Bilateral assignment and optimal desegregation

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Abstract

We develop a model of bilateral assignment under capacity constraints, under the relevant version of the celebrated Consistency axiom: every submatrix of an optimal assignment matrix must be optimal as well.

When the unconstrained proportional assignment is the ideal assignment, the only consistent approximation under our constraints minimizes the entropy of the assignment matrix; exactly like the Mutual Information index of segregation ([9]).

More general consistent assignment rules minimize a common additive welfare function over the entries of the matrix.

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1 Introduction and the punchlines

1.1 Optimal desegregation

Reducing segregation is an important policy issue that requires a measurement tool. The typical instance involves a school district with students of various backgrounds (ethinicity, gender, academic status, etc..) and several schools.[1] The assignment of students to schools involve *no segregation* if the relative distribution of student types is the same in every school of the district. This ideal situation is unlikely, so the practical question is to decide whether segregation increases or decreases when the assignment changes. Inspired by the theory of inequality indices, a large literature proposes a great variety of segregation indices ([11] and [9] are two good surveys.) providing different systematic answers to this question. Such an index is null only for the *proportional assignment,* and increases to 1, or sometimes ∞, for maximally segregated assignments (for instance, a single ethnic group per school).

Rather than focusing on the ordering of assignment matrices, we consider the choice of one assignment under exogenous constraints, specifically lower

[1] School segregation is but one paramount example; others include the gender distribution in jobs of hierarchically ranked status, of education or health level relative to ethnicity of income class , and so on.

and upper bounds on the number of students of a certain type enrolled at a certain school. If the segregation-minded public authority can freely assign the ethnic groups in the district to its schools, it will do so proportionally; what assignment is optimal if some capacity constraints rule out the ideal proportional assignment? We allow arbitrary lower and upper bounds on each entry of the assignment matrix: for instance a school has to re-enroll a certain amount of students of each group; geographic constraints prevent it from enrolling more than a certain amount of students in certain groups; and so on .

When we fix the school capacities and the sizes of the different groups of students, most existing segregation indices proposed in the literature are a convex function of the feasible assignment matrices, with a unique minimum at the proportional assignment. This is true for the Dissimilarity and Gini indices, as well as the family of Akinson's indices. Therefore each such index answers the question above by choosing the assignment minimizing under constraints its own measure of segregation. The only restriction is that the constraints define a convex set of feasible assignments, which is certainly the case for the capacity constraints we discuss here.

We single out a particular measure of segregation known as the *Mutual Information* index, introduced by Theil [16] and axiomatized by Frankel and Volij [9], with the help of two axioms. These axioms are familiar choice-theoretic statements that convey no intuition of what it means for an assignment to be more or less segregated.

The first requirement is that the exogenous capacity constraints should not have any normative meaning, which leads to the property we call *constraint neutrality.* If the assignment optimal under certain constraints satisfies tighter constraints, it should still be optimal under the latter constraints; moreover we can ignore a non binding constraint. Note that minimizing a strictly convex index as two paragraphs above, is a constraint-neutral assignment rule.

Our second axiom is novel in the context of the segregation problem, but very familiar to the fair allocation literature, where it has played a central role for the last three decades (see [21], [18], [19]). In the bilateral assignment problem the celebrated *Consistency* principle means that if we fix the capacity constraints, and an assignment matrix is "optimal" given those constraints, then any submatrix is an optimal assignment for the corresponding subproblem. As usual, Consistency by itself conveys no judgment of fairness, it is only a decentralization property: every part of an optimal division should be optimal.[2]

Theorem 1 states that the combination of Constraint Neutrality and Consistency, together with standard Symmetry and Continuity conditions, captures a single assignment rule. It maximizes under the capacity constraints the total entropy of the assignment matrix. This is precisely what the Mutual Information segregation index recommends. In addition, the optimal assignment takes a simple multiplicative form similar to that of the unconstrained case, and described in the next subsection.

[2] In the words of Balinski and Young [3]: "every part of a fair division should be fair"; yet fairness is not relevant here; CSY applies to any choice method, however unfairly it treats the participants, and it is satisfied by some very unfair rules.

1.2 Capacited proportional assignment

The two rationality axioms driving the result, Constraint Neutrality and Consistency, are not related to any test of fairness. Our only normative requirement is the uncontroversial postulate that full proportionality is optimal when feasible. Therefore our Theorem 1 can be applied in many other contexts than segregation where proportionality is a compelling ideal. For instance a chain store assigns consumer goods of different "quality" to different stores in the chain, and fairness demands that every store gets the same proportion of high, medium and low quality goods. Similarly for the distribution of jobs between workers, of itemized funds (for infrastructure, for health, etc) by a public authority between different communities; and so on. Exogenous capacity constraints are natural in those new contexts as well: some stores may have a limited storage capacty for certain goods; some communities may limit spending on some items, they may have incompressible needs for other items; and so on.

We propose a simple way to adapt the proportional ideal in the presence of exogenous capacity constraints. To fix ideas we speak of a set of *agents* , labeled $i, i \in N$, and a set of *resources* $a, a \in A$ (both finite). In the school example, each agent i is a different type of students (e.g. an ethnic group), and each resource a is a different school. Also given are the total capacity x_i of agent i (the number of type i students) and r_a of resource a (the number of students at school a). In the funding example i is a community and x_i its total budget, while r_a is the total amount to spend on item a.

We assume *budget-balance*, $\sum_N x_i = \sum_A r_a = b$, and must choose a non negative *assignment* $[y_{ia}]$ of the resources to the agents: that is, we require $\sum_A y_{ia} = x_i$ for all i, and $\sum_N y_{ia} = r_a$ for all a. The *proportional* assignment $y_{ia} = \frac{x_i \cdot r_a}{b}$ achieves the ideal of equal relative representation of each resource in the basket of each agent. We adapt as follows this multiplicative form in the presence of exogenous capacity constraints $q_{ia}^- \leq y_{ia} \leq q_{ia}^+$ for each i, a. The *capacited proportional assignment* is the only feasible assignment that can be written, for some positive numbers λ_i, μ_a, as

$$y_{ia} = med\{(\lambda_i \cdot \mu_a), q_{ia}^-, q_{ia}^+\} \text{ for all } i, a \tag{1}$$

(where med is the median operator). Equivalently, this assignment minimizes the entropy function $\sum_{(i,a) \in N \times A} y_{ia} \cdot \ln(y_{ia})$ over all feasible assignments: Proposition 1.

1.3 Consistent bilateral assignment

The bilateral assignment problem addresses the more general question of dividing the resources $a \in A$, between the agents $i \in N$, given the capacities x_i, r_a, and budget balanced as above, but does not necessarily regards proportionality as ideal. For instance if the entry y_{ia} is the load of a truck from the warehouse a to the retailer i, and we may want to equalize the loads across all trucks, perhaps for efficiency reasons; as this ideal is typically unfeasible, we will instead maximize the leximin ordering over all feasible assignments.

We ask two questions.

We consider first *unconstrained* assignment problems (N, A, x, r), where we can choose any flow $y = (y_{ia}) \in \mathbb{R}_+^{N\times A}$ such that $\sum_A y_{ia} = x_i$ for all i, and $\sum_N y_{ia} = r_a$ for all a. Which choice rules respect the Consistency principle (any submatrix of an optimal assgnment matrix is optimal too), together with the standard Symmetry and Continuity conditions? Our Theorem 2 provides the answer under an additional monotonicity restriction on the choice rule. *Srict Resource Monotonicity* says that a transfer from resource a to resource b, ceteris paribus, increases y_{ia} and decreases y_{ib} for all i. Theorem 2, our most challenging result, shows that a consistent and strictly resource monotonic assignment rule must be "welfarist": it selects the assignment minimizing a separably additive function $\sum W(y_{ia})$ where W is a smooth and strictly convex function such that $W'(0) = -\infty$. We speak of *consistent welfarist* assignment rules. This result is a relative of Young's Theorem [20] on rationing rules[3], where Consistency, Symmetry and Continuity force the minimization of a convex objective function, separably additive in individual shares. Our proof uses critically Young's result.

Then we ask how we can adapt consistently such a welfarist rule in the presence of exogenous capacity constraints? Our answer generalizes Theorem 1: we show that the only way is to minimize the same welfarist objective under the additional constraints (Proposition 3), and give a parametric representation of the corresponding assignment similar to (1) (Proposition 2).

Contents Section 2 reviews the literature. Section 3 introduces the model and basic definitions. Section 4 discusses the capacited proportional assignment and the corresponding choice rule (Theorem 1). Section 5 is devoted to the more general capacited welfarist assignments (and choice rules). Section 6 states their partial characterization as consistent asignment rules (Theorem 2). Section 7 gathers some concluding comments and open questions.

2 Related literature

2.0.1 On the Mutual Information index

In addition to providing a very useful survey of the main indices in the literature, Frankel and Volij [9] characterize the Mutual Information index as an ordering of assignment matrices (their Theorem 2). Thus their axioms imply that on any convex subset of feasible assignments, one should simply minimize this index, whereas our Theorem 1 implies the same conclusion only for "rectangular" subsets (defined by a lower and an upper bound on each entry). On the other hand our characterization is much leaner than theirs.

Both results share Symmetry and Continuity requirements. Frankel and Volij assume that we select an assignment minimizing some ordering, whereas our Constraint Neutrality requirement is implied by, but does not imply, the

[3] A rationing problem, in our notations, is a bilateral assignment problem with a single resource a, and budget deficit: $r_a \leq \sum_N x_i$.

existence of such an ordering. Our central Consistency axiom is similar in spirit and in bite to their Independence axiom, requiring that if two assignment matrices differ only in a submatrix, what happens in the submatrix is all we need to order the two matrices. Though CSY and Independence are not logically related, Independence is the classic tool to derive a separably additive utility representation of the ordering, just like CSY, in our model (Theorem 2) and in rationing problems, forces the minimization of such a utility.

Frankel and Volij need three more substantial requirements to capture the Mutual Information index. One is the familiar Scale Invariance. Next when we split a school (resp. a group of students) in two smaller schools with identical representations of the student groups (resp. in two subgroups equally distributed across schools), the index does not change. And finally the above splitting operations can only increase segregation (when the two new distributions differ). The latter imediately implies that the proportional assignment is on top of the ordering.

2.0.2 On Fair Representation

Apportioning seats in a parliament to voting districts in proportion to their population poses an interesting rounding problem because seats are indivisible. Balinski and Young's in their classic book [3] argue for a particular rounding method on the basis of the Consistency axiom itself (Uniformity in their terminology).

In order to allocate seats to political parties and townships, Balinski and Demange [2] generalize the apportionment problem in two dimensions: given an arbitrary exogenous assignment matrix $[z_{ia}]$, and capacity constraints on the sums of rows and of columns (but not on the entries of the matrix), they look for a feasible assignment $[y_{ia}]$ "as proportional as possible" to the initial matrix z. By contrast in our model we fix the sums of rows x_i and of columns r_a, and look for an assignment as close as possible to the ideal proportional matrix $[z_{ia}]$, given exogenous constraints on the entries of the matrix.

Balinski and Demange allow for real valued assignments,and rely on Consistency as well as Scale Invariance, and Monotonicity (of $z \rightarrow y$), to derive a solution minimizing a weighted entropy, much like our capacited proportional assignment. Their parametric representation of the solution is a relative of our Proposition 1. See comment 2 in section 7.

2.0.3 On Rationing methods

In the standard rationing problem, we must divide r units of a single commodity a, given individual claims x_i such that $\sum_i x_i > r$, so that we have excess demand. Combining Consistency, Symmetry and Continuity, Young's Theorem ([20]) characterizes the rules minimizing a separably additive welfare function $\sum_i W(x_i, y_i)$ where W is strictly convex in y_i. Our welfarist rules minimize a similar but simpler sum $\sum_{i,a} W(y_{ia})$, because the domain of W is of dimension 1. Indeed Consistency in the bilateral assignment problem works in both dimensions of the matrix, and so has more bite. We stress that Young's Theorem is a key ingredient in the proof of both Theorems 1 and 2.

Moulin and Sethuraman ([14], [15]), two direct inspirations for the current work, consider a bilateral rationing problem with individual claims x_i and a supply r_a of the "type a" resources. Capacity constraints are binary: either agent i can eat any amount of resource a, or he cannot eat any a at all. Theorem 1 in [14] characterizes the "bipartite proportional rationing" method, that coincides with the Capacited Proportional assignment here when the budget is balanced (so there is no need to ration the claims). Our Theorem 1 is easily adapted to a generalization of that result for rationing problems with general capacity constraints. See Remark 1 ending section 4, and comment 3 in section 7.

We note that bilateral rationing with binary capacity constraints was introduced in [5], [6], where agents are endowed with single-peaked preferences over their share and must report them truthfully. See also [8].

Finally [4], and [10] discuss rationing methods under lower and upper capacity constraints on individual shares similar to ours. They give normative content to these constraints, thus contradicting our Constraint Neutrality requirement.

2.0.4 On Probabilistic Assignment

In the special case where $x_i = r_a = 1$ (and $|N| = |A|$) our matrix $[y_{ia}]$ is doubly stochastic, hence it can be interpreted as a probabilistic assignment of rows to columns. But the capacity constraints on entries are hard to motivate, and Consistency as we define it does not apply because it requires arbitrary sums in rows and columns. In the unconstrained version of that model, Chambers [7] uses a stronger version of Consistency to capture the uniform assignment, a special case of the proportional one.

3 Model and basic definitions

To fix ideas, and to stress the connection with the rationing model that plays an important role in the proofs, we speak of the sets N of agents and A of resources, rather than sources and sinks. They are both finite and of cardinality at least three, and their generic elements are i and a respectively. We use repeatedly the notation $z_V = \sum_{v \in V} z_v$.

3.1 Assignment problems

An *assignment problem* $P = (N, A, x, r, Q)$ *specifies*

- *the total allocation (capacity)* x_i *of each agent, so* $x \in \mathbb{R}_+^N$
- *the endowment* r_a *of the resource of type* a*, so* $r \in \mathbb{R}_+^A$
- *for each pair* $(i, a) \in N \times A$*, a closed capacity interval* $Q_{ia} = [q_{ia}^-, q_{ia}^+] \subseteq \mathbb{R}_+$

meeting the budget balance equation $x_N = r_A = b$.

If $Q_{ia} = \mathbb{R}_+$ for all $(i,a) \in N \times A$, we speak of an *unconstrained* problem, and we write simply $P = (N, A, x, r)$.

We write $Q = \Pi_{N\times A} Q_{ia} \subseteq \mathbb{R}_+^{N\times A}$ for the profile of capacity constraints. A *feasible assignment* is a matrix $y \in Q$ such that $y_{\{i\}\times A} = x_i$ for all $i \in N$, and $y_{N\times\{a\}} = r_a$ for all $a \in A$. Such an assignment always exists if P is unconstrained, but in general its existence is not guaranteed.

Lemma 1 *The problem $P = (N, A, x, r, Q)$ has at least one feasible assignment if and only if for all $S, \varnothing \subseteq S \subseteq N$, and $T, \varnothing \subseteq T \subseteq A$, we have*

$$r_T + q^-_{S\times(A\diagdown T)} \leq x_S + q^+_{(N\diagdown S)\times T} \tag{2}$$

Then we call P a feasible problem.

The proof is a standard application of the max-flow min-cut theorem. It is omited for brevity.

We write $\mathcal{P}$ for the set of feasible assignment problems, and $\Phi(P)$, or simply $\Phi(x, r; Q)$ if no confusion can arise, for the set of feasible assignments of a feasible problem. We write $\mathcal{P}^u$ for the subset of $\mathcal{P}$ containing the unconstrained problems ($Q_{ia} = \mathbb{R}_+$ for all ia).

Let $\tau(P)$ be the set of all entries ia such that y_{ia} is constant in $\Phi(P)$. This contains all ia such that $|Q_{ia}| = 1$, and possibly more. For instance take $N = \{1, 2\}$, $A = \{a, b\}$ and check that $r_b = x_1 + q^+_{2b}$ and/or $x_1 = q^-_{1b}$ implies $y_{1a} = 0$, irrespective of q^+_{1a}. More generally, suppose for some $S \subseteq N, T \subseteq A$, inclusions not both equalities, we have $r_T + q^-_{S\times(A\diagdown T)} = x_S + q^+_{(N\diagdown S)\times T}$.[4] Combining this with the two accounting identities $r_T = y_{S\times T} + y_{(N\diagdown S)\times T}$ and $x_S = y_{S\times T} + y_{S\times(A\diagdown T)}$, we get

$$y_{(N\diagdown S)\times T} + q^-_{S\times(A\diagdown T)} = y_{S\times(A\diagdown T)} + q^+_{(N\diagdown S)\times T}$$

$$\iff (y_{S\times(A\diagdown T)} - q^-_{S\times(A\diagdown T)}) + (q^+_{(N\diagdown S)\times T} - y_{(N\diagdown S)\times T}) = 0$$

which, together with $q^- \leq y \leq q^+$, implies $y = q^-$ on $S \times (A\diagdown T)$ and $y = q^+$ on $(N\diagdown S) \times T$.

It is sometimes useful to restrict attention to problems where such implied constraints do not appear, i.e., the only constant entries result from the capacity constraints.

Definition 1 *A problem $P \in \mathcal{P}$ is irreducible if (2) is an equality only if $S = N$ and $T = A$. This implies $\tau(P) = \{ia||Q_{ia}| = 1\}$.*

The simple proof is omitted for brevity.

3.2 Assignment rules

A *bilateral assignment rule* selects a feasible assignment y for every problem $P \in \mathcal{P}$. We restrict attention to rules treating all agents, and all resources,

[4] Check that the simple 2×2 examples above is of this type.

symmetrically. We also require that a small change in the demands x_i, r_a should have only a small influence on the solution.

If σ is a bijection of N, from the new name i to the old name $\sigma(i)$, and $y \in \Phi(P)$ is an assignment with the old names, the same assignment with the new names is y^σ: $y^\sigma_{ia} = y_{\sigma(i)a}$; define similarly x^σ, Q^σ, and $P^\sigma = (N, A, x^\sigma, r, Q^\sigma)$.

Definition 2 *An assignment rule F chooses for every feasible problem $P = (N, A, x, r, Q) \in \mathcal{P}$ a feasible assignment $F(P) = y \in \Phi(P)$. Moreover the mapping F meets the following properties:*

- **Symmetry in** N: for any $P \in \mathcal{P}$ and bijection σ of N, $F(P)^\sigma = F(P^\sigma)$
- **Symmetry in** A: same property after exchanging the roles of N and A
- **Continuity** of the mapping $\mathcal{P} \ni (x, r, Q) \to F(P)$, for any fixed N, A

We write $\mathcal{F}$ for the set of bilateral assignment rules.

The next two properties are critical. We start with the familiar Consistency. Given a rule F, the constraints Q, and a feasible matrix $y \in Q$, we say that the matrix y is *F-fair for Q* if the rule F chooses y in the problem (N, A, x, r, Q) where $x_i = y_{\{i\}\times A}$ for all i, and $r_a = y_{N\times\{a\}}$ for all a.

- **Consistency (CSY):** every submatrix of a matrix F-fair for Q is F-fair for the restriction of Q.

Note the following consequence of CSY: if $x_i = 0$ for some i, then $y_{ia} = 0$ for all a, so the submatrix after deleting row i has all the same sums in rows and columns; therefore we can simply delete i altogether; similarly if $r_a = 0$. Thus we can always assume when convenient $x, r \gg 0$.

The second key property conveys the idea that exogenous constraints have no ethical meaning, they are *normatively neutral.* We use the notation $Q[-ia]$ for the profile of capacity constraints Q' such that $Q'_{ia} = \mathbb{R}_+$ and $Q'_{jb} = Q_{jb}$ for any $jb \neq ia$.

- **Constraint Neutrality (NEUT)**: fix any $P = (N, A, x, r, Q)$ and set $y = F(P)$; *i*) for any $Q' \subset Q$, if $y \in Q'$ then $y = F(P')$ where $P' = (N, A, x, r, Q')$; *ii*) for any $ia \in N \times A$ such that $q^-_{ia} < y_{ia}$ (resp. $y_{ia} < q^+_{ia}$), then $y = F(P')$ where $Q'_{ia} = [0, q^+_{ia}]$ (resp. $[q^-_{ia}, \infty[$) and $P' = P$ otherwise

Both statements express a version of "independence of irrelevant constraints", the first one with respect to a tightening of the constraints from Q to Q', the second one with respect to a losening from Q at ia, when the Q_{ia} constraint does not bind. Satement *i*) holds whenever $F(P)$ is the unique assignment matrix minimizing over $\Phi(P)$ some numerical index (or ordinal ordering), possibly depending on N, A, x and r, but not on Q. Statement *ii*) holds when the numerical index (or ordering) is convex over the set of feasible assignments of (N, A, x, r), as is the case for the rules discussed in the next two sections.

4 Capacited proportional assignment

In the discussion of segregation, the optimality of the proportional assignment $y_{ia} = \frac{x_i \cdot r_a}{b}$ is the seminal postulate. Note that it can also be characterized by a single axiom capturing the informational parsimony of this matrix: the entry y_{ia} depends only upon x_i, r_a, and b.[5]

In an unconstrained problem $P = (N, A, x, r) \in \mathcal{P}^u$, the proportional flow is the unique minimum of a great variety of functions $\mathcal{W}^P$, strictly convex over $\Phi(P)$. An example is the Atkinson segregation index $\mathcal{W}^P(y) = 1 - \sum_N \Pi_A(\frac{y_{ia}}{r_a})^{\frac{1}{|A|}}$, and its dual $\mathcal{W}^P(y) = 1 - \sum_A \Pi_N(\frac{y_{ia}}{x_i})^{\frac{1}{|N|}}$. For any parameter $p > 1$, the function $\mathcal{W}^P(y) = \sum_{N \times A}(\frac{y_{ia}}{x_i r_a})^p$ is another example.[6]

All assignment rules minimizing the above functions $\mathcal{W}^P$ over $\Phi(P)$ for capacity constrained problems $P \in \mathcal{P}$ meet Definition 1 and Constraint Neutrality.[7] Symmetry is clear, and Continuity follows from Berge's Theorem. But none of them meet Consistency, that is only achieved by minimizing the following (negative of the) entropy function $\mathcal{W}^{en}$:

$$\mathcal{W}^{en}(y) = \sum_{N \times A} y_{ia} \ln(y_{ia}) \tag{3}$$

To check that the proportional flow minimizes $\mathcal{W}^{en}$ in an unconstrained problem $P \in \mathcal{P}^u$, note that $\mathcal{W}^{en}$ is strictly convex, and$(\mathcal{W}^{en})'(0) = +\infty$. Therefore the optimal flow y has $y_{ia} > 0$ whenever x_i, r_a are both positive, and the KKT conditions imply for all i, j, a, b (with positive corresponding row and column)

$$\ln(y_{ia}) + \ln(y_{jb}) = \ln(y_{ib}) + \ln(y_{ja}) \iff \frac{y_{ia}}{y_{ja}} = \frac{y_{ib}}{y_{jb}}$$

The key to Consistency is that $\mathcal{W}^{en}$ depends on P only through N and A. Fix N, A, a set of constraints Q, and an assignment matrix y. The latter is fair for Q if it minimizes $\mathcal{W}^{en}$ when we freeze all the sums $y_{\{i\} \times A}$ and $y_{N \times \{a\}}$. If $\widetilde{y}$ is a submatrix of y, the sum $\mathcal{W}^{en}(\widetilde{y})$ is simply the subset of the sum $\mathcal{W}^{en}(y)$ corresponding to the coordinates of $\widetilde{y}$, and the constraints $\widetilde{Q}$ are the same on those coordinates, so that $\widetilde{y}$ is still fair. For the other functions $\mathcal{W}^P(y)$ mentioned above, the minimization program solved by $\widetilde{y}$ is not comparable to that solved by y because the parameters x_i, r_a, have changed.

[5] By Symmetry we can write $y_{ia} = f(b, x_i, r_a)$. Fixing x_i and b we have: $\forall r \in \mathbb{R}_+^A$, $r_A = b \Longrightarrow \sum_A f(b, x_i, r_a) = x_i$. By classic functional equation results this gives the form $f(b, x_i, r_a) = \frac{1}{|A|}x_i + \Lambda(b, x_i)(r_a - \frac{1}{|A|}b)$; finally $y_{ia} = 0$ for $r_a = 0$ proves the claim.

[6] Indeed if we set $\delta_{ia} = (\frac{y_{ia}}{x_i r_a})^{p-1}$, the KKT conditions for this program are $\delta_{ia} + \delta_{jb} = \delta_{ib} + \delta_{ja}$, so δ_{ia} is additively separable: $\delta_{ia} = \alpha_i + \beta_a$ for some numbers α_i, β_a. This implies $y_{ia} = x_i r_a(\alpha_i + \beta_a)^{\frac{1}{p-1}}$, from which the proportional form follows easily.

[7] We can also use some convex, but not strictly convex, functions $\mathcal{W}^P$ derived from other segregation indices such as the Dissimilarity index $\mathcal{W}^P(y) = \sum_{N \times A}|y_{ia} - \frac{x_i \cdot r_a}{b}|$, and the Gini index $\mathcal{W}^P(y) = \sum_{A \times N \times N} x_i x_j |\frac{y_{ia}}{x_i} - \frac{y_{ja}}{x_j}|$: the proportional flow is still the unique minimum in an unconstrained problem, but in a constrained problem we need to deal with possibly multiple minima.

We write F^{en} for the assignment rule corresponding to $\mathcal{W}^{en}$, i.e., for all $P \in \mathcal{P}$, $F^{en}(P)$ is the unique minimum of $\mathcal{W}^{en}$ over $\Phi(P)$. We give first a parametric representation of the assignment $F^{en}(P)$, that is critical to the axiomatic characterization of the rule F^{en} in Theorem 1.

We use the notation $z * I$ for the projection of z on the closed real interval I; so if $I = [s, t]$, $z * I$ is simply the median of z, s, and t. Recall that $\tau(P)$ is the set of entries ia such that y_{ia} is constant in $\Phi(P)$. It contains all ia such that $|Q_{ia}| = 1$, and only those if P is irreducible.

Proposition 1
For any $P \in \mathcal{P}$, the assignment $y = F^{en}(P)$ is the only one in $\Phi(P)$ that can be written, for some positive numbers λ_i, μ_a, as

$$y_{ia} = (\lambda_i \cdot \mu_a) * Q_{ia} \text{ for all } i, a \notin \tau(P) \tag{4}$$

We speak of the capacited proportional assignment at P.
If P is irreducible (Definition 1), equation (4) holds for all ia.

Theorem 1
The assignment rule F^{en} is characterized, among all rules in $\mathcal{F}$, by the combination of three properties: it picks the proportional flow for an unconstrained problem; Consistency; and Constraint Neutrality.
We call F^{en} the capacited proportional assignment rule.

Both Proposition 1 and Theorem 1 follow the more general Propositions 2 and 3 in the next section.

Remark 1 In the related bilateral rationing model of [14], [15], the capacity constraints take the simple form $Q_{ia} = \{0\}$ or $\mathbb{R}_+$ for all ia: agents may not be able to consume all resources. On the other hand the budget balance typically does not hold so that the demands x_i cannot be fully met by the resources r_a and the question is how to ration the resources fairly. The constrained proportional flow defined there coincides with ours for balanced problems. Statement *ii*) of Theorem 1 in [14], in the special case of balanced problems, follows from Proposition 1 above. Statement *iii*) of that same Theorem 1 is a characerization very similar to Theorem 1 above, however it is not a special case of the present result because it requires the axioms to hold on the larger domain containing unbalanced rationing problems. It is easy to adapt the proof of our Theorem 1 to bilateral rationing problems with general capacity constraints. The capacited proportional rationing allocation minimizes $\sum_{N \times A} En(y_{ia}) + \sum_N En(x_i - y_{iA})$ over our rectangular constraints Q; it is characterized by the same three axioms as in Theorem 1.

5 Capacited welfarist assignments

From now on we do not assume any more that the fair assignment for an unconstrained problem must be proportional. We construct a large family of assignment rules meeting CSY and NEUT, by maximizing a separably additive welfare objective.

Fix a strictly convex and smooth function W on $]0,\infty[$, of which the derivative W' is continuous and strictly increasing on $]0,\infty[$. Define $W(0)=\lim_{z\to 0}W(z)$, which could be $+\infty$. Loosely speaking, our *W-welfarist rule* selects the assignment minimizing the strictly convex objective $\sum_{N\times A}W(y_{ia})$ over all feasible assignments. The special case $W(y_{ia})=y_{ia}\ln(y_{ia})$ yields the capacited proportional assignment..

Proposition 2 below generalizes Proposition 1 by providing the parametrization (7) of the capacited W-welfarist assignment entirely similar to (4) for the proportional one. Proposition 3 generalizes Theorem 1 by showing that this rule is the only extension of the W-welfarist rule to unconstrained problems meeting CSY and NEUT. However Proposition 3 applies only to welfare functions W such that $W'(0)=-\infty$.

Because $W(0)=\infty$ is possible, we must define the W-welfarist assignment as the solution of the program

$$\min_{y\in\Phi(P)}\sum_{(N\times A)\diagdown\tau(P)}W(y_{ia}) \tag{5}$$

We check it has a unique solution. If $W(0)$ is finite, it is the same program as

$$\min_{y\in\Phi(P)}\sum_{N\times A}W(y_{ia}) \tag{6}$$

and the strictly convex function W (defined on $[0,\infty[$) has a unique minimum over the convex compact $\Phi(P)$.

If $W(0)=\infty$, there is some feasible assignment $\overline{y}$ such that $\overline{y}_{ia}>0$ for all $(i,a)\in(N\times A)\diagdown\tau(P)$ (by definition of $\tau(P)$ and because $\Phi(P)$ is convex) therefore program (5) is finite, and is equivalent to minimizing the strictly convex function $\sum_{(N\times A)\diagdown\tau(P)}W(y_{ia})$ over the compact, convex set $\{y\in\Phi(P)|\sum_{(N\times A)\diagdown\tau(P)}W(y_{ia})\leq\sum_{(N\times A)\diagdown\tau(P)}W(\overline{y}_{ia})\}$. The claim follows.

Program (5) defines a rule F^W that we call the *W-welfarist assignment rule*, or W-rule for short. Continuity follows from Berge's Theorem. It is enough to consider, for fixed N,A, sequences $P^t\in\mathcal{P}$ such that P^t converges to P^∞ and $\tau(P^t)$ is constant in t; then $\tau(P^\infty)\supseteq\tau(P^t)$ and for $ia\in\tau(P^\infty)\diagdown\tau(P^t)$ the convergence of y^t_{ia} to 0 is guaranteed. Symmetry is clear. CSY and NEUT follow exactly as for the capacited proportional rule.

Understanding the structure of the assignment $F^W(P)$ for any P is critical to our results. It turns out that this flow has a very simple additive structure, derived from the familiar KKT conditions for the program (5).

We define the following extension Γ of the inverse of W'. The domain of W' is $\mathbb{R}_+$ and its range an interval $[W'(0),W'(\infty)[$ such that $W'(0)\geq-\infty$ and $W'(\infty)\leq\infty$. We set: $\Gamma(\alpha)=0$ if $\alpha\leq W'(0)$; $\Gamma(\alpha)=(W')^{-1}(\alpha)$ if $W'(0)\leq\alpha<W'(\infty)$; $\Gamma(\alpha)=\infty$ if $\alpha\geq W'(\infty)$.

Proposition 2

i) Fix a problem $P\in\mathcal{P}$, and a feasible assignment $y\in\Phi(P)$. Then $y=F^W(P)$ (y solves program (5)) if and only if there exists two vectors $\alpha\in\mathbb{R}^N,\beta\in\mathbb{R}^A$, *such that for all $i \in N, a \in A$:*

$$y_{ia} = \Gamma(\alpha_i + \beta_a) * Q_{ia} \text{ for all } ia \notin \tau(P) \tag{7}$$

ii) If P is irreducible, then $F^W(P)$ meets (7) for all ia.

iii) If $P \in \mathcal{P}^u$ is an unconstrained problem, then $\tau(P) = \{ia | x_i = 0$ and/or $r_a = 0\}$, and system (7) reduces to

$$y_{ia} = \max\{\Gamma(\alpha_i + \beta_a), 0\} \text{ for all } ia \text{ such that } x_i, r_a > 0 \tag{8}$$

If, in addition, $W'(0) = \lim_{z \to 0} W'(z) = -\infty$, then

$$y_{ia} = \Gamma(\alpha_i + \beta_a) \text{ for all } ia \text{ such that } x_i, r_a > 0 \tag{9}$$

Proof of Proposition 2

Statement i) Fix P and note that program (5) is equivalent to minimizing $\sum_{(N \times A) \setminus \tau(P)} W(y_{ia})$. We restrict attention to the submatrix of y such that $(N \times A) \setminus \tau(P)$ intersects each row and each column, i.e., we delete any row or column contained in $\tau(P)$. For simplicity we still write the sets of rows and columns of the submatrix as N and A. Keep in mind that $(N \times A) \setminus \tau(P)$ has at least two entries in each row and in each column. We adjust the capacities of rows and columns accordingly: we set $x_i^* = x_i - \sum_{a: ia \in \tau(P)} y_{ia}$, and $r_a^* = r_a - \sum_{i: ia \in \tau(P)} y_{ia}$

The assignment $y = F^W(P)$ minimizes $\sum_{(N \times A) \setminus \tau(P)} W(y_{ia})$ under the following equality and inequality constraints:

$$y_{\{i\} \times A \setminus \tau(P)} = x_i^* \text{ , } y_{N \times \{a\} \setminus \tau(P)} = r_a^* \text{ ; } y_{ia} - q_{ia}^+ \leq 0, \; q_{ia}^- - y_{ia} \leq 0 \text{ for } ia \notin \tau(P)$$

The Lagrangien of the problem is $\mathcal{L}(y, \alpha, \beta, \theta^+, \theta^-)$, where $\alpha \in \mathbb{R}^N, \beta \in \mathbb{R}^A$ and $\theta^+, \theta^- \in \mathbb{R}_+^{N \times A \setminus \tau(P)}$:

$$\mathcal{L}(y, \alpha, \beta, \theta^+, \theta^-) = \sum_{(N \times A) \setminus \tau(P)} W(y_{ia}) - \sum_N \alpha_i (y_{\{i\} \times A} - x_i) - \sum_A \beta_a (y_{N \times \{a\}} - r_a) +$$

$$+ \sum_{N \times A \setminus \tau(P)} \theta_{ia}^+ (y_{ia} - q_{ia}^+) + \sum_{N \times A \setminus \tau(P)} \theta_{ia}^- (q_{ia}^- - y_{ia})$$

We check the qualification constraints. From $|N \times A \setminus \tau(P)| \geq 2 \max\{|N|, |A|\}$ we see that the linear mapping $\mathbb{R}^{N \times A \setminus \tau(P)} \ni y \to (y_{\{i\} \times A}, y_{N \times \{a\}}) \in \mathbb{R}^{N \cup A}$ is of maximal rank. Second, by definition of $\tau(P)$ and convexity of $\Phi(P)$ there exist $y \in \tau(P)$ such that $q_{ia}^- < y_{ia} < q_{ia}^+$ for all $ia \in N \times A \setminus \tau(P)$. Therefore there exist some KKT multipliers $\alpha, \beta, \theta^+, \theta^-$, such that

$$\min_{y \in \Phi^*(P)} \sum_{(N \times A) \setminus \tau(P)} W(y_{ia}) = \min_{y \in \mathbb{R}^{N \times A \setminus \tau(P)}} \mathcal{L}(y, \alpha, \beta, \theta^+, \theta^-)$$

where $\Phi^*(P)$ is the projection of $\Phi(P)$ on $\mathbb{R}^{N\times A\diagdown\tau(P)}$. Moreover y solves the Left Hand program above if and only if 1) y solve the Right Hand program; 2) y is in $\Phi^*(P)$; 3) and the complementarity properties $\theta^+_{ia}(y_{ia}-q^+_{ia}) = \theta^-_{ia}(q^-_{ia}-y_{ia}) = 0$ hold for all $ia \in N\times A\diagdown\tau(P)$.

The first order conditions for statement 1) are, for all $ia \in N\times A\diagdown\tau(P)$:

$$W'(y_{ia}) = \alpha_i + \beta_a - \theta^+_{ia} + \theta^-_{ia}$$

If $q^-_{ia} < y_{ia} < q^+_{ia}$, this reduces to $W'(y_{ia}) = \alpha_i + \beta_a \iff y_{ia} = \Gamma(\alpha_i+\beta_a) = \Gamma(\alpha_i+\beta_a) * Q_{ia}$. If $y_{ia} = q^-_{ia}$ we get $W'(y_{ia}) = \alpha_i+\beta_a+\theta^-_{ia} \Longrightarrow y_{ia} \geq \Gamma(\alpha_i+\beta_a) \Longrightarrow y_{ia} = \Gamma(\alpha_i+\beta_a)*Q_{ia}$. Similarly $y_{ia} = q^+_{ia}$ gives $y_{ia} \leq \Gamma(\alpha_i+\beta_a) = \Gamma(\alpha_i+\beta_a)*Q_{ia}$.

This proves the "only if" statement.

For the "if" statement we start from $y \in \Phi(P)$ and $\alpha \in \mathbb{R}^N, \beta \in \mathbb{R}^A$ meeting (7). We set $\theta^+_{ia} = \theta^-_{ia} = 0$ if $q^-_{ia} < y_{ia} < q^+_{ia}$; $\theta^+_{ia} = 0$ and $\theta^-_{ia} = W'(y_{ia}) - \alpha_i - \beta_a$ if $y_{ia} = q^-_{ia}$; $\theta^-_{ia} = 0$ and $\theta^+_{ia} = -W'(y_{ia}) + \alpha_i + \beta_a$ if $y_{ia} = q^+_{ia}$. It is then easy to check that the projection of y on $\mathbb{R}^{N\times A\diagdown\tau(P)}$ minimizes $\mathcal{L}(y,\alpha,\beta,\theta^+,\theta^-)$ in the entire space.

Statement ii) If P is irreducible, every entry $ia \in \tau(P)$ corresponds to a singleton Q_{ia}, where equation (7) is automatically true.

Statement iii) The fact that (7) reduces to (8) if there are no capacity constraints is clear. For the second statement, note that if $W'(0) = \lim_{z\to 0} W'(z) = -\infty$, then for any $P \in \mathcal{P}^u$ each entry y_{ia} is strictly positive provided $x_i, r_a > 0$. Indeed in equation (8) $y_{ia} = 0$ would imply $\alpha_i + \beta_a = -\infty$. (it is also easy to construct a perturbation of y improving the objective function if some $y_{ia} = 0$).■

We give some examples of W-rules. A natural subfamily is defined by the familiar **Scale Invariance (SI)** property. For any two problems $P = (N,A,x,r,Q)$ and $P' = (N,A,\delta x,\delta r,\delta Q)$, SI requires $F(P') = \delta\cdot F(P)$.

Lemma 2: *The one dimensional family $F^{W^q}, q \in \mathbb{R}$, contains all scale invariant W-welfarist assignment rules:*
i) $W^q(z) = z^q$ for $q > 1$ and for $q < 0$
ii) $W^q(z) = -z^q$ for $0 < q < 1$
ii) $W^0(z) = -\ln(z)$
ii) $W^1(z) = z\ln(z)$

Proof sketch That all rules F^{W^p} are SI is clear. Conversely observe that for any F^W the assignment matrix $\begin{matrix} a & b \\ c & d \end{matrix}$ where all entries are positive and $W'(a)+W'(d) = W'(b)+W'(c)$ is fair (in the unconstrained problem). Therefore SI implies, for all positive a,b,c,d,δ:

$$W'(a)+W'(d) = W'(b)+W'(c) \iff W'(\delta a)+W'(\delta d) = W'(\delta b)+W'(\delta c)$$

Assume for simplicty that W' is differentiable (it must be so almost everywhere; we omit the details of the general argument). Fix a, c, and δ; for any small enough ε there exists $\lambda > 0$ such that $W'(a+\varepsilon) - W'(a) = W'(c+\lambda\varepsilon) - W'(c)$. By the equivalence above, this amounts to $W'(\delta a+\delta\varepsilon) - W'(\delta a) = W'(\delta c+\lambda\delta\varepsilon) -$

$W'(\delta c)$. This implies $W''(a) - \lambda W''(c) = o(\varepsilon)$ and $W''(\delta a) - \lambda W''(\delta c) = o(\varepsilon)$, where λ depends on ε but must converge by differentiability of W'. We get $W''(a) \cdot W''(\delta c) = W''(c) \cdot W''(\delta a)$, from which follows, after rescaling W'', $W''(ab) = W''(a) \cdot W''(b)$ for all $a, b > 0$. Thus W'' is a power function by standard arguments.■

The capacited proportional rule F^{en} is F^{W^1}; Γ is the exponential function, hence the additive parametrization (7) is equivalent to the multiplicative one in (4).

For F^{W^2}, Γ is the identity and (7) says that y is the projection on Q of a separably additive assignment. In an unconstrained problem, the rule chooses

$$y_{ia} = \frac{1}{|A|}x_i + \frac{1}{|N|}r_a - \frac{1}{|A| \times |N|}b$$

whenever this expression is non negative for all i, a.

The rule F^{W^0} picks the assignment maximizing the Nash product $\Pi_{N \times A} y_{ia}$. It is the only feasible assignment that we can write as $y_{ia} = \frac{1}{\alpha_i + \beta_a} * Q_{ia}$.

When q goes to $-\infty$, the assignment $F^{W^q}(P)$ converges to the *egalitarian assignment* $F^{eg}(P)$, maximizing the leximin ordering of $\mathbb{R}_+^{N \times A}$. This assignment is defined recursively by comparing first $\frac{\min_i x_i}{|A|}$ and $\frac{\min_a r_a}{|N|}$. If $\frac{\min_i x_i}{|A|} \leq \frac{\min_a r_a}{|N|}$, we assign $y_{ia} = \frac{\min_i x_i}{|A|}$ to all i achieving the minimum. and all a; we proceed similarly if $\frac{\min_a r_a}{|N|} \leq \frac{\min_i x_i}{|A|}$; we repeat this operation on the reduced problem among the unassigned rows and columns, adjusting accordingly the capacities x_i, r_a. The egalitarian assignment rule is consistent.

Our last result in this section is about the consistent extension, from unconstrained to constrained problems, of those welfarist rules such that $W'(0) = \lim_{z \to 0} W'(z) = -\infty$. The key fact (see the proof of statement *iii*) of Proposition 2) is that this guarantees that F^W gives strictly positive shares in an unconstrained problem:

- **Strict Positivity** (SP)**:** $\{x_i, r_a > 0\} \Longrightarrow y_{ia} > 0$, for any $P \in \mathcal{P}^u$ and $y = F(P)$

Proposition 3 *Fix a function W, with a derivative W' continuous and strictly increasing on* $]0, \infty[$, and such that $W'(0) = \lim_{z \to 0} W'(z) = -\infty$. *Then the W-welfarist assignment rule F^W for constrained problems in $\mathcal{P}$ is the unique extension of F^W for unconstrained problems satisfying Consistency and Constraint Neutrality.*

Among the scale invariant functions in Lemma 2, the assumption $W'(0) = -\infty$ selects all W^q such that $q \leq 1$. Theorem 1 is just Proposition 3 for the function W^1. I do not know whether or not Proposition 3 holds without the assumption $W'(0) = -\infty$.

Proof

Step 1 To any $F \in \mathcal{F}$ meeting CSY and NEUT we associate a *rationing rule* to divide any amount of a single resource according to individual claims or demands. This construction is critical in the proof of Theorem 2 as well.

Formally a rationing problem is (N, x, t), where $x \in \mathbb{R}_+^N$ is the profile of demands, $t \geq 0$ is the amount to be divided, and $t \leq x_N$. A rationing rule h associates to every problem a division $y = h(N, x, t)$ of t among N such that $0 \leq y \leq x$ and $y_N = t$.[8] Although a rationing problem is not a special case of an assignment problem (there is a single resource but budget balance does not hold), it is straightforward to adapt the properties of Symmetry w.r.t. N, Continuity, and Consistency.

Fix $F \in \mathcal{F}$ and construct h as follows. Given the rationing problem (N, x, t) consider the assignment problem $P = (N, \{a, b\}, x, r)$ where $r_a = t$ and $r_b = x_N - t$, and define $h(N, x, t)$ to be the a-column of $F(P)$. By SYM the choice of a, b, does not matter; SYM, CONT, and CSY for F imply the same for h. By Young's Theorem h is parametrized by a continuous function $\theta(x_i, \lambda)$, non decreasing in $\lambda \in \mathbb{R} \cup \{-\infty, +\infty\}$ and such that $\theta(x_i, -\infty) = 0$ and $\theta(x_i, \infty) = x_i$. This means that for any (N, x, t)

$$h(N, x, t) = y \Longleftrightarrow \{\exists \lambda : y_i = \theta(x_i, \lambda) \text{ for all } i, \text{ and } \sum_N \theta(x_i, \lambda) = t\} \quad (10)$$

Note that by SYM the rule h is *self-dual*

$$h(N, x, t) + h(N, x, x_N - t) = x \text{ for all } x, t$$

Step 2 Still fixing $F \in \mathcal{F}$ meeting CSY and NEUT, we extend the rationing rule h just defined to *capacited rationing problems* (N, x, t, q), where $q_i = [q_i^-, q_i^+] \subseteq \mathbb{R}_+$ constrains agent i's share. A division y of t is feasible iff $q_i^- \leq y_i \leq \min\{q_i^+, x_i\}$, and such division exists iff $q_N^- \leq \sum_N \min\{q_i^+, x_i\}$, which we assume.

As in Step 1 we consider the assignment problem $P = (N, \{a, b\}, x, r, Q)$ where $r = (t, x_N - t)$ and $Q_{ia} = Q_i$, $Q_{ib} = \mathbb{R}_+$ for all i. Then $h(N, x, t, Q)$ is defined as the a-column of $F(P)$. The parametrization θ of h for ordinary (unconstrained) rationing problems in step 1, determines its extension to capacited problems as follows. For any (N, x, t, Q)

$$h(N, x, t, Q) = y \Longleftrightarrow \{\exists \lambda\text{: } y_i = \theta(x_i, \lambda) * Q_i \text{ and } \sum_N \theta(x_i, \lambda) * Q_i = t\} \quad (11)$$

The proof is by induction on the number k of non trivial constraints $Q_i \neq \mathbb{R}_+$. The case $k = 0$ is covered, so we assume (11) holds up to $k - 1$. Fix (N, x, t, Q) with k non trivial constraints, for instance Q_1 is non trivial. Replacing Q_1 by $\mathbb{R}_+$ leaves us with the constraints Q^{-1}.

If $\overline{y} = h(N, x, t, Q^{-1})$ meets Q_1, NEUT (statement i)) implies $\overline{y} = h(N, x, t, Q)$, while by the inductive assumption there exists λ such that $\overline{y}_i = \theta(x_i, \lambda) * Q_i$ for $i \geq 2$, and $\overline{y}_1 = \theta(x_1, \lambda) = \theta(x_1, \lambda) * Q_1$.

[8] See [13], [17], for two surveys on axiomatic properties of rationing rules.

If $\overline{y} \notin Q_1$, we assume for instance $\overline{y}_1 > q_1^+$, and we show first $h_1(N,x,t,Q) = q_1^+$. Define $Q_1^* = [q_1^-, \overline{y}_1]$ and $Q_i^* = Q_i$ for $i \geq 2$. NEUT $i)$ implies $h(N,x,t,Q^*) = \overline{y}$. Assume $y_1 = h_1(N,x,t,Q) < q_1^+$ and invoke NEUT $ii)$ to get $y = h(N,x,t,Q) = h(N,x,t,Q\prime)$ where $Q_1' = [q_1^-, \infty[$ and $Q_i' = q_i$ for $i \geq 2$. By NEUT $i)$ again, we have then $y = h(N,x,t,Q^*)$, a contradiction. The similar proof that $\overline{y}_1 < q_1^-$ implies $h_1(N,x,t,Q) = q_1^-$ is omitted.

We still assume $\overline{y}_1 > q_1^+$, and we compare $\overline{y} = h(N,x,t,Q^{-1})$ and $y = h(N,x,t,Q)$. We apply the inductive assumption twice: to (N,x,t,Q^{-1}) it gives some λ such that $\overline{y}_i = \theta(x_i,\lambda) * Q_i$ for all $i \geq 2$ and $\overline{y}_{N\diagdown 1} = t - \overline{y}_1$; to the problem $(N\diagdown 1, x_{-1}, t - q_1^+, Q^{-1})$ obtained from (N,x,t,Q) by dropping agent 1, we get μ such that (by CSY) $y_i = \theta(x_i,\mu) * Q_i$ for all $i \geq 2$ and $y_{N\diagdown i} = t - q_1^+$. Hence $\mu > \lambda$, implying $\theta(x_1,\mu) \geq \theta(x_1,\lambda) = \overline{y}_1 > q_1^+$, so that $y_1 = q_1^+ = \theta(x_1,\mu) * Q_1$. This completes the proof of (11) for the case $\overline{y}_1 > q_1^+$, and the case $\overline{y}_1 < q_1^-$ is treated similarly.

Step 3 Fix a function W as in the statement of Proposition 3, and assume from now on that, for unconstrained problems, F is the welfarist rule F^W defined before Proposition 2. Then the corresponding rationing rule h^W defined in Step 1, and denoted h for simplicity, takes the following form for every $x \gg 0, t > 0$, and $t < x_N$:

$$h(N,x,t) = \arg \min_{y \in \Phi(N,x,t)} (\sum_N W(y_i) + W(x_i - y_i))$$

where $\Phi(N,x,t) = \{y \in \mathbb{R}_+^N | y \leq x \text{ and } y_N = t\}$. Strict Positivity guarantees $0 < y_i < x_i$ for all i, and the KKT conditions for this program are

$$\exists \lambda \in \mathbb{R} \text{ such that } \forall i \in N \ W'(y_i) - W'(x_i - y_i) = \lambda$$

Thus a parametric representation of h is the function $\theta(u,\lambda)$ defined as follows

$$v = \theta(u,\lambda) \Longleftrightarrow W'(v) - W'(u - v) = \lambda$$

where θ is continuous and strictly increasing in both variables. Define now a second function $\pi(z,\lambda)$ by solving the following equation in v for $z \geq 0$ and $\lambda \in \mathbb{R}$:

$$v = \pi(z,\lambda) \Leftrightarrow v = \theta(z+v,\lambda) \Longleftrightarrow W'(v) - W'(z) = \lambda$$
$$\Longleftrightarrow \pi(z,\lambda) = \Gamma(W'(z) + \lambda) \quad (12)$$

Recalling our definition of Γ, we have $\pi(0,\lambda) = 0$, $\pi(z,\lambda) = \infty$ if $\lambda + W'(z) \geq W'(\infty)$, and $\pi(z,\lambda)$ strictly increases in both variables as long as $z > 0$ and $\lambda + W'(z) < W'(\infty)$. We check finally that for any z, λ, and any closed interval $Q \subseteq \mathbb{R}_+$

$$v = \pi(z,\lambda) * Q \Longleftrightarrow v = \theta(z+v,\lambda) * Q \quad (13)$$

This follows from (12) if v is interior to Q. If $v \geq Q$ then $v = \theta(z+v,\lambda) * Q$ implies $v \leq \theta(z+v,\lambda)$; pick $z' \leq z$ such that $v = \theta(z'+v,\lambda)$, so that $v = \pi(z',\lambda) \leq \pi(z,\lambda)$ and we get $y = \pi(z,\lambda) * Q$ as desired. The other case is similar.

Step 4 We fix W as in step 3, and $F \in \mathcal{F}$ meeting CSY and NEUT, and equal to F^W on $\mathcal{P}^u$. We choose an arbitrary problem $P = (N, A, x, r, Q) \in \mathcal{P}$, and to show $F(P) = F^W(P)$ we construct for all $\varepsilon \geq 0$ an augmented problem $P^\varepsilon = (N, A \cup \{a^*\}, x^\varepsilon, r^\varepsilon, Q^\varepsilon)$ with one more resource a^* as follows:

$$x_i^\varepsilon = x_i + \frac{\varepsilon}{|N|} \text{ for all } i \text{ ; } r^\varepsilon = (r, \varepsilon) \text{ ; } Q_{ia^*}^\varepsilon = \mathbb{R}_+, \; Q_{ia}^\varepsilon = Q_{ia} \text{ otherwise}$$

Set $y = F(P)$ and $(y^\varepsilon, z^\varepsilon) = F(P^\varepsilon)$ where z_i^ε is the ia^* coordinate of $F(P^\varepsilon)$ and y^ε is of dimension $N \times A$. By CSY we have $F(P^0) = (y, 0)$ and by CONT $\lim_{\varepsilon \to 0}(y^\varepsilon, z^\varepsilon) = (y, 0)$.

Choose any $a \in A$ and reduce P^ε by dropping all resources except a, a^*, and adjusting x_i^ε to $x_i' = y_{ia}^\varepsilon + z_i^\varepsilon$. We have $y_{ia}^\varepsilon = h(N, x', r_a, Q_a)$ where h is the capacited rationing method defined in steps 1, 2. Combining (11), (13), and (12), there exists λ_a such that

$$y_{ia}^\varepsilon = \theta(x_i', \lambda_a) * Q_{ia} = \theta(z_i^\varepsilon + y_{ia}^\varepsilon, \lambda_a) * Q_{ia} = \pi(z_i^\varepsilon, \lambda_a) * Q_{ia}$$

$$\Longrightarrow y_{ia}^\varepsilon = \Gamma(W'(z_i^\varepsilon) + \lambda_a) * Q_{ia}$$

The equality above holds for all $ia \in N \times A$; moreover $z_i^\varepsilon = \Gamma(W'(z_i^\varepsilon)) = \Gamma(W'(z_i^\varepsilon) + 0) * Q_{ia^*}$. We invoke now Proposition 2 because $(y^\varepsilon, z^\varepsilon)$ takes the form (7): thus $(y^\varepsilon, z^\varepsilon) = F^W(P^\varepsilon)$. Finally $\lim_{\varepsilon \to 0}(y^\varepsilon, z^\varepsilon) = F^W(P^0) = (F^W(P), 0)$, by CONT and CSY of F^W.■

6 Consistent assignment rules: a partial characterization

Our last result concerns consistent assignment rules defined for unconstrained problems. We write $\mathcal{F}^u$ the set of such rules.

We already noted that the W-welfarist assignment rule $F^W \in \mathcal{F}^u$ meets Strict Positivity whenever $W'(0) = -\infty$. A related property looks at the impact of transfering resources while leaving everything else unchanged. Fix an assignment rule $F \in \mathcal{F}^u$:

- **Strict Resource Monotonicity** (SRM): for any $P, P' \in \mathcal{P}^u$ and $a \in A$: $\{r_a' > r_a, \; r_b' \leq r_b$ for all $b \neq a$, P and P' identical otherwise$\} \Longrightarrow \{y_{ia}' > y_{ia}$ whenever $x_i > 0\}$; and a similar statement by exchanging the roles of N and A

Under CSY, SRM implies SP. Indeed fix any $P \in \mathcal{P}^u$ and i, a such that $x_i, r_a > 0$; augment P to P^* by adding a resource a^* with $r_{a^*} = 0$. By CSY $F(P^*)$ is $F(P)$ augmented by a null column; when we next transfer half of r_a from a to a^*, the entry y_{ia} decreases strictly by SRM.

Theorem 2 *For any assignment rule $F \in \mathcal{F}^u$ the two following statements are equivalent:*

i) F meets CSY, and SRM
ii) F is a W-welfarist rule in $\mathcal{P}^u$, and $W'(0) = -\infty$.

Combining Theorem 2 and Proposition 3 gives the

Corollary *For any assignment rule $F \in \mathcal{F}$ the two following statements are equivalent:*
i) F meets NEUT, CSY, and SRM
ii) F is a W-welfarist rule in $\mathcal{P}$, and $W'(0) = -\infty$.

Proof of Theorem 2

Statement ii) $\Longrightarrow$ i)

We need only to check hat F^W meets SRM whenever $W'(0) = -\infty$. Fix P, P' and a^* as in the premises of SRM. We can assume $x_i > 0$ for all i because CSY allows us to delete a null row, and $r_b > 0$ for all b because CSY allows us to delete a null column (that remains null in P and P'). If $r_{a^*} = 0$, the desired conclusion $y'_{ia^*} > 0$ follows from Strict Positivity, so we assume $r_{a^*} > 0$ as well.

Apply (9) to P and P': $y_{ia} = \Gamma(\alpha_i + \beta_a), y'_{ia} = \Gamma(\alpha'_i + \beta'_a)$ for all i, a. We write $\partial_{ia} = y'_{ia} - y_{ia}$, and note that for any distinct $i, j \in N$ and distinct $a, b \in A$, the following implication holds

$$\{\partial_{ia}, \partial_{jb} \geq 0 \text{ and } \partial_{ib}, \partial_{ja} \leq 0\} \Longrightarrow \text{ all inequalities are equalities} \tag{14}$$

This follows easily from $\partial_{ia} \geq 0 \iff \alpha_i + \beta_a \leq \alpha'_i + \beta'_a$ and $\partial_{ia} > 0 \iff \alpha_i + \beta_a > \alpha'_i + \beta'_a$.

We show $\partial_{1a^*} > 0$ (where 1 is arbitrary) by contradiction: we assume $\partial_{1a^*} \leq 0$ and distinguish two cases. In Case 1 we have $\partial_{1a} \leq 0$ for all $a \neq a^*$. Then $\partial_{1a} = 0$ for all a (because $x_1 = x'_1$). As r_{a^*} increases there is an agent noted 2 such that $\partial_{2a^*} > 0$. Applying (14) to $1, 2, a^*, b$, for any $b \neq a^*$, gives $\partial_{2b} > 0$. This contradicts $x_2 = x'_2$. In the remaining Case 2 there is some $a_2 \neq a^*$ such that $\partial_{1a_2} > 0$. Because r_{a_2} decreases weakly, there is an agent 2 such that $\partial_{2a_2} < 0$. Applying (14) to $1, 2, a^*, a_2$, gives $\partial_{2a^*} < 0$. Then $\partial_{2a^*}, \partial_{2a_2} < 0$ and $x_2 = x'_2$ imply the existence of a_3 such that $\partial_{2a_3} > 0$. Applying (14) to $1, 2, a_2, a_3$, gives $\partial_{1a_3} > 0$. Then we find agent 3 such that $\partial_{3a_3} < 0$, and use (14) repeatedly to show $\partial_{3a^*}, \partial_{3a_2} < 0$. The induction argument is now clear: we find agents $1, \cdots, k$ and resources $a^* = a_1, a_2, \cdots, a_k$, such that

$$\partial_{ia_k} > 0 \text{ if } i < k \text{ ; } \partial_{ia_k} < 0 \text{ if } i \geq k \text{ with the exception of } \partial_{1a^*} \leq 0$$

We reach a contradiction when $k = \inf\{|N|, |A|\}$. If $k = |N| \leq |A|$ we have $y'_{ia^*} < y_{ia^*}$ for all i, whereas $r'_{a^*} > r_{a^*}$; if $k = |A| \leq |N|$ we have $y'_{ka} < y_{ka}$ for all $a \in A$, whereas $x'_k = x_k$.

Statement i) $\Longrightarrow$ ii)

We fix $F \in \mathcal{F}^u$ meeting CSY, and SRM. Recall that F meets SP, and note that it meets also

- **Strict Ranking** (SRK)**:** for any $P \in \mathcal{P}^u$ and any i, j, a : $\{x_i > x_j$ and $r_a > 0\} \Longrightarrow y_{ia} > y_{ib}$; and a similar statement after exchanging the roles of N and A

To check this we construct the problem P' where $x'_i = x'_j = \frac{x_i+x_j}{2}$, and everything else is as in P. By Symmetry $y'_{ia} = y'_{ib}$; P obtains from P' by transferring $\frac{x_i-x_j}{2}$ from resource b to resource a, so by SRM $y_{ia} > y'_{ia} = y'_{ib} > y_{ib}$.

The proof consists of 6 steps.

Step 1 We associate to F the rationing rule $h(N,x,t)$ as in Step 1 of the proof of Proposition 3. In this step we discuss properies of h, and a parametrization $\theta(x_i,\lambda)$ of h as in (10). Recall $\lambda \in \mathbb{R} \cup \{-\infty,+\infty\}$, θ is continuous and weakly increasing in λ, and $\theta(x_i,-\infty) = 0$ and $\theta(x_i,\infty) = x_i$. In particular $\theta(0,\lambda) = 0$ for all λ. Without loss we assume that θ is "clean" in the sense that the two functions $\theta(\cdot,\lambda)$ and $\theta(\cdot,\lambda')$ are different if $\lambda \neq \lambda'$.

SRM implies at once that $t \to h_i(N,x,t)$ is strictly increasing for all i such that $x_i > 0$. This in turn means that $\theta(z,\lambda)$ increases strictly in λ if $z > 0$. Suppose, on the contrary, $\theta(z,\lambda) = \theta(z,\lambda')$ and $\lambda < \lambda'$. Because θ is clean there is some z' such that $\theta(z',\lambda) < \theta(z',\lambda')$, but then SRM is violated going from $(\{1,2\},(z,z'),\theta(z,\lambda)+\theta(z',\lambda))$ to $(\{1,2\},(z,z'),\theta(z,\lambda')+\theta(z',\lambda'))$.

SRK implies similarly $\{x_i > x_j \ , \ t > 0\} \Longrightarrow h_i(N,x,t) > h_j(N,x,t)$, from which we see that $\theta(z,\lambda)$ increases strictly in z (we omit the easy argument). Recall that h is self-dual: $h(N,x,t) + h(N,x,x_N - t) = x$ for all x,t. Thus SRK yields $\{x_i > x_j \ , \ t < x_N\} \Longrightarrow x_i - h_i(N,x,t) > x_j - h_j(N,x,t)$, and this implies that $\theta(z,\lambda)$ is 1-contracting in z when $\lambda \neq -\infty,+\infty$:

$$z < z' \Longrightarrow \theta(z',\lambda) - \theta(z,\lambda) < z' - z \text{ for all } z,z' \text{ and all } \lambda \neq -\infty,+\infty \quad (15)$$

Fix any N,x,λ and apply self duality to the problem $(N,x,t = \sum_N \theta(x_i,\lambda))$: there is some λ' such that $\theta(x_i,\lambda)+\theta(x_i,\lambda') = x_i$ for all i. Fixing λ and varying the pair N,x, we see that λ' does not depend on N,x (recall θ increases strictly in λ). This defines the "inverse" λ^{-1} of λ by the identity

$$\theta(z,\lambda) + \theta(z,\lambda^{-1}) = z \text{ for all } z \quad (16)$$

For instance $\infty^{-1} = -\infty$. Moreover self duality gives $h(N,x,\frac{x_N}{2}) = \frac{1}{2}x$, which means there is some λ^* such that $\theta(z,\lambda^*) = \frac{z}{2}$ for all z. Thus $(\lambda^*)^{-1} = \lambda^*$. Finally $\lambda \leq \lambda^* \Longleftrightarrow \lambda^* \leq \lambda^{-1}$.

Step 2 We define now an alternative parametrization $\pi(z,\lambda)$ of the rationing rule h, exactly as we did in Step 3 of the proof of Proposition 3. That is, for $z \geq 0$ and $\lambda \neq -\infty,+\infty$, $\pi(z,\lambda)$ solves the following equation in v :

$$v = \pi(z,\lambda) \Leftrightarrow v = \theta(z+v,\lambda) \quad (17)$$

Because $\theta(z,\lambda)$ is 1-contracting in z, this equation has at most one solution. In particular $\pi(0,\lambda) = 0$ for all λ, and $\pi(z,\lambda^*) = z$ for all z.

We check that $\pi(z,\lambda)$ is always defined if $\lambda \leq \lambda^*$. We have $0 \leq \theta(z+0,\lambda)$ and $\theta(z+v,\lambda) \leq \theta(z+v,\lambda^*) = \frac{z+v}{2} \leq v$ for $v \geq z$, so the claim follows by continuity of $z \to \theta(z,\lambda)$. For $\lambda \geq \lambda^*$ we set

$$\kappa_\lambda = \lim_{x\to\infty} \{x - \theta(x,\lambda)\}$$

which is positive and possibly infinite. We claim that (17) has a solution if and only if $z < \kappa_\lambda$. Fix such a pair z, λ, and choose x such that $z < x - \theta(x, \lambda) \iff \theta(z+v, \lambda) < v$ for $v = x - z$, so the "if" statement follows as above by continuity of θ in z. If $z \geq \kappa_\lambda$, then $x - \theta(x, \lambda) < z$ for all x as $x - \theta(x, \lambda)$ increases strictly. Hence $v < \theta(z + v, \lambda)$ for $v = x - z$, where v ranges over $\mathbb{R}_+$.

To sum up, $\pi(z, \lambda)$ is defined for $\lambda \in \mathbb{R}$ and $z \in [0, \kappa_\lambda[$, and we keep in mind $\kappa_\lambda = \infty$ whenever $\lambda \leq \lambda^*$. Moreover $\lambda \to \kappa_\lambda$ is weakly decreasing.

Continuity and monotonicity properties. By (17) and continuity of θ, the graph of π is closed hence π itself is continuous. Moreover $z \to \pi(z, \lambda)$ increases strictly for all λ; and $\lambda \to \pi(z, \lambda)$ increases strictly for $z > 0$.

For the former claim fix λ, z, z' such that $z < z' < \kappa_\lambda$. Set $v = \pi(z, \lambda), v' = \pi(z', \lambda)$. As θ increases strictly in z, we have $v < \theta(z' + v, \lambda)$, while $v' = \theta(z' + v', \lambda)$; so $v' = v$ is impossible, and $v' < v$ contradicts the fact that θ is 1-contracting in z (as $t \to t - \theta(z' + t, \lambda)$ decreases from v' to v).

For the latter claim fix λ, λ' such that $\lambda < \lambda'$, and z, $0 < z < \kappa_{\lambda'}$. Set $v = \pi(z, \lambda), v' = \pi(z, \lambda')$. As θ increases strictly in λ because $z > 0$, we have $v < \theta(z+v, \lambda')$, while $v' = \theta(z+v', \lambda')$. Thus $v' \leq v$ contradicts the 1-contracting property of θ exactly as in the previous paragraph.

*Self-duality properties.*We check first

$$\lim_{z \to \kappa_\lambda} \pi(z, \lambda) = \kappa_{\lambda^{-1}} \tag{18}$$

Indeed (16) implies $\kappa_{\lambda^{-1}} = \lim_{x \to \infty} \theta(x, \lambda)$, and if $\kappa_\lambda = \infty$ the equality $\lim_{z \to \infty} \pi(z, \lambda) = \lim_{x \to \infty} \theta(x, \lambda)$ is clear by definition of π. If on the other hand $\kappa_\lambda < \infty$, then $\lambda \geq \lambda^*$ and $\kappa_{\lambda^{-1}} = \infty$, and we must check $\lim_{z \to \kappa_\lambda} \pi(z, \lambda) = \infty$. Fix an arbitrary large $w > 0$, an $x > w + \kappa_\lambda$, and z such that $x - \theta(x, \lambda) < z < \kappa_\lambda$. Then $\pi(z, \lambda)$ is well defined and $\theta(z + (x - z), \lambda) > (x - z)$ implying $\pi(z, \lambda) > (x - z)$ because θ is 1-contracting in z, and in turn $\pi(z, \lambda) > w$.

Now (16) gives for all $z < \kappa_\lambda$:

$$v = \pi(z, \lambda) \Leftrightarrow z = \pi(v, \lambda^{-1}) \tag{19}$$

where $y < \kappa_{\lambda^{-1}}$ is a consequence of (18).

Step 3 We use now the representation π of the rationing rule h to describe the property of F-fairness of a matrix $y \in \mathbb{R}_+^{N \times A}$(see the definition of CSY in section 2). We write the a-column of y as $y_{[a]}$, and prove first

Fact 1: *If y is F-fair and $r_b > 0$ for all $b \in A$ (no null column in y), then for any $a \in A$ there is a unique parameter $\lambda_b \in \mathbb{R}$, one for each $b \neq a$, such that*

$$y_{ib} = \pi(y_{ia}, \lambda_b) \text{ for all } i \in N \text{ , all } b \in A \diagdown \{a\} \tag{20}$$

Indeed for each $b \neq a$ the reduced $N \times \{a, b\}$ matrix $[y_{[a]}, y_{[b]}]$ is F-fair, therefore $y_{[b]}$ is just $h(N, x^{ab}, r_b)$ for the profile of demands $x_i^{ab} = y_{ia} + y_{ib}$. By definition of the parametrization θ, there is some λ_b such that $y_{ib} = \theta(x_i^{ab}, \lambda_b)$. We cannot have $\lambda_b = \pm\infty$ because no column of y is null. Thus $y_{ib} = \theta(x_i^{ab}, \lambda_b) \iff y_{ib} = \pi(y_{ia}, \lambda_b)$ for all i, as claimed.

System (20) is written below as $y_{[b]} = \pi(y_{[a]}, \lambda_b)$. Now we prove the converse statement of Fact 1:

Fact 2: *Fix $a \in A$ and for each $b \neq a$ some parameter $\lambda_b \in \mathbb{R}$. Choose a column $y_{[a]}$ such that $0 \leq y_{ia} < \min_{b \in A \setminus \{a\}} \kappa_{\lambda_b}$ for all i, and define $y_{[b]} = \pi(y_{[a]}, \lambda_b)$. Then the matrix $y = [y_{[a]}, y_{[b]}]_{b \in A \setminus \{a\}}$ is fair.*

Proof If $y_{[a]} = 0$ the statement is trivial, so we assume without loss $y_{[a]} \neq 0$. Therefore $y_{[b]} \neq 0$ for all b as well (because $z > 0 \Longrightarrow \pi(z, \lambda) > 0$). Let x, r be the sums of rows and columns of y, and $\widetilde{y} = F(x, r)$. We show $y = \widetilde{y}$. By Fact 1 there are parameters $\widetilde{\lambda}_b$ such that $\widetilde{y}_{[b]} = \pi(\widetilde{y}_{[a]}, \widetilde{\lambda}_b)$ for all $b \neq a$. Assume first $y_{[a]} = \widetilde{y}_{[a]}$. As y and $\widetilde{y}$ have the same column sums, this implies $\sum_N \pi(y_{ia}, \lambda_b) = \sum_N \pi(y_{ia}, \widetilde{\lambda}_b)$ for all $b \neq a$, hence $\lambda_b = \widetilde{\lambda}_b$ because π increases strictly in λ for $y_{ia} > 0$ (and is constant if $y_{ia} = 0$), and we are done. Assume next $y_{[a]} \neq \widetilde{y}_{[a]}$ and partition N as $N^+ = \{i \in N | y_{ia} > \widetilde{y}_{ia}\}$ and $N^- = \{i \in N | y_{ia} \leq \widetilde{y}_{ia}\}$, both non empty. Define similarly $A^+ = \{b \in A \setminus \{a\} | \lambda_b \geq \widetilde{\lambda}_b\}$ and $A^- = \{b \in A \setminus \{a\} | \lambda_b < \widetilde{\lambda}_b\}$, where one set could be empty. Write $\delta_{ib} = y_{ib} - \widetilde{y}_{ib}$. The monotonicity properties of π imply $\delta_{ib} > 0$ if $i \in N^+, b \in A^+$, and $\delta_{ib} \leq 0$ if $i \in N^-, b \in A^-$. Therefore

$$\sum_{N^+} \delta_{ia} > 0 \ , \ \sum_{N^-} \delta_{ia} < 0 \ , \ \sum_{N^+ \times A^+} \delta_{ib} > 0 \ ; \ \sum_{N^- \times A^-} \delta_{ib} \leq 0$$

We derive first a contradiction if $A^+ \neq \varnothing$, by summing up all columns in A^+

$$\sum_{N^+ \times A^+} \delta_{ib} + \sum_{N^- \times A^+} \delta_{ib} = 0 \Longrightarrow \sum_{N^- \times A^+} \delta_{ib} < 0$$

then summing all rows in N^-

$$\sum_{N^- \times A^- \cup \{a\}} \delta_{ib} + \sum_{N^- \times A^+} \delta_{ib} = 0 \Longrightarrow \sum_{N^- \times A^+} \delta_{ib} > 0$$

Thus $A^+ = \varnothing$ and $A^- = A \setminus \{a\}$. Now we sum the columns in A^-, then the rows in N^+

$$\sum_{N^+ \times A^-} \delta_{ib} + \sum_{N^- \times A^-} \delta_{ib} = 0 \Longrightarrow \sum_{N^+ \times A^-} \delta_{ib} \geq 0$$

$$\text{and} \sum_{N^+} \delta_{ia} + \sum_{N^+ \times A^-} \delta_{ib} = 0 \Longrightarrow \sum_{N^+ \times A^-} \delta_{ib} < 0$$

another contradiction.

Step 4 We define an inner product for the parameters $\lambda \in \mathbb{R}$, related to the inverse operation discussed in steps 1,2. Fix two such parameters λ, μ, not necessarily distinct. We will define the product $\lambda * \mu$ by the equality $\pi(\pi(z, \lambda), \mu) = \pi(z, \lambda * \mu)$, for all z such that this expression is well defined.

We claim that $\pi(\pi(z, \lambda), \mu)$ is well defined if and only if $z < \pi(\kappa_\mu, \lambda^{-1})$, where we use the convention $\pi(z, \lambda) = \kappa_{\lambda^{-1}}$ if $z \geq \kappa_\lambda$. Supose first $\lambda \leq \lambda^*$ so that $\pi(z, \lambda)$ is well defined for all z and we only need $\pi(z, \lambda) < \kappa_\mu$. We saw in step

2 that $\lim_{z\to\infty}\pi(z,\lambda)=\kappa_{\lambda^{-1}}$, so there are no restrictions on z if $\kappa_\mu \geq \kappa_{\lambda^{-1}}$, just as we claim: $\pi(\kappa_\mu,\lambda^{-1})=\kappa_\lambda=\infty$; if on the other hand $\kappa_\mu<\kappa_{\lambda^{-1}}$, then the only restriction is $\pi(z,\lambda)<\kappa_\mu \iff z<\pi(\kappa_\mu,\lambda^{-1})$. Suppose next $\lambda>\lambda^*$, then we need $z<\kappa_\lambda$ and $\pi(z,\lambda)<\kappa_\mu$. We know from step 2 that $\lim_{z\to\kappa_\lambda}\pi(z,\lambda)=\infty$. If $\kappa_\mu<\infty$ we have $\pi(z,\lambda)<\kappa_\mu \iff z<\pi(\kappa_\mu,\lambda^{-1})$, and $\pi(\kappa_\mu,\lambda^{-1})<\lim_{z\to\infty}\pi(z,\lambda^{-1})=\kappa_\lambda$, proving the claim. If $\kappa_\mu=\infty$, we only need $z<\kappa_\lambda$, and with our convention $\pi(\kappa_\mu,\lambda^{-1})=\kappa_\lambda$ the proof is complete.

We fix λ,μ and we write $J(\lambda,\mu)=]0,\pi(\kappa_\mu,\lambda^{-1})[$ for the interior of the interval just discussed. We also fix a,b,c distinct in A. For an arbitrary profile $z=(z_i)\in J(\lambda,\mu)^N$ we construct the $N\times\{a,b,c\}$ assignment matrix y with strictly positive entries:

$$y_{ia}=z_i\ ,\ y_{ib}=\pi(z_i,\lambda)\ ,\ y_{ic}=\pi(\pi(z_i,\lambda),\mu)\text{ for all }i$$

Setting $z'_i=y_{ib}$, an equivalent description of the matrix is

$$y_{ia}=\pi(z'_i,\lambda^{-1})\ ,\ y_{ib}=z'_i\ ,\ y_{ic}=\pi(z'_i,\mu)\text{ for all }i$$

The matrix y is F-fair by Fact 2 in step 3 applied to the latter description. By Fact 1 applied to the former description, there exists a parameter ρ such that $y_{ic}=\pi(y_{ia},\rho)$. Thus we have $\pi(\pi(z_i,\lambda),\mu)=\pi(z_i,\rho)$ for all z_i. Clearly ρ is unique, and in fact it does not depend at all on the choice of the z_i-s: if we take two such profiles z,z' overlapping at z_1, say, then $\pi(z_1,\rho)$ is the same for both profiles, implying that ρ did not change. Thus the definition of $\lambda*\mu\in\mathbb{R}$ is complete and we have

$$\pi(\pi(z,\lambda),\mu)=\pi(z,\lambda*\mu)\text{ for all }z<\pi(\kappa_\mu,\lambda^{-1}) \tag{21}$$

The identity $\pi(z,\lambda^*)=z$ (step 2) means that λ^* is the neutral element of this operation, and (19) implies $\lambda*\lambda^{-1}=\lambda^*$.

Step 5 We show that $\lambda*\mu$ is an associative product, and derive an additive representation of this operation from the well known Associativity Theorem ([1], [12]).

For any three parameters λ,μ,v, associativity follows the repeated application of (21):

$$\pi(z,(\lambda*\mu)*v)=\pi(\pi(z,(\lambda*\mu),v)=\pi(\pi(\pi(z,\lambda),\mu),v)=\pi(\pi(z,\lambda),\mu*v)=\pi(z,\lambda*(\mu*v))$$

where those expressions are all well defined for z in a positive interval $[0,K[$. For instance $\pi(\pi(\pi(z,\lambda),\mu),v)$ is well defined whenever $\pi(z,\lambda)$ is well defined, and $\pi(z,\lambda)<\pi(\kappa_v,\mu^{-1})$, which amounts to $z<\kappa_\lambda$ and $z<\pi(\pi(\kappa_v,\mu^{-1}),\lambda^{-1})$. We omit the similar arguments for the other four terms.

Associativity of $*$ implies the identity $(\lambda*\mu)*(\mu^{-1}*\lambda^{-1})=\lambda^*$, therefore $(\lambda*\mu)^{-1}=\mu^{-1}*\lambda^{-1}$.

The next two critical properties of $*$ are the continuity and strict monotonicity of the function f: $f(\lambda,\mu)=\lambda*\mu$. For continuity pick any two $\overline{\lambda},\overline{\mu}$ and observe that in a small enough neighborhood of $(\overline{\lambda},\overline{\mu})$, equation (21) in z holds on a non empty open interval $]0, K[$ (because $\pi(\kappa_\mu, \lambda^{-1})$ decreases in both λ and μ). As π is strictly monotonic in both variables, this means that f is defined in this neighborhood by equation (21) at a single value z, therefore its graph is closed by continuity of π. The same local argument, and strict monotonicity of π in λ, show that f is strictly monotonic.

By Aczel's Theorem (section 6.2 in [1]) an associative, continuous, and strictly monotonic product $*$ in $\mathbb{R}$ is represented as follows by a continuous and strictly increasing function g on $\mathbb{R}$:

$$\lambda * \mu = g^{-1}(g(\lambda) + g(\mu)) \text{ for all } \lambda, \mu \in \mathbb{R}$$

In particular $g(\lambda^*) = 0$ is the neutral element, and $\lambda * \lambda^{-1} = \lambda^*$ becomes $g(\lambda^{-1}) = -g(\lambda)$.

Thus g is an homeomorphism of $\mathbb{R}$ into its range, and its range must be $\mathbb{R}$: it is an interval stable by addition and symmetry around 0. We use now the new variable $\beta = g(\lambda)$ to parametrize the rationing rule h: we set $\widetilde{\theta}(z, \beta) = \theta(z, g^{-1}(\beta))$ for $\beta \in \mathbb{R}$, and $\widetilde{\theta}(z, -\infty) = \theta(z, -\infty) = 0$, $\widetilde{\theta}(z, \infty) = \theta(z, \infty) = z$. The rule h is still represented by $\widetilde{\theta}$ through property (10), and the entire discussion of steps 1 to 4, including the definition of $\widetilde{\pi}$ through (17), the domain restriction $\widetilde{\kappa}_\beta = \kappa_{g^{-1}(\beta)}$, and the regularity properties of $\widetilde{\theta}$ and $\widetilde{\pi}$, are preserved. The advantage of this parametrization is that the equation (21) takes the form

$$\widetilde{\pi}(\widetilde{\pi}(z, \beta), \gamma) = \widetilde{\pi}(z, \beta + \gamma) \text{ for all } z < \widetilde{\pi}(\widetilde{\kappa}_\gamma, -\beta)$$

Step 6 We derive finally the desired representation of F as a W-welfarist assignment rule. We will construct the functions $W, \Gamma = W'^{-1}$ in statement statement *iii)* of Proposition 2.

First we check that the supremum of $\widetilde{\pi}(1, \beta)$ over all β for which it is defined (i.e., such that $1 < \widetilde{\kappa}_\beta$), is ∞. By continuity of $\widetilde{\theta}$, for any $\Delta > 0$ there exists β such that $\Delta - \widetilde{\theta}(\Delta + 1, \beta) = 0$, because this expression is $\Delta > 0$ at $\beta = -\infty$ and -1 at $\beta = \infty$. This equality is just $\Delta = \widetilde{\pi}(1, \beta)$. Thus the range of $\beta \to \widetilde{\pi}(1, \beta)$ is $\mathbb{R}_+$.

We fix now an arbitrary problem (N, A, x, r) with all $x_i, r_a > 0$, and set $y = F(N, A, x, r)$. Fix an arbitrary $a \in A$ and use Fact 1 in step 3: for each $b \neq a$ there is a parameter β_b such that $y_{[b]} = \widetilde{\pi}(y_{[a]}, \beta_b)$. On the other hand the argument in the previous paragraph shows that for each i there is a parameter α_i such that $y_{ia} = \widetilde{\pi}(1, \alpha_i)$. Combining these equations gives $y_{ib} = \widetilde{\pi}(y_{ia}, \beta_b) = \widetilde{\pi}(\widetilde{\pi}(1, \alpha_i), \beta_b) = \widetilde{\pi}(1, \alpha_i + \beta_b)$ for all i, all $b \in A \diagdown \{a\}$. Setting $\beta_a = 0$ implies finally

$$y_{ib} = \widetilde{\pi}(1, \alpha_i + \beta_b) \text{ for all } i \in N, b \in A \tag{22}$$

Let $\overline{\beta}$ be the upper bound of the domain of definition of $\widetilde{\pi}(1, \beta)$. If $\overline{\beta} < \infty$ we have $\overline{\beta} = \min\{\beta | \widetilde{\kappa}_\beta \leq 1\}$, because $\widetilde{\kappa}_\beta$ is continuous and weakly increasing when finite. Define the function Γ on $]-\infty, \overline{\beta}[$ as follows

$$\Gamma(-\infty) = 0 \text{ ; } \Gamma(\alpha) = \widetilde{\pi}(1, \alpha) \text{ if } -\infty < \alpha < \overline{\beta} \text{ ; } \Gamma(\alpha) = \infty \text{ if } \alpha \geq \overline{\beta}$$

It is an increasing homeomorphism from $[-\infty, \overline{\beta}]$ into $[0, \infty]$, hence its inverse W' is also an homeomorphism from $[0, \infty]$ into $[-\infty, \overline{\beta}]$; in particular $\lim_{z \to 0} W'(z) = -\infty$, $\lim_{z \to \infty} W'(z) = \overline{\beta}$. This is precisely the connexion between W' and Γ in Proposition 2, and the system (22) is exactly (9). We conclude from statement *iii*) in Proposition 2 that y is W-welfarist whenever all $x_i, r_a > 0$; if some rows or columns are null CSY allows us to drop them and maintain the statement that y is the W-welfarist assignment.■

7 Concluding comments

1. Within the set of unconstrained assignments, Theorem 2 is a partial characterization of the W-welfarist rules, limited to those ruling out null shares, which amounts to impose $W'(0) = -\infty$ on the welfare function. This excludes a whole range of interesting welfarist rules, such as the scale invariant W^q in Lemma 2 for $q > 1$. An obvious next step is to relax the assumption $W'(0) = -\infty$ and look for a more general characterization result. Ideally one would even capture consistent rules such as the *egalitarian rule*, equalizing the entries of the assignment matrix by maximizing the leximin ordering; however the latter rule cannot be described as minimizing a convex objective function
2. Proposition 3 is an extension result, from rules defined for unconstrained problems in $\mathcal{P}^u$ to capacited problems in $\mathcal{P}$. As above it is important to understand if and how the extension result is preserved when we drop the assumption $W'(0) = -\infty$. A more difficult question is to deal with more general types of constraints than the "rectangular" ones we assume here. For instance Balinski and Demange [2] consider bounds on the sums in rows and columns of the assignment matrix; one could also think of linear constraints cutting across the entries. A plausible conjecture is that, again, minimizing the welfare sum $\sum_{N \times A} W(y_{ia})$ is the only consistent extension.
3. The bilateral rationing model developed in [14], [15] is a direct generalization of the present model, if one incorporates our more general capacity constraints. In these papers, like here, the grand goal is to capture the structure of consistent bilateral rationing rules, but Moulin and Sethuraman impose an axiom, Merging of Identically connected Resource-types (MIR), that has no compelling counterpart in the capacited problem, and moreover forces the proportional rule for unconstrained problems. Thus a promising reserach question after dropping MIR is to explore the rich family of consistent bilateral rationing rules, first in unconstrained problems, then with more general capacity constraints.

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