0$ small enough so that the ϵ-balls $B_\epsilon(a_1)$, $B_\epsilon(a_2)$, and $B_\epsilon(a_3)$ are disjoint. R and R^{-1} being both continuous are homeomorphisms. So, one can find $\epsilon > \delta_1 > 0$ and $\epsilon > \delta_2 > 0$ such that $R(B_{\delta_1}(a_1)) \subset B_\epsilon(a_2)$ and $R^{-1}(B_{\delta_2}(a_1)) \subset B_\epsilon(a_3)$. Let $\delta = \min\{\delta_1, \delta_2\}$ and $A_1 := B_\delta(a_1)$. We next define a particular deviation k. Let the function k be defined as

$$k(x, \theta) = k(R\,x, R\,\theta) = k(R^{-1}\,x, R^{-1}\,\theta) = \begin{cases} \gamma\,[\delta - d((x, \theta), a_1)]\,, & \text{if } (x, \theta) \in A_1 \\ 0, & \text{otherwise.} \end{cases} \tag{E.2}$$

where $\gamma > 0$ is a scalable constant. This is possible because A_1, $R(A_1)$ and $R^{-1}(A_1)$ are disjoint sets. Constructed this way, k is a bounded, continuous, and measurable function[59]. Substitute k into equation (E.1). By construction, the only set on which k is positive is the set A_1 which is itself a subset of A, the set of elements of $\mathbf{X} \times \mathbf{\Theta}$ for which the expression (E.1) is positive. Hence, evaluated with the constructed deviation function k, $J'(0) > 0$, a contradiction to the fact that the policy function h was optimal. Therefore the statement of the lemma must be true. This concludes the proof. ■

This implies that

$$\begin{aligned} h(x_{T-1}, \theta_T) \quad &= \quad (\alpha_1 + \alpha_2 + 4\alpha_3)^{-1} \Big(\alpha_1 x^0_{T-1} + \alpha_2 \theta^0_{T-1} \\ &\quad + 2\alpha_3\, h(R^{-1}\,x_{T-1}, R^{-1}\,\theta_T) + 2\alpha_3\, h(R\,x_{T-1}, R\,\theta_T) \Big) \end{aligned} \tag{E.3}$$

As in the proof of existence, the operator induced by (E.3) is a contraction on the Banach space of bounded, continuous, measurable functions with the supnorm, whose unique fixed is in G, defined in (A.3). Therefore, one can fit the following solution

$$h(x_{T-1}, \theta_T) = \sum_a c^a_P\, x^a_{T-1} + \sum_a d^a_P\, \theta^a_T$$

[58] Endowed with the product topology, the space $\mathbf{X} \times \mathbf{\Theta}$ is metrizable by the metric d. See footnote 55. Product topology and the associated metric allows us to choose positive measure proper subsets of X for choices of near-by agents and the whole sets X and Θ for far-away agents, staying at the same time in the close vicinity of the point $(\bar{x}, \bar{\theta})$.

[59] We endow the range space, the real line, with the Borel σ-field hence any continuous function into the real line is automatically measurable.

substituting, we get

$$\sum_a c_P^a\, x_{T-1}^a + \sum_a d_P^a\, \theta_T^a \;=\; (\alpha_1+\alpha_2+4\alpha_3)^{-1}\Bigg[\alpha_1\, x_{T-1}^0 + \alpha_2\, \theta_T^0$$
$$+2\alpha_3\left(\sum_a c_P^a\, x_{T-1}^{a-1} + \sum_a d_P^a\, \theta_T^{a-1}\right) + 2\alpha_3\left(\sum_a c_P^a\, x_{T-1}^{a+1} + \sum_a d_P^a\, \theta_T^{a+1}\right)\Bigg]$$

By matching coefficients, we get for all $a \in \mathbb{A}$

$$c_P^a = (\alpha_1+\alpha_2+4\alpha_3)^{-1}\left[2\alpha_3\, c_P^{a-1} + 2\alpha_3\, c_P^{a+1} + \alpha_1 \mathbf{1}_{\{a=0\}}\right]$$
$$d_P^a = (\alpha_1+\alpha_2+4\alpha_3)^{-1}\left[2\alpha_3\, d_P^{a-1} + 2\alpha_3\, d_P^{a+1} + \alpha_2 \mathbf{1}_{\{a=0\}}\right]$$

The same method as in the proof of Theorem 2 yields for any $a \in \mathbb{A}$,

$$c_P^a = r_P^{|a|}\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)\left(\frac{1-r_P}{1+r_P}\right) \quad \text{and} \quad d_P^a = r_P^{|a|}\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)\left(\frac{1-r_P}{1+r_P}\right) \tag{E.4}$$
$$r_P = \left(\frac{\Delta_P}{2\,\alpha_3}\right) - \sqrt{\left(\frac{\Delta_P}{2\,\alpha_3}\right)^2 - 1} \quad \text{with} \quad \Delta_P = \alpha_1+\alpha_2+4\,\alpha_3. \tag{E.5}$$

We next compare the equilibrium policy sequence in Theorem 2 with the planner's optimal choice coefficient sequence. Notice that

$$\left(\frac{\Delta_P}{2\alpha_3}\right) = \frac{\alpha_1+\alpha_2+4\alpha_3}{2\alpha_3} = 2 + \frac{\alpha_1+\alpha_2}{2\alpha_3} < 2 + \frac{\alpha_1+\alpha_2}{\alpha_3} = \left(\frac{\Delta_1}{\alpha_3}\right)$$

which implies that $r_P > r_1$ since r_P is decreasing in Δ_P by (E.5). Thus, the planner's optimal policy coefficient sequence converges to zero slower than the equilibrium policy coefficient sequence. Moreover, the equilibrium policy cannot satisfy the FOC of the planner's problem. Therefore, the equilibrium is inefficient for finite-horizon economies.

Infinite-Horizon: The argument here is very similar to the one in the finite horizon case. We know from Theorem 1 that the equilibrium has the following structure

$$g(x_{T-1}, \theta_T) = \sum_a c^a\, x_{T-1}^a + \sum_a d^a\, \theta_T^a + e\,\bar{\theta}$$

We argue that this solution cannot satisfy the planner's problem's optimality condition. For a given function $h \in G$ (see (A.3)), define $H : \mathbf{X} \times \boldsymbol{\Theta} \to \mathbf{X}$ as

$$H(x_{T-1}, \theta_T) := \Big(h\left(R^a\, x_{T-1}, R^a\, \theta_T\right)\Big)_{a \in \mathbb{A}} \tag{E.6}$$

Let V^h be the continuation value of using the function h in the future, defined recursively as

$$V^h(x_{T-1},\theta_T) = u\left(x^0_{T-1}, h(x_{T-1},\theta_T), h(R^{-1}x_{T-1}, R^{-1}\theta_T), h(Rx_{T-1}, R\theta_T), \theta^0_T\right) + \beta\int V^h\left(H(x_{T-1},\theta_T), \theta_{T+1}\right)\mathbb{P}(d\theta_{T+1}) \tag{E.7}$$

where u is as in Assumption 1. Since the policy $h\in G$ is linear and the utility function is continuously differentiable and strictly concave with respect to all arguments, elementary dynamic programming techniques (see e.g. Stokey and Lucas (1989)) guarantee that the value function V^h exists, it is bounded, continuous, strictly concave and continuously differentiable. Denote by V^h_a the partial derivative of V^h with respect to agent a's initial choice. Given an initial absolutely continuous distribution π_{T-1} on the space of previous period's choice profiles with positive density, the planner maximizes agent 0's expected discounted utility. So, the planner's problem is

$$\max_{\{h\in G\}} \int\Big[u\left(x^0_{T-1}, h(x_{T-1},\theta_T), h(R^{-1}x_{T-1}, R^{-1}\theta_T), h(Rx_{T-1}, R\theta_T), \theta^0_T\right) + \beta\int V^h\left(H(x_{T-1},\theta_T), \theta_{T+1}\right)\Big]\mathbb{P}(d\theta_T)\,\mathbb{P}(d\theta_{T+1})\,\pi_{T-1}(dx_{T-1})$$

Once again, the solution exists and it is unique thanks to the compactness (with respect to the product topology) of the underlying space $\mathbf{X}\times\mathbf{\Theta}$ and the continuity and strict concavity of the utility and value functions. A straightforward modification of the first order condition argument in the finite case yields, for any $(x_{T-1},\theta_T)\in\mathbf{X}\times\mathbf{\Theta}$

$$\begin{aligned} 0 &= u_2(x^0_{T-1}, h(x_{T-1},\theta_T), h(R^{-1}x_{T-1}, R^{-1}\theta_T), h(Rx_{T-1}, R\theta_T), \theta^0_T) \\ &+ u_3(x^1_{T-1}, h(Rx_{T-1}, R\theta_T), h(x_{T-1},\theta_T), h(R^2x_{T-1}, R^2\theta_T), \theta^1_T) \\ &+ u_4(x^{-1}_{T-1}, h(R^{-1}x_{T-1}, R^{-1}\theta_T), h(R^{-2}x_{T-1}, R^{-2}\theta_T), h(x_{T-1},\theta_T), \theta^{-1}_T) \\ &+ \beta\sum_a\int V^h_a\left(H\left(R^{-a}x_{T-1}, R^{-a}\theta_T\right), \theta_{T+1}\right)\mathbb{P}(d\theta_{T+1}) \end{aligned}$$

But the FOC for equilibrium is

$$\begin{aligned} 0 &= \int u_2(x^0_{T-1}, h(x_{T-1},\theta_T), h(R^{-1}x_{T-1}, R^{-1}\theta_T), h(Rx_{T-1}, R\theta_T), \theta^0_T) \\ &+ \beta\int V^h_a\left(H(x_{T-1},\theta_T), \theta_{T+1}\right)\mathbb{P}(d\theta_{T+1}) \end{aligned}$$

For the equilibrium policy to be efficient, it needs to satisfy both FOCs for any $(x_{T-1},\theta_T)\in\mathbf{X}\times\mathbf{\Theta}$. For the quadratic specification, this entails

$$\begin{aligned} 0 &= 2\alpha_3\left(h(x_{T-1},\theta_T) - h(Rx_{T-1}, R\theta_T)\right) + 2\alpha_3\left(h(x_{T-1},\theta_T) - h(R^{-1}x_{T-1}, R^{-1}\theta_T)\right) \\ &+ \beta\sum_{a\neq 0}\int V^h_a\left(H\left(R^{-a}x_{T-1}, R^{-a}\theta_T\right), \theta_{T+1}\right)\mathbb{P}(d\theta_{T+1}) \end{aligned}$$

Substituting the equilibrium policy function g for h and recollecting terms

$$\begin{aligned}&2\alpha_3\left[\sum_a c^a\left(2x^a_{T-1}-x^{a-1}_{T-1}-x^{a+1}_{T-1}\right)+\sum_a d^a\left(2\theta^a_{T-1}-\theta^{a-1}_{T-1}-\theta^{a+1}_{T-1}\right)\right]\\&+\beta\sum_{a\neq 0}\int V^h_a\left(H\left(R^{-a}\,x_{T-1},R^{-a}\,\theta_T\right),\theta_{T+1}\right)\mathbb{P}\left(d\theta_{T+1}\right)=0\end{aligned}\tag{E.8}$$

The next lemma says that there exists a positive measure subset of the underlying space on which the expression in (E.8) is non-zero.

Lemma 9 *Let $\left(\hat{x},\hat{\theta}\right)\in\mathbf{X}\times\mathbf{\Theta}$ be the point where $\hat{x}_a=\bar{x}$ and $\hat{\theta}_a=\bar{x}$, for all $a\in\mathbb{A}$*[60]*. The expression in (E.8) is negative on a positive measure subset $E\subset\mathbf{X}\times\mathbf{\Theta}$, that includes $\left(\hat{x},\hat{\theta}\right)$.*

Proof: Using (E.7) iteratively, one can write for any $(x_T,\theta_{T+1})\in\mathbf{X}\times\mathbf{\Theta}$

$$\begin{aligned}V^h_a\left(x_T,\theta_{T+1}\right) &= \int\sum_{t=T+1}^{T+N}\beta^{t-T-1}\Bigg[2\alpha_1\left(x^0_t-x^0_{t-1}\right)\frac{\partial}{\partial x^a_T}\left(x^0_{t-1}-x^0_t\right)+2\alpha_2\left(\theta^0_t-x^0_t\right)\frac{\partial}{\partial x^a_T}x^0_t\\&+2\alpha_3\left(x^{-1}_t-x^0_t\right)\frac{\partial}{\partial x^a_T}\left(x^{-1}_t-x^0_t\right)+2\alpha_3\left(x^1_t-x^0_t\right)\frac{\partial}{\partial x^a_T}\left(x^1_t-x^0_t\right)\\&+\beta^{N+1}\,V^h_a\left(x_{t+N},\theta_{t+N+1}\right)\Bigg]\prod_{i=1}^{N}\mathbb{P}\left(d\theta_{T+1+i}\right)\end{aligned}\tag{E.9}$$

where x_t is written as, using iterations of the policy function g and Lemma 2 (i) with x_T instead of x_1

$$\begin{aligned}x^a_t &= \sum_{b_1\in\mathbb{A}}\cdots\sum_{b_{t-T}\in\mathbb{A}}c^{b_1}\cdots c^{b_{t-T}}\,x_T^{a+b_1+\cdots+b_{t-T}}\\&+\sum_{s=1}^{t-T}\sum_{b_1\in\mathbb{A}}\cdots\sum_{b_{s-1}\in\mathbb{A}}c^{b_1}\cdots c^{b_{s-1}}\left(\sum_{b_s\in\mathbb{A}}d^{b_s}\theta^{a+b_1+\cdots+b_s}_{t-(s-1)}+e\,\overline{\theta}\right)\end{aligned}\tag{E.10}$$

At the point $\left(\hat{x},\hat{\theta}\right)$, $x^a_T=\bar{x}$ for all $a\in\mathbb{A}$. So, the first part after the equality sign in (E.10) is the same for all agents. Since the preference shocks are i.i.d., the second part will be the same for all agents in expectations, which eliminates the terms in the second line after the equality sign in (E.9). Thanks to Lemma 2 (i), $\frac{\partial}{\partial x^a_T}x^0_t>0$ for any $a\in\mathbb{A}$, and for all $t=T+1,\ldots,T+N$. But then, the second term in (E.9) after the first bracket is negative in expectations. This is because using (E.10) $E[x^0_t\,|\,(x_T,\theta_{T+1})]=C^{t-T}\,\bar{x}+(1-C^{t-T})\,\bar{\theta}>\bar{\theta}$, where $C=\sum_a c^a$. The first term after the bracket sign too is negative in expectations. Here is why: The term

$$\begin{aligned}E[\left(x^0_t-x^0_{t-1}\right)|\,(x_T,\theta_{T+1})] &= C^{t-T}\,\bar{x}+(1-C^{t-T})\,\bar{\theta}-C^{t-1-T}\,\bar{x}-(1-C^{t-1-T})\,\bar{\theta}\\&= C^{t-1-T}\,(1-C)\left(\bar{\theta}-\bar{x}\right)\;<\;0\end{aligned}$$

[60]Recall from Assumption 1 that $\bar{x}$ is the upper boundary of the feasible action and type sets X and Θ.

for any $t = T+1, \ldots, T+N$. So, one can write

$$\begin{aligned} E\left[2\alpha_1\left(x_t^0 - x_{t-1}^0\right)\frac{\partial}{\partial x_T^a}\left(x_{t-1}^0 - x_t^0\right) \mid (x_T, \theta_{T+1})\right] &< E\left[2\alpha_1\left(x_t^0 - x_{t-1}^0\right)\frac{\partial}{\partial x_T^a} x_{t-1}^0 \mid (x_T, \theta_{T+1})\right] \\ &< 0 \end{aligned}$$

which shows that the summand in (E.9) is negative in expectations in every period. In turn, the whole sum, then, until the last line of (E.9), is negative in expectations for any arbitrary N. The choice of a was arbitrary and that V_a^h is continuous on $\mathbf{X} \times \mathbf{\Theta}$ for any $a \in \mathbb{A}$. The latter is compact with respect to the product topology. Hence, V_a^h is bounded. So, one can choose an N large enough to make the $\beta^{N+1} V_a^h(x_{t+N}, \theta_{t+N+1})$ term arbitrarily small. This implies that the whole expression in (E.9) is negative, which in turn means that $V_a^h\left(\hat{x}, \hat{\theta}\right) < 0$ for any $a \in \mathbb{A}$.

At the point $\left(\hat{x}, \hat{\theta}\right)$, the first line of (E.8) is zero and the second line is negative, as we just showed, which makes the whole expression in (E.8) negative. Since the first line in (E.8) is continuous and so are V_a for any $a \in \mathbb{A}$, the whole expression in (E.8) is continuous. Hence, as in the proof of Lemma 8, there exists a $(\pi \times \mathbb{P})$-positive measure neigborhood $E \subset \mathbf{X} \times \mathbf{\Theta}$ around $(\bar{x}, \bar{\theta})$ such that the above expression stays negative for all $(x_{T-1}, \theta_T) \in E$. This concludes the proof. ∎

The statement of Lemma 9 leads to a contradiction since it means that the planner's optimal rule and the equilibrium policy function g does not agree on E. Therefore, g is inefficient. This concludes the proof. ∎

F Identification

F.1 Rationality vs. Myopia

We prove here Proposition 2. We showed in the proof of Proposition 1 (Tail Convergence Monotonicity) that, for $T = 1$, the ratio $\left(\frac{\rho_{a+1,T}}{\rho_{a,T}}\right)$ is necessarily monotonically decreasing in a for any underlying preference parameter vector α, converging eventually, at the tail, to the rate r_1 given in Theorem 2 (i). Moreover, as we showed in Section 4.2, the cross-sectional covariances at the stationary distribution can be written recursively given the weights of the policy function. For the myopic policy function, they take the form

$$Cov\left(x^0, x^a\right) = \sum_{a_1 \in \mathbb{A}} \sum_{b_1 \in \mathbb{A}} c_1^{a_1} c_1^{b_1} Cov\left(x^{a_1}, x^{a+b_1}\right) + Var(\theta) \sum_{a_1 \in \mathbb{A}} d_1^{a_1} d_1^{a_1 - a}, \tag{F.1}$$

Since the c_1 and d_1 sequences are exponential at the rate r_1 from Theorem 2 (i), by straightforward modifications of the arguments in the first part of the proof of Proposition 1, the ratio of consecutive covariances for the myopic, $\left(\frac{Cov(x^0, x^{a+1})}{Cov(x^0, x^a)}\right)$ converges monotonically as a gets large.

However, the above ratio for the stationary policy function is non-monotonic for a set of parameter values. See the table below where we plot the above ratio for any parameter vector $\bar{\alpha}$ where $\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha}_3$.

$r_a := \left(\frac{\rho_{a+1}}{\rho_a}\right)$

r_1	...	r_5	...	r_9	...	r_∞
.4757	...	.3543	...	.3571	...	.3680

The values dip first then come back up which proves non-monotonicity of $\left(\frac{\rho_{a+1}}{\rho_a}\right)$ in a for the chosen parameter vector $\bar{\alpha}$. Moreover, the map in (5) that generates the policy weights as fixed points is continuous in the parameters $(\alpha_1, \alpha_2, \alpha_3)$ of the utility function. Thus, there is an open-neighborhood around $\bar{\alpha}$ such that for each element $\hat{\alpha}$ of that neighborhood, the same non-monotonicity property obtains. This concludes the proof. ■

We conjecture that this non-monotonicity property should hold for the entire admissible parameter space. As an illustration we present in Figure 11, the ratio $\left(\frac{\rho_{a+1,T}}{\rho_{a,T}}\right)$ (*y-axis*) as a function of a (*x-axis*), at the stationary distribution, for different levels of strength of interaction proxied by the ratio $\left(\frac{2\alpha_3}{\Delta_1}\right)$.[61] Clearly, for a large set of parameters, non-monotonicity obtains at the stationary distribution. The limit auto-correlation function for the myopic model, on the contrary, inherits the behavior of its one-step transition counterpart: it converges at a monotonically decreasing rate.

F.2 Lack of Identification in Infinite Horizon Economies

We prove here Proposition 4. In the case of complete information, the policy function is:

$$x_t^a = \sum_{b \in \mathbb{A}} c^b(\alpha)\, x_{t-1}^{a+b} + \sum_{b \in \mathbb{A}} d^b(\alpha)\, \theta_t^{a+b} + e\, \bar{\theta}$$

As we saw in Lemma 2, one can obtain by iteration the reduced form[62]

$$\begin{aligned} x_t^a \;=\; & \sum_{b_1} \cdots \sum_{b_t} c(\alpha)^{b_1} \cdots c(\alpha)^{b_t}\; x_0^{a+b_1+\cdots+b_t} \\ & + \sum_{s=1}^{t} \sum_{b_1} \cdots \sum_{b_{s-1}} c(\alpha)^{b_1} \cdots c(\alpha)^{b_{s-1}} \left(\sum_{b_s} d(\alpha)^{b_s}\; \theta_{t-(s-1)}^{a+b_1+\cdots+b_s} + e(\alpha)\, \bar{\theta} \right) \end{aligned}$$

[61] More precisely, we set $\left(\frac{\alpha_1}{\alpha_2}\right) = 1$, $\left(\frac{2\alpha_3}{\Delta_1}\right) \in \{0.1, 0.2, 0.75, 0.9\}$ and $\beta = .95$. Note that the results are independent of particular values of α_i as long as the ratio $\left(\frac{2\alpha_3}{\Delta_1}\right)$ is the same.

[62] In Lemma 2, the iteration stops once it reaches period 1. But, since a stationary MPE exists by Theorem 1, we iterate here once more on the form in Lemma 2 using the stationary policy function and write equilibrium choices as a function of the initial conditions x_0.

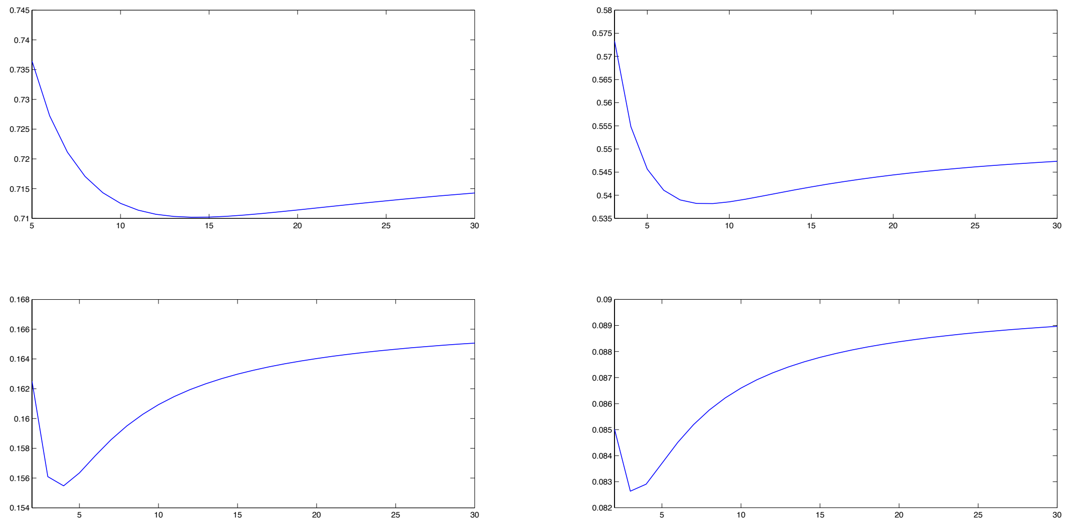

Figure 11: The ratio r_a as a function of a for different values of $\left(\frac{2\alpha_3}{\Delta_1}\right)$ at the stationary distribution. Namely, all parameter vectors α such that $\left(\frac{2\alpha_3}{\Delta_1}\right) = 0.9, 0.75, 0.2, 0.1$, respectively for the *Top-Left*, *Top-Right*, *Bottom-Left*, and *Bottom-Right* panels.

Consider now the alternative specification with *no interactions* between agents ($\hat{\alpha}_3 = 0$) and *no habits* ($\hat{\alpha}_1 = 0$), a preference shock process $\{\hat{\theta}_t^a\}_{t\geq 1}^{a\in\mathbb{A}}$ and *own type effects* with $\hat{\alpha}_2 > 0$. For this economy, equilibrium choice of agent a at time t is given by

$$x_t^a = \hat{\theta}_t^a$$

Defining the new preference shock process $\{\hat{\theta}_t^a\}_{t\geq 1}^{a\in\mathbb{A}}$ as

$$\begin{aligned}\hat{\theta}_t^a : \quad = & \quad \sum_{b_1} \cdots \sum_{b_t} c(\alpha)^{b_1} \cdots c(\alpha)^{b_t} \; x_0^{a+b_1+\cdots+b_t} \\ & + \sum_{s=1}^{t} \sum_{b_1} \cdots \sum_{b_s} c(\alpha)^{b_1} \cdots c(\alpha)^{b_{s-1}} \left(d(\alpha)^{b_s} \; \theta_{t-(s-1)}^{a+b_1+\cdots+b_s} + e(\alpha)\, \overline{\theta} \right)\end{aligned}$$

would imply that for an arbitrary initial distribution π_0 for x_0, the joint probability distributions that the two specifications (with and without interactions)) generate on the process $\{x_t^a\}_{t\geq 1}^{a\in\mathbb{A}}$, are identical. Moreover, if one allows for infinite histories, one can define the preference shock process $\{\hat{\theta}_t^a\}_{t\geq 1}^{a\in\mathbb{A}}$ as before

$$\hat{\theta}_t^a := \frac{e(\alpha)\, \overline{\theta}}{1 - C(\alpha)} + \sum_{s=1}^{\infty} \sum_{b_1} \cdots \sum_{b_s} (c(\alpha))^{b_1} \cdots (c(\alpha))^{b_{s-1}} \left((d(\alpha))^{b_s} \; \theta_{t-(s-1)}^{a+b_1+\cdots+b_s} \right)$$

and obtain observational equivalence once again. Hence, we conclude that identification is not possible. This concludes the proof. ■

F.3 Identification in Finite-Horizon Economies

At the final period ($T < \infty$), the policy weights c and d do not change. At period $T-1$ (and before) though, we claim that the policy functions change in order to take into account the given information on $n-1$ past realization of the stochastic process, namely

Lemma 10 *For any $a \in \mathbb{A}$ and any period $t \leq T-1$,*

$$x_t^a = \sum_b c_{T-t+1}^b \, x_{t-1}^{a+b} + \sum_{s=0}^{n-1} \sum_b d_{2,s}^b \, \theta_{t-s}^{a+b}.$$

if $\alpha_3 > 0$ and

$$x_t^a = \hat{c}_{T-t+1} \, x_{t-1}^a + \sum_{s=0}^{n-1} \sum_b \hat{d}_{2,s}^b \, \theta_{t-s}^{a+b}.$$

if $\alpha_3 = 0$. For the final period ($t = T$), since there is no continuation there is no expectation and one gets

$$x_t^a = \sum_b c_1^b \, x_{T-1}^{a+b} + \sum_b d_1^b \, \theta_T^{a+b}.$$

if $\alpha_3 > 0$ and

$$x_t^a = \hat{c}_1 \, x_{t-1}^a + \hat{d}_1 \, \theta_T^{a+b}.$$

if $\alpha_3 = 0$.

Proof: Equation (A.10) is linear in all variables. Moreover, expectations are assumed to be linear too in Section 6.2.2. So, the same arguments used to get (A.12) can be used to obtain

$$0 \quad = \quad -x_1^0 \, \Delta_T + \alpha_1 \, x_0^0 + \alpha_2 \, \theta_1^0 + \sum_{a \neq 0} \gamma_T^a \, x_1^a + \sum_{s=0}^{n-1} \sum_b \mu_{T,s}^b \, \theta_{1-s}^b \tag{F.2}$$

where $\Delta_T := \alpha_1 + \alpha_2 + \sum_{a \neq 0} \gamma_T^a + \sum_{s=0}^{n-1} \sum_b \mu_{T,s}^b$ and

$$\begin{aligned} \mu_{T,s}^b \quad = \quad & \alpha_2 \, I_{\{b=0,s=0\}} \\ & + \frac{\partial}{\partial \theta_{1-s}^b} \, E\left[\sum_{\tau=2}^{T} \beta^{\tau-1} \left(-\alpha_1 \, \left(x_{\tau-1}^0 - x_\tau^0\right) \frac{\partial}{\partial x_1^0} \left(x_{\tau-1}^0 - x_\tau^0\right) + \alpha_2 \, \left(\theta_\tau^0 - x_\tau^0\right) \frac{\partial}{\partial x_1^0} \, x_\tau^0 \right.\right. \\ & \left.\left. -\alpha_3 \, \left(x_\tau^{-1} - x_\tau^0\right) \frac{\partial}{\partial x_1^0} \left(x_\tau^{-1} - x_\tau^0\right) - \alpha_3 \, \left(x_\tau^1 - x_\tau^0\right) \frac{\partial}{\partial x_1^0} \left(x_\tau^1 - x_\tau^0\right) \right) \middle| \left(x^{t-1}, \theta^t\right) \right] \end{aligned} \tag{F.3}$$

As in (A.17), the above equation induces a contraction operator on the space of coefficients. If $\alpha_3 > 0$, the fixed point satisfies, for $b \in \mathbb{A}$ and $s = 0, \ldots, n-1$,

$$d^b_{T,s} = \Delta_T^{-1} \left[\sum_{a \neq 0} \gamma^a_T \, d^{b-a}_T + \mu^b_{T,s} \right] \tag{F.4}$$

Since $\Delta_T < \infty$, $\mu^b_{T,s} \to 0$ as $b \to \infty$. The weights on history c_T are the same as in the model with i.i.d. shocks since the weights (γ^a_T) do not change. If $\alpha_3 = 0$, $\gamma^a_T = 0$ for all $a \neq 0$ and the fixed point satisfies, for $b \in \mathbb{A}$ and $s = 0, \ldots, n-1$,

$$\hat{d}^b_{T,s} = \Delta_T^{-1} \left[\mu^b_{T,s} \right] \tag{F.5}$$

For the last period, for $\alpha_3 > 0$, we get the policy of the i.i.d. case. For $\alpha_3 = 0$, equation (A.2) becomes

$$x^0_1 = \Delta_1^{-1} \left(\alpha_1 \, x^0_0 + \alpha_2 \, \theta^0_1 \right)$$

with $\Delta = \alpha_1 + \alpha_2$. Thus, $\hat{c}_1 = \frac{\alpha_1}{\alpha_1 + \alpha_2}$ and $\hat{d}_1 = \frac{\alpha_2}{\alpha_1 + \alpha_2}$ yield the form in the statement. ■

G Extensions

Proof of Proposition 6

The analogue for the expression (A.2) implied by the FOC in the final period of the economy with a general network structure is

$$x^a_1 = \Delta_{a,1}^{-1} \left(\alpha_{a,1} \, x^a_0 + \alpha_{a,2} \, \theta^a_1 + \sum_{b \in N(a)} \alpha_{a,b} \, x^b_1 \right) \tag{G.1}$$

where $\Delta_{a,1} := \alpha_{a,1} + \alpha_{a,2} + \sum_{b \in N(a)} \alpha_{a,b} > 0$. Hence, the best-response for agent a, $BR^a_1 : G^{\mathbb{A} \setminus \{a\}} \to G$, is uniformly continuous for all $a \in \mathbb{A}$, which implies that so is the induced best-response profile $BR_1 = (BR^a_1) : G^{\mathbb{A}} \to G^{\mathbb{A}}$. The convex set $G^{\mathbb{A}}$ is also compact thanks to Lemma 4. Hence a fixed point exists (not necessarily unique). Thus, an equilibrium exists for a one-period economy. If the uniform boundedness condition holds, the rate at which each one of the maps in (G.1) contracts is uniformly bounded. Hence, BR_1 becomes a contraction operator on $G^{\mathbb{A}}$ implying uniqueness of the equilibrium for a one-period economy. For a $T < \infty$ period economy, one mimicks the Induction arguments of the general existence proof to obtain the analogue of the expression (A.12) in Lemma 3, i.e.,

$$0 \quad = \quad -x^a_1 \, \Delta_{a,T} + \alpha_{a,1} \, x^a_0 + \alpha_{a,2} \, \theta^a_1 + \sum_{b \neq a} \gamma^{a,b}_T \, x^b_1 + \mu^a_T \, \bar{\theta} \tag{G.2}$$

where $\Delta_{a,T} := \alpha_{a,1} + \alpha_{a,2} + \sum_{b\neq a} \gamma_T^{a,b} + \mu_T^a$. Applying the above arguments to the best-response profile induced by the infinite system of equations (G.2) for each a gives existence of an equilibrium for a $T < \infty$ period economy. Since $\gamma_T^{a,b}$ is the total effect of a change in x_1^b ($b \neq a$) on the expected discounted marginal utility of agent a (as in expression (A.13)), the sum of these effects is uniformly bounded across agents if the peer effects are uniformly bounded across agents. Hence, once again, the best-response profile for a T-period economy, BR_T, becomes a contraction on $G^{\mathbb{A}}$, implying uniqueness of equilibrium. Moreover, using the same arguments in the third step of the general existence proof, the sequence of unique equilibria approximates a stationary equilibrium as the horizon length becomes arbitrarily large. ■

Proof of Proposition 7

Policy function for a one-period economy does not change since there is no expectations. Suppose now that the form in the proposition is true up to $T-1$ period economies. FOCs given (x_0, θ_1) give rise to an analogue of the map in (A.16) for agent a

$$x_1^a \quad = \quad \Delta_{a,T}^{-1} \left(\alpha_1 \, x_0^a + \alpha_2 \, \theta_1^a + \sum_{b\neq 0} \gamma_T^b \, x_1^{a+b} + \mu_{T-(t-1)}^a(\theta_t) \right) \tag{G.3}$$

where thanks to the linearity of the problem, the only change is in the coefficient for the expectations: $\mu_{T-(t-1)}\,\bar{\theta}$ is replaced by $\mu_{T-(t-1)}^a(\theta_t)$. The latter represents the total expected discounted (with the future policy weights) effect of future preference shocks on agent a's current optimal choice, i.e.,

$$\begin{aligned}\mu_{T-(t-1)}^a(\theta_t) \quad := \quad & E_Q \Bigg[\sum_{s=t+1}^{T} \beta^{s-1} \Bigg(-\alpha_1 \left(\eta_{s-1}^a - \eta_s^a\right) \frac{\partial}{\partial x_t^a} \left(x_{s-1}^a - x_s^a\right) + \alpha_2 \left(\theta_s^a - \eta_s^a\right) \frac{\partial}{\partial x_t^a} x_s^a \\ & -\alpha_3 \left(\eta_s^{a-1} - \eta_s^a\right) \frac{\partial}{\partial x_t^a} \left(x_s^{a-1} - x_s^a\right) - \alpha_3 \left(\eta_s^{a+1} - \eta_s^a\right) \frac{\partial}{\partial x_t^a} \left(x_s^{a+1} - x_s^a\right) \Bigg) \Bigg| (x_{t-1}, \theta_t) \Bigg]\end{aligned}$$

where the η_s^b term represents, similarly, the stochastic part (see the second line of equation (A.7) in Lemma 2) of the total discounted effect of agent b on agent a's t-th period optimal choice, i.e., for $s = t+1, \ldots, T$

$$\begin{aligned}\eta_s^b \quad := \quad & \sum_{\tau=1}^{s-t+1} \sum_{b_1\in\mathbb{A}} \cdots \sum_{b_{\tau-1}\in\mathbb{A}} c_{T-(s-1)}^{b_1} \cdots c_{T-(s-(\tau-1))}^{b_{\tau-1}} \Bigg(\sum_{b_\tau\in\mathbb{A}} d_{T-(s-\tau)}^{b_\tau} \theta_{s-(\tau-1)}^{b+b_1+\cdot s+b_\tau} \\ & + e_{T-(s-\tau)}(\theta_{s-(\tau-1)}, b + b_1 + \cdots + b_{\tau-1}) \Bigg)\end{aligned}$$

The rest of the proof uses the same arguments as in the proof of Proposition 6 along with the extra fact that now the normalized coefficients in (G.3) all depend on θ_t since the last one $\frac{\mu_{T-(t-1)}^a(\theta_t)}{\Delta_{a,T}}$

does. ■

Proof of Proposition 8

The proof uses straightforward modifications of the arguments in Section 5 of Bisin, Horst, and Özgür (2006) to our environment. Interested reader should consult that work. ■

Proof of Proposition 9

Similar to the proof of Proposition 6, the expressions (A.2) and (A.12) change this time to

$$x_1^a = \Delta_{a,1}^{-1} \left(\alpha_{a,1}\, S_1^a + \alpha_{a,2}\, \theta_1^a + \sum_{b \in N(a)} \alpha_{a,b}\, x_1^b \right) \tag{G.4}$$

and

$$0 \quad = \quad -x_1^a\, \Delta_{a,T} + \alpha_{a,1}\, S_1^a + \alpha_{a,2}\, \theta_1^a + \sum_{b \neq a} \gamma_T^{a,b}\, x_1^b + \mu_T^a\, \bar{\theta} \tag{G.5}$$

respectively. The rest is a direct application of the arguments in the proof of Proposition 6, because the coefficients in the new system of maps above are the same as those in (G.1) and (G.2). ■

H Details about the simulations

We build an artificial economy that consists of a large number of agents ($|\mathbb{A}| = 1300, 2500$, and 5000, depending on the treatment) distributed on the one-dimensional integer lattice. At both ends, "buffer" agents that act randomly are added to smooth boundary effects. Depending on the treatment, we start the economy with the following initial configuration of choices: (i) the highest action for all agents; (ii) the lowest action for all agents, (iii) the action equal to the mean shock for all agents.

The core engine behind the simulations is a Matlab code, **g.m**, which computes the equilibrium policy weights recursively as outlined in Section 3.2 of the paper. The code is posted on Özgür's webpage, http://www.sceco.umontreal.ca/onurozgur/, at the Université de Montréal; the code contains also detailed explanations. The correlation computations use two other codes, **cor.m** and **cor_s.m**, also available on Özgür's webpage.

All three codes use as input parameters values of the preference parameters α_i, $i = 1, 2, 3$, the discount factor β, the horizon for the economy T, the number of agents $|\mathbb{A}|$, and the longest distance between agents for which the equilibrium correlation is computed M.

For the limit distributions results, once **g.m** computes the policy weights, we let the computer draw $(\theta_t^a)_{a=1}^{|\mathbb{A}|}$ from the interval $[-D, D]$ according to the uniform distribution (this is for simplicity since all results in the paper are distribution-free).