Tests for a Broken Trend with Stationary or Integrated Shocks

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Job Market Paper

November 8, 2013

Abstract

We consider testing for the presence of a structural break in the trend of a univariate time-series where the date of the break is unknown. The tests we propose are robust as to whether the shocks are generated by a stationary or an integrated process. From an empirical standpoint, our robust tests can be quite useful in policy analysis: these tests can easily be employed to evaluate the impact of a one time policy change or a new regulation on a trending variable. Following Harvey et al. (2010), we utilize two different test statistics; one is appropriate for the stationary alternative and the other is for the unit root alternative. Our approach exploits the under-sizing property of each test statistic under the other alternative, and is based on the union of rejections approach proposed by Harvey et al. (2009a). Two trend break models are considered: the first one is a joint broken trend model where there is a break in the trend holding the level fixed, while the latter allows for a simultaneous break in level and trend which we call the disjoint broken trend model. The behavior of the proposed tests is studied through Monte Carlo experiments.The simulation results suggest that our robust tests perform well in small samples, showing good size control and displaying very decent power regardless of the degree of persistence of the data.

Keywords: Trend Breaks, Robust tests, stationarity, unit root
JEL classification: C12, C22

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1 Introduction

Many macroeconomic and financial time series can be characterized by shocks fluctuating around a broken trend function. It is very important to account for changes in the trend function and failure to model the trend function properly will lead to inconsistent parameter estimates and poor forecasts. Perron (1989) has shown that an unaccounted break in the trend function can bias unit root tests toward the nonrejection of the unit root hypothesis. Similarly, unmodeled trend breaks also lead to spurious rejections in stationarity tests as documented by Lee et al. (1997). In fact, there are many interesting economic applications for which the existence of structural change in the trend function can be of interest itself. For example, the empirical debate that convergence in incomes per capita among U.S. regions has leveled off in the mid-1970s can be explored by modeling the trend function in time series of incomes per capita in each of the U.S. regions as having a slope shift in the mid-1970s (see Sayginsoy and Vogelsang (2004)).

Tests of a break in level and/or trend of a time series exist in the literature. However, for a reliable inference on the existence of a break these tests assume a priori that the data is generated either by a stationary or an integrated process (see Bai (1994), Bai and Perron (1998) and Hansen (1992)). As it is well known, many macroeconomic and financial variables are highly persistent and it is very difficult to tell a priori that these series are stationary or integrated. Hence the structural break tests mentioned above have limitations in practice. One way to overcome this problem is to pretest the series with unit root or stationarity tests to determine if the series has a unit root. However, unit root and stationarity tests are known to have poor size and power properties in the presence of a structural break (Perron (1989), Lee et al. (1997)). We thus have a circular testing problem between tests of structural breaks and unit root/stationarity tests.

In this paper, we provide a solution to this dilemma by proposing new tests for the presence of a structural break in the trend function at an unknown date that are valid in the presence of stationary (I(0)) or integrated (I(1)) shocks. We consider tests based on two statistics, one appropriate for the case of I(0) shocks and the other appropriate for I(1) shocks. These statistics are the supremum of the trend break t-statistics, calculated for all permissible break dates, from a regression in levels and a regression in growth rates. We derive the asymptotic distributions of these statistics in both I(0) and (local to) I(1) environments. The asymptotic properties demonstrate that the test statistic appropriate for the case of I(1) shocks converges in probability to zero under I(0) shocks, while the test appropriate for I(0) shocks is always under-sized when the shocks are (local to) I(1). These properties make it possible the construction of a size-controlled union of rejections testing approach whereby we reject the null hypothesis of no trend break if either of the two tests rejects.

Recently, a few other structural break tests that are valid regardless of whether the shocks are stationary or integrated have been proposed in the econometrics literature. Sayginsoy and Vogelsang (2011)[SV] propose a Mean Wald and a Sup Wald statistic using the fixed-b asymptotic framework of Kiefer and Vogelsang (2005) in conjunction with the scaling factor approach of Vogelsang (1998) to smooth the discontinuities in the asymptotic distributions of the test statistics as the shocks go from I(0) to I(1). Harvey et al. (2009b) [HLT] employ tests that are formed as a weighted average of the supremum of the regression t-statistics, calculated for all permissible break dates, for a broken trend appropriate for the case of I(0) and I(1) shocks. The weighting function they utilize is based on the KPSS stationarity test applied to the levels and first-differenced data. Perron and Yabu (2009) [PY] consider an alternative approach based on a Feasible Generalized Least Squares procedure that uses a super-efficient estimate of the sum of the autoregressive parameters α when $\alpha = 1$. This allows tests of basically the same size with stationary or integrated shocks regardless of whether the break is known or unknown, provided that the Exp functional of Andrews and Ploberger (1994) is used in the latter case.

Our robust tests compare favorably to the other robust tests mentioned above. The advantage of our method over those of SV and HLT is that it does not involve any random scaling so that the test used is more prone to have higher power and less size distortions, as documented under the finite sample results in Section 5. Our method does well also in comparison with the method of PY; first PY assume the normality of shocks to derive pivotal asymptotic distributions for certain broken trend models, and second their structural break tests don't allow the estimation of the break date simultaneously due to the use of Exp functional. On the other hand, the test statistics we propose have pivotal asymptotic distributions without any stringent assumptions on the shocks and our method allows us to simultaneously test for a break and estimate the break date.

It is also straightforward to extend our method to test for multiple trend breaks or to sequentially test for l versus l+1 breaks in the trend function. However, the main focus of this paper is to test the null hypothesis that there is no structural break in the trend against the alternative that there exists a break in the trend. This type of hypothesis is more relevant if one wants to evaluate the impact of a one time policy change or a new regulation on a trending variable. For example, Sidneva and Zivot (2007) consider the question of whether there was a break in the trends of two air pollutants, nitrogen oxides and volatile organic compounds, emissions around the time the Clean Air Act Amendments of 1970 were passed. The tests we propose in this paper can easily be employed to answer this question without the need to know whether the shocks are stationary or integrated.

The rest of the paper is organized as follows. Section 2 reviews the two trend break models considered and the main assumptions used in the paper. In sections 3 and 4, we investigate each trend break model separately. Section 3 outlines the testing procedure we propose for the joint broken trend model. We introduce our tests based on two statistics with their asymptotic properties under the null and the alternative and explain how the union of rejections approach works in principle. In section 4, the same steps are carried out for the disjoint broken trend model. Section 5 provides some Monte Carlo studies that demonstrate the finite sample properties of our test. Section 6 concludes.

2 Trend Break Models

We consider two trend break models: "Model 1" is a joint broken trend model where there is a break in the trend holding the level fixed, while "Model 2" allows for a simultaneous break in level and trend which we call the disjoint broken trend model. These are the same models considered by HLT, corresponds to models (3) and (4) in SV and to models II and III in PY respectively. We consider the following trend break data generating processes (DGP) to model broken trends:

$$y_t = \alpha + \beta t + \gamma DT_t(\tau^*) + u_t, \qquad t = 1, ..., T \tag{1}$$

$$y_t = \alpha + \beta t + \delta DU_t(\tau^*) + \gamma DT_t(\tau^*) + u_t, \qquad t = 1, ..., T \tag{2}$$

$$u_t = \rho u_{t-1} + \varepsilon_t, \qquad t = 2, ..., T, \quad u_1 = \varepsilon_1, \tag{3}$$

where $DU_t(\tau^*) = 1(t > T^*)$ and $DT_t(\tau^*) = 1(t > T^*)(t - T^*)$, with $T^* = \lfloor \tau^* T \rfloor$ the (potential) trend break date with associated break fraction $\tau^* \in (0, 1)$. Equation (1) refers to the joint broken trend model, while equation (2) defines the disjoint broken trend model.

We assume that ε_t in (3) satisfies Assumption 1 of SV:

Assumption 1. The stochastic process $\{\varepsilon_t\}$ is such that

$$\varepsilon_t = c(L)\eta_t, \qquad c(L) = \sum_{i=0}^{\infty} c_i L^i,$$

with $c(1)^2 > 0$ and $\sum_{i=0}^{\infty} i|c_i| < \infty$, and where $\{\eta_t\}$ is a martingale difference sequence with unit conditional variance and $sup_t E(\eta_t^4) < \infty$.

The error process $\{u_t\}$ is $I(0)$ when $|\rho| < 1$ in (3). Alternatively, $\{u_t\}$ can be modeled as a nearly $I(1)$ process by defining $\rho = \rho_T = 1 - c/T$, where $c = 0$ corresponds to the pure $I(1)$ case. We are interested in testing if there is a trend break in y_t. Our interest in this paper therefore centers on testing the null hypothesis $H_0 : \gamma = 0$ against the two-sided alternative hypothesis $H_1 : \gamma \neq 0$, independently of whether u_t is $I(0)$ or $I(1)$[1].

Remark 1. Under the conditions of Assumption 1, the long-run variance of ε_t is given by $\omega_\varepsilon^2 = lim_{T\to\infty} T^{-1} E(\sum_{t=1}^{T} \varepsilon_t)^2 = c(1)^2$. Moreover, in the $I(0)$ case the long-run variance of u_t is given by $\omega_u^2 = lim_{T\to\infty} T^{-1} E(\sum_{t=1}^{T} u_t)^2 = \omega_\varepsilon^2/(1-\rho)^2$. Both these long-run variances play important roles in our subsequent analysis.

3 Joint Broken Trend Model

We start by considering a time-series process y_t with a first-order linear trend and one possible change in the slope such that the trend function is always joined at the time of the break, which we call the joint broken trend model:

$$y_t = \alpha + \beta t + \gamma DT_t(\tau^*) + u_t, \qquad t = 1, ..., T \tag{4}$$

$$u_t = \rho u_{t-1} + \varepsilon_t, \qquad t = 2, ..., T, \quad u_1 = \varepsilon_1, \tag{5}$$

where $DT_t(\tau^*) = 1(t > T^*)(t - T^*)$ captures the eventual break in the slope occurring at date $T^* = \lfloor \tau^* T \rfloor$ with associated break fraction $\tau^* \in (0, 1)$. The slope coefficient changes from β to $\beta + \gamma$ at time T^*. However, notice that the trend function is continuous in every period including the date at which the slope change occurs. The discontinuous case is considered in section 4.

3.1 Known Break Fraction

We start considering the case where the true break fraction,τ^*, is known. The unknown break fraction case will be subsequently discussed in Section 3.3. We also assume that the long-run variances, ω_u^2 and ω_ε^2, are known in the following analysis. We will relax this assumption in Section 3.3.

Suppose one knows that u_t is $I(0)$, with $\rho = 0$ in (5) and ε_t is Gaussian white noise. Then the uniformly most powerful unbiased (optimal) test of H_0 against H_1 is the standard t-test associated with γ when (4) is estimated using OLS. The t-ratio $t_0(\tau^*)$, corrected for serial correlation in errors, can

[1]Hereafter $I(1)$ implies ρ=$1 - c/T$, with c=0 corresponding to the pure $I(1)$ case

be expressed as[2]

$$t_0(\tau^*) = \frac{\hat{\gamma}(\tau^*)}{\sqrt{\omega_u^2 \left[\left\{\sum_{t=1}^{T} x_{DT,t}(\tau^*) x_{DT,t}(\tau^*)'\right\}^{-1}\right]_{33}}} \tag{6}$$

$$\hat{\gamma}(\tau^*) = \left[\left\{\sum_{t=1}^{T} x_{DT,t}(\tau^*) x_{DT,t}(\tau^*)'\right\}^{-1} \sum_{t=1}^{T} x_{DT,t}(\tau^*) y_t\right]_3,$$

with $x_{DT,t}(\tau^*) = \{1, t, DT_t(\tau^*)\}'$.

On the other hand, if u_t is known to be pure $I(1)$, so that $\rho = 1$ in (5), and Δu_t is a Gaussian white noise process, then the optimal test is based on the t-statistic associated with γ when (4) is estimated with OLS in first differences. That is, writing

$$\Delta y_t = \beta + \gamma DU_t(\tau^*) + \Delta u_t, \qquad t = 2, ..., T, \tag{7}$$

where $DU_t(\tau^*) = 1(t > T^*)$, the optimal test rejects for large values of $|t_1(\tau^*)|$, where

$$t_1(\tau^*) = \frac{\hat{\gamma}(\tau^*)}{\sqrt{\omega_\varepsilon^2 \left[\left\{\sum_{t=2}^{T} x_{DU,t}(\tau^*) x_{DU,t}(\tau^*)'\right\}^{-1}\right]_{22}}} \tag{8}$$

$$\hat{\gamma}(\tau^*) = \left[\left\{\sum_{t=2}^{T} x_{DU,t}(\tau^*) x_{DU,t}(\tau^*)'\right\}^{-1} \sum_{t=2}^{T} x_{DU,t}(\tau^*) \Delta y_t\right]_2,$$

with $x_{DU,t}(\tau^*) = \{1, DU_t(\tau^*)\}'$.

Theorem 1 establishes the asymptotic properties of the $|t_0(\tau^*)|$ and $|t_1(\tau^*)|$ statistics under both I(0) and I(1) errors.

Theorem 1. Let the time series process $\{y_t\}$ be generated according to (4) and (5) under $H_0 : \gamma = 0$, and let Assumption 1 hold.

(i) If u_t in (5) is $I(0)$ (*i.e.*, $|\rho| < 1$), then (*a*) $|t_0(\tau^*)| \xrightarrow{d} |J_0(\tau^*)|$, *where*
$$J_0(\tau^*) = \frac{\int_0^1 RT(r, \tau^*) dW(r)}{\{\int_0^1 RT(r, \tau^*)^2 dr\}^{1/2}},$$
and (*b*) $|t_1(\tau^*)| = O_p(T^{-1/2})$.

[2] The notation $[.]_{jj}([.]_j)$ is used to denote the jj'th (j'th) element of the matrix (vector) within the square brackets.

(ii) If u_t in (5) is $I(1)$, then (a) $|t_0(\tau^*)| = O_p(T)$, *and* (b) $|t_1(\tau^*)| \xrightarrow{d} |J_1(\tau^*, c)|$, *where* $J_1(\tau^*, c) = \dfrac{\int_0^1 RU(r, \tau^*) dW_c(r)}{\{\int_0^1 RU(r, \tau^*)^2 dr\}^{1/2}}$, *where* $W(r)$ *is a standard Brownian motion, and* $W_c(r) = \int_0^r e^{-(r-s)c} dW(s)$ *denotes a standard Ornstein-Uhlenbeck* (OU) *process on [0,1],* $RT(r, \tau^*)$ *is the continuous-time residual from the projection of* $(r - \tau^*)1(r > \tau^*)$ *onto the space spanned by* $\{1, r\}$*, and* $RU(r, \tau^*)$ *is the residual from the projection of* $1(r > \tau^*)$ *onto* $\{1\}$.

3.2 Unknown Break Fraction

In this section, we consider tests of a structural break in the trend function at an unknown break fraction. Following Andrews (1993) we consider statistics based on the maximum of the sequences of statistics $\{|t_0(\tau)|, \tau \in \Lambda\}$ and $\{|t_1(\tau)|, \tau \in \Lambda\}$, where $\Lambda = [\tau_L, \tau_U]$, with $0 < \tau_L < \tau_U < 1$, and the quantities τ_L and τ_U will be referred to as the trimming parameters, and it is assumed throughout that $\tau^* \in \Lambda$. The supremum functional allows us to test for a break and determine the break date simultaneously. These statistics are given by

$$t_0^* = \sup_{\tau \in \Lambda} |t_0(\tau)| \tag{9}$$

and

$$t_1^* = \sup_{\tau \in \Lambda} |t_1(\tau)|, \tag{10}$$

with associated break fraction estimators of τ^* given by $\hat{\tau} = arg \sup_{\tau \in \Lambda} |t_0(\tau)|$ and $\tilde{\tau} = arg \sup_{\tau \in \Lambda} |t_1(\tau)|$, respectively, such that $t_0^* \equiv |t_0(\hat{\tau})|$ and $t_1^* \equiv |t_1(\tilde{\tau})|$. The break fraction estimators,$\hat{\tau}$ and $\tilde{\tau}$, are asymptotically equivalent to the corresponding minimum sum of squares break fraction estimator of Perron and Zhu (2005) and hence consistent under the alternative of fixed break magnitude.

In the following theorem we establish the asymptotic properties of t_0^* and t_1^* under both $I(0)$ and $I(1)$ environments.

Theorem 2. Let the time series process $\{y_t\}$ be generated according to (4) and (5) under $H_0 : \gamma = 0$, and let Assumption 1 hold.

(i) If u_t is $I(0)$, then $(a) t_0^* \xrightarrow{d} \sup_{\tau \in \Lambda} |J_0(\tau)|$, and $(b) t_1^* = O_p(T^{-1/2})$.

(ii) If u_t is $I(1)$, then $(a) t_0^* = O_p(T)$, and $(b) t_1^* \xrightarrow{d} \sup_{\tau \in \Lambda} |J_1(\tau, c)|$.

Remark 2. From the results of part (i) of Theorem 2 it can be seen that when u_t is $I(0)$, t_1^* converges in probability to zero. Similarly, from the results in part (ii) of Theorem 2 it can be seen that when u_t is $I(1)$, t_0^* diverges.

In view of the large sample results in Theorem 2, we would encounter two problems in practice. First, under H_0, the appropriate test statistic (either t_0^* or t_1^*) to obtain a non-degenerate limiting distribution and second the choice of long run variance standardization (either ω_ε^2 or ω_u^2) to establish a pivotal limiting null distribution both depend on whether the errors are $I(1)$ or $I(0)$, which is not known in practice. Moreover, in practice we would also need to estimate either ω_ε^2 or ω_u^2 in order to yield a feasible testing procedure. In the next section we will explore solutions to these issues.

3.3 Feasible Robust Tests for a Broken Trend

In this section we address the practical issues that exist in developing a feasible test for a broken trend outlined at the end of the previous section. In Section 3.3.1, we first consider the issue of long run variance estimation, and examine the behavior of the estimators under both $I(1)$ and $I(0)$ errors. In Section 3.3.2, we then use the results from section 3.3.1 to develop an operational test against a broken trend in model (4)-(5) for the situation where the order of integration is unknown. Section 3.3.2 also presents an analysis of the asymptotic size properties of the proposed tests.

3.3.1 Long Run Variance Estimation

We now consider estimation of the long run variances ω_ε^2 (relevant under $I(1)$ errors) and ω_u^2 (relevant under $I(0)$ errors). Estimation of ω_ε^2 is standard and is explained first. However, in order to develop a feasible test we estimate ω_u^2 using the Berk(1974)-type autoregressive spectral density estimator. The large sample properties of these estimators are provided under both $I(0)$ and $I(1)$ environments.

Estimation of ω_ε^2

First we consider estimating ω_ε^2 when the errors are known to be $I(1)$. Let $\hat{\tau}$ be a consistent estimator of the true break fraction τ^*. We first estimate the following equation with OLS to get the residuals $\hat{\varepsilon}_t(\hat{\tau})$

$$\Delta y_t = \beta + \gamma DU_t(\hat{\tau}) + \varepsilon_t, \qquad t = 2, ..., T, \tag{11}$$

Nonparametric long-run variance estimator $\hat{\omega}_{\varepsilon}^2(\hat{\tau})$ is then given by

$$\hat{\omega}_{\varepsilon}^2(\hat{\tau}) = \hat{\gamma}_0(\hat{\tau}) + 2\sum_{j=1}^{T-2} k(j/l)\hat{\gamma}_j(\hat{\tau}), \tag{12}$$

$$\hat{\gamma}_j(\hat{\tau}) = (T-1)^{-1} \sum_{t=j+2}^{T} \hat{\varepsilon}_t(\hat{\tau})\hat{\varepsilon}_{t-j}(\hat{\tau})$$

where $k(.)$ is a kernel function with associated bandwidth parameter l. In what follows we shall make use of the Bartlett kernel for $k(.)$, such that $k(j/l) = 1 - j/(l+1)$, with bandwidth parameter $l = O(T^{1/4})$.

In Theorem 3, we now establish the large sample properties of $\hat{\omega}_{\varepsilon}^2(\hat{\tau})$; since in practice, the order of integration is unknown we detail the asymptotic behavior of the estimator under both $I(1)$ and $I(0)$ errors.

Theorem 3. Let the conditions of Theorem 1 hold. Then

(i) If u_t is $I(1)$, then $\hat{\omega}_{\varepsilon}^2(\hat{\tau}) \xrightarrow{p} \omega_{\varepsilon}^2$

(ii) If u_t is $I(0)$, then $\hat{\omega}_{\varepsilon}^2(\hat{\tau}) = O_p(l^{-1})$.

Estimation of ω_u^2

We now consider estimating ω_u^2 in the case where the errors are known to be $I(0)$. Here we focus on Berk-type autoregressive spectral density estimators in the estimation of ω_u^2. The Berk-type estimator is the key in developing a feasible test that works under both $I(0)$ and $I(1)$ errors. As will be seen in the next section, estimation of ω_u^2 using an autoregressive framework allows the t_0^* statistic to be stochastically bounded under both $I(0)$ and $I(1)$ errors and enables us to utilize the *union of rejections* principle to develop a feasible test.

We estimate ω_u^2 under the null $H_0 : \gamma = 0$ and hence no trend break dummy is included in the following regression

$$y_t = \alpha + \beta t + u_t, \qquad t = 1, ..., T \tag{13}$$

We estimate equation (13) via OLS and get the residuals $\hat{u}_t$.The Berk-type estimator $\hat{\omega}_u^2$ is then given by

$$\hat{\omega}_u^2 = \frac{\hat{\sigma}^2}{\hat{\pi}^2}$$

where $\hat{\pi}$ and $\hat{\sigma}$ are obtained from the OLS regression

$$\Delta\hat{u}_t = \hat{\pi}\hat{u}_{t-1} + \sum_{j=1}^{k-1} \hat{\psi}_j \Delta\hat{u}_{t-j} + \hat{e}_t, \quad t = k+2, ..., T \tag{14}$$

with $\hat{\sigma}^2 = (T - 2k - 1)^{-1} \sum_{t=k+2}^{T} \hat{e}_t^2$. As is standard, we require that the lag truncation parameter, k, in (14) satisfies the condition that, as $T \to \infty$, $1/k + k^3/T \to 0$.

In Theorem 4, we now establish the asymptotic properties of $\hat{\omega}_u^2$ under both $I(0)$ and $I(1)$ environments.

Theorem 4. Let the conditions of Theorem 1 hold. Then under the null $H_0 : \gamma = 0$

(i) If u_t is $I(1)$, then $T^{-2}\hat{\omega}_u^2 \xrightarrow{d} \omega_\varepsilon^2 \Phi(c)$

(ii) If u_t is $I(0)$, then $\hat{\omega}_u^2 \xrightarrow{p} \omega_u^2$

where

$$\Phi(c) = \frac{\left\{\int_0^1 Q(r,c)^2 dr\right\}^2}{\left\{\int_0^1 Q(r,c)^2 dW_c(r)\right\}^2}$$

and $Q(r, c)$ is a continuous-time residual from the projection of $W_c(r)$ onto the space spanned by $\{1, r\}$.

Remark 3. Observe from part (ii) of Theorem 4 that $\hat{\omega}_u^2$ is a consistent estimator of ω_u^2 under $I(0)$ errors. It is also seen from part (i) of Theorem 4 that the Berk-type estimator diverges at a rate of T^2 which is crucial in our construction of a feasible test, as explained in the next section.

3.3.2 Feasible Tests and Asymptotic Size

Having proposed suitable long run variance estimators and having established their asymptotic properties, we are now in a position to define feasible statistics for detecting a break in the trend function. The results of Theorem 2, along with the properties of the long run variance estimators described in Theorems 3 and 4, suggest the following statistics, appropriate under $I(1)$ and $I(0)$ errors, respectively:

$$S_1 = \sup_{\tau \in \Lambda} |\hat{t}_1(\tau)| \tag{15}$$

$$S_0 = \sup_{\tau \in \Lambda} |\hat{t}_0(\tau)| \tag{16}$$

where

$$\hat{t}_1(\tau) = \frac{\hat{\gamma}(\tau)}{\sqrt{\hat{\omega}^2_{\varepsilon}(\tau)\left[\left\{\sum_{t=2}^{T} x_{DU,t}(\tau)x_{DU,t}(\tau)'\right\}^{-1}\right]_{22}}}$$

$$\hat{\gamma}(\tau) = \left[\left\{\sum_{t=2}^{T} x_{DU,t}(\tau)x_{DU,t}(\tau)'\right\}^{-1}\sum_{t=2}^{T} x_{DU,t}(\tau)\Delta y_t\right]_2,$$

and

$$\hat{t}_0(\tau) = \frac{\hat{\gamma}(\tau)}{\sqrt{\hat{\omega}^2_{u}\left[\left\{\sum_{t=1}^{T} x_{DT,t}(\tau)x_{DT,t}(\tau)'\right\}^{-1}\right]_{33}}}$$

$$\hat{\gamma}(\tau) = \left[\left\{\sum_{t=1}^{T} x_{DT,t}(\tau)x_{DT,t}(\tau)'\right\}^{-1}\sum_{t=1}^{T} x_{DT,t}(\tau)y_t\right]_3.$$

In the following lemma we now establish the asymptotic properties of the S_1 and S_0 statistics of (15) and (16), respectively, in both $I(1)$ and $I(0)$ environments.

Lemma 1. Let the time series process $\{y_t\}$ be generated according to (4) and (5) under $H_0 : \gamma = 0$, and let Assumption 1 hold.

(i) If u_t is $I(0)$, then

  (*a*) $S_0 \xrightarrow{d} \sup_{\tau\in\Lambda} |J_0(\tau)|$,

  (*b*) $S_1 = O_p\left\{(l/T)^{1/2}\right\}$.

(ii) If u_t is $I(1)$, then

  (*a*) $S_0 \xrightarrow{d} \dfrac{\sup_{\tau\in\Lambda}|K_0(\tau,c)|}{\Phi^{1/2}(c)}$, *where*

$$K_0(\tau,c) = \frac{\int_0^1 RT(r,\tau)W_c(r)dr}{\{\int_0^1 RT(r,\tau)^2dr\}^{1/2}}$$

  (*b*) $S_1 \xrightarrow{d} \sup_{\tau\in\Lambda} |J_1(\tau,c)|$.

Asymptotic null critical values for S_1 under $I(1)$ errors with $c = 0$, and S_0 under $I(0)$ errors, are reported in Table 1, for the settings τ_L=0.1 and τ_U=0.9, and for the significance levels $\xi = 0.10, 0.05$ and 0.01. The numerical results were obtained by simulation of the appropriate limiting distributions using discrete approximations for $T = 1,000$ and 10,000 replications using normal $IID(0,1)$ random .

Critical Values			
	S_1	S_0	κ_ξ
$\xi = 0.10$	2.741	2.268	1.0238
$\xi = 0.05$	3.024	2.570	1.0065
$\xi = 0.01$	3.565	3.139	1.0048

Note: The critical values for S_0 and S_1 are for the $I(0)$ and $I(1)(c = 0)$ cases, respectively.

Table 1 Asymptotic critical values for nominal ξ-level S_0 and S_1 tests, and asymptotic κ_ξ values for U.

It is also of interest, given lack of knowledge concerning the order of integration, to examine the size properties of S_1 when $c > 0$, and also S_0 under both $c = 0$ and $c > 0$. These results are presented in Table 2, again obtained via direct simulation of the limit distributions in Lemma 1. The S_1 test becomes increasingly under-sized as c increases. Therefore, employing critical values which are appropriate for $c = 0$ will result in an under-sized test when $c > 0$. Also, notice from part (i)-(b) of Lemma 1 that S_1 converges in probability to zero in $I(0)$ case, and thus it is automatically under-sized under $I(0)$ errors.

Of particular interest is the behavior of S_0 in the (local to) $I(1)$ case. For all significance levels, we see that the asymptotic size of S_0 remains well below the nominal size across all c. Therefore, employing critical values which are appropriate for $I(0)$ case will result in an under-sized test in the (local to) $I(1)$ case.

We now turn to consideration of a feasible test that can be applied in the absence of knowledge concerning the order of integration. Our approach deliberately exploits the under-sizing phenomenon seen in the S_0 test in the (local to) $I(1)$ world, and is based on the *union of rejections* approach advocated by Harvey et al. (2009a) in a unit root testing context. Specifically, we consider the union of rejections decision rule

$$U : \text{Reject } H_0 \quad \text{if } \{S_1 > \kappa_\xi cv_\xi^1 \text{ or } S_0 > \kappa_\xi cv_\xi^0\}$$

where cv_ξ^1 and cv_ξ^0 denote the ξ significance level asymptotic critical values of S_1 under $I(1)$ $(c = 0)$ errors and S_0 under $I(0)$ errors, respectively, and κ_ξ is a positive scaling constant whose role is made precise below.

	S_1	S_0	U
Panel A,$\xi = 0.10$			
$I(1), c = 0$	0.1000	0.0331	0.100
$I(1), c = 10$	0.0054	0.0215	0.025
$I(1), c = 20$	0.0000	0.0174	0.017
$I(1), c = 40$	0.0000	0.0141	0.014
$I(0)$	0.0000	0.1000	0.088
Panel B,$\xi = 0.05$			
$I(1), c = 0$	0.050	0.009	0.050
$I(1), c = 10$	0.001	0.006	0.006
$I(1), c = 20$	0.000	0.004	0.004
$I(1), c = 40$	0.000	0.003	0.003
$I(0)$	0.000	0.050	0.047
Panel C,$\xi = 0.01$			
$I(1), c = 0$	0.01	0.0007	0.0100
$I(1), c = 10$	0.00	0.0004	0.0004
$I(1), c = 20$	0.00	0.0001	0.0001
$I(1), c = 40$	0.00	0.0001	0.0001
$I(0)$	0.00	0.0100	0.0096

Note: The rejections for S_1 are computed using critical values for S_1 under $I(1), c = 0$ errors; the rejections for S_0 are computed using critical values for S_0 under $I(0)$ errors.

Table 2 Asymptotic sizes of nominal ξ-level tests under $I(1)$ and $I(0)$ errors.

If the U decision rule was to be applied with $\kappa_\xi = 1$ (i.e. without any adjustment to the asymptotic critical values used for the constituent tests in U), then the testing strategy would be asymptotically correctly sized under $I(0)$ errors, as $S_1 \xrightarrow{p} 0$. In the $I(1)$ case, the Bonferroni inequality along with the size results for S_1 and S_0 reported in Table 2, show that such a strategy could only ever be (modestly) asymptotically over-sized when c=0; indeed, the maximum possible asymptotic sizes at the $0.10, 0.05$ and 0.01 nominal significance levels are, respectively, 0.133, 0.059 and 0.0107 ,such that the size distortions will be small. However, to ensure that U is an asymptotically conservative testing strategy (i.e. asymptotically exactly correctly sized in the case of $I(0)$ errors and $I(1)$ errors when $c = 0$, and always asymptotically under-sized elsewhere), we can avoid any size distortions by suitably choosing κ_ξ.

Noting that the maximum size of U is realized when $c = 0$, choosing κ_ξ such that U has an asymptotic size of ξ in this case ensures that the procedure will be conservative. We therefore obtain κ_ξ by simulating the limit distribution of $\max\{S_1, (cv_\xi^1/cv_\xi^0)S_0\}$, calculating the ξ-level critical value for this distribution, say cv_ξ^{max}, and then computing $\kappa_\xi = cv_\xi^{max}/cv_\xi^1$. Values of κ_ξ for different ξ are shown in Table 1. Hereafter, reference to the decision rule U assumes the κ_ξ adjustment values from Table 1 are used.

Table 2 also provides asymptotic size results for U. As expected, the testing strategy is correctly sized for $I(1)$ errors when $c = 0$. When the errors are $I(1)$ with $c > 0$, U is conservative, in line with the size properties of the constituent tests S_1 and S_0 discussed above. It is only slightly conservative when the errors are $I(0)$.

Remark 4. It is important to note that the union of rejections procedure is only rendered viable due to specific behavior of the Berk-type estimator $\hat{\omega}_u^2$ under $I(1)$ errors, in that it diverges at a rate T^2; see Theorem 4(i). This ensures that S_0 is $O_p(1)$. If a typical kernel-based (e.g. Bartlett) long run variance estimator with bandwidth l, say, growing at rate smaller than T was used, then under $I(1)$ errors, it is easy to show that $\hat{\omega}_u^2$ diverges at a rate less than T^2, so that S_0 diverges to ∞. In such a case, a union of rejections approach is clearly precluded, because, regardless of the choice of κ_ξ, its size would approach one in the limit under $I(1)$ errors.

3.3.3 Asymptotic Power

In this section, we first investigate the local asymptotic power properties of our feasible test. In this case, we model the break magnitude as the appropriate Pitman drifts under $I(0)$ and $I(1)$ errors respectively. The local asymptotic power results demonstrate that the conservativeness of our test under $I(1)$ errors with $c > 0$ does not necessarily imply a loss of power. Next, we show that our feasible test is consistent under the fixed break magnitude by establishing the divergence rates of the constituent tests.

Consider first the case where the break magnitude, γ, is modeled as a local alternative. In this case, we may partition H_1 into two scaled components: $H_{1,0} : \gamma = \omega_u T^{-3/2}\vartheta$ when u_t is $I(0)$, and $H_{1,1} : \gamma = \omega_\varepsilon T^{-1/2}\vartheta$ when u_t is $I(1)$, where in each case ϑ is a finite non-negative constant. As we shall see below, these provide the appropriate Pitman drifts on γ under $I(0)$ and $I(1)$ errors, respectively, to obtain nondegenerate and pivotal (except ϑ) asymptotic distributions under the alternative hypothesis.

In the following theorem we establish the large sample behavior of the S_1 and S_0 statistics of (15) and (16), respectively, under the local alternative in both $I(1)$ and $I(0)$ environments.

Theorem 5. Let the time series process $\{y_t\}$ be generated according to (4) and (5), and let Assumption 1 hold.

(i) If u_t is $I(0)$, then under $H_{1,0} : \gamma = \omega_u T^{-3/2}\vartheta$

(a) $S_0 \xrightarrow{d} \sup_{\tau\in\Lambda} |L_{00}(\tau, \vartheta)|$,

where

$$L_{00}(\tau, \vartheta) = \vartheta\{\int_0^1 RT(r, \tau)^2 dr\}^{1/2} + \frac{\int_0^1 RT(r, \tau)dW(r)}{\{\int_0^1 RT(r, \tau)^2 dr\}^{1/2}}$$

(b) $S_1 = O_p \left\{(l/T)^{(1/2)}\right\}$.

(ii) If u_t is $I(1)$, then under $H_{1,1} : \gamma = \omega_\varepsilon T^{-1/2}\vartheta$

(a) $S_0 \xrightarrow{d} \dfrac{\sup_{\tau\in\Lambda} |K_{01}(\tau, \vartheta, c)|}{\Phi^{1/2}(c)}$,

where

$$K_{01}(\tau, \vartheta, c) = \vartheta\{\int_0^1 RT(r, \tau)^2 dr\}^{1/2} + \frac{\int_0^1 RT(r, \tau)W_c(r)dr}{\{\int_0^1 RT(r, \tau)^2 dr\}^{1/2}}$$

(b) $S_1 \xrightarrow{d} \sup_{\tau\in\Lambda} |L_{11}(\tau, \vartheta, c)|$,

where

$$L_{11}(\tau, \vartheta, c) = \vartheta\{\int_0^1 RU(r, \tau)^2 dr\}^{1/2} + \frac{\int_0^1 RU(r, \tau)dW_c(r)}{\{\int_0^1 RU(r, \tau)^2 dr\}^{1/2}}$$

Table 3 shows asymptotic local powers of S_1,S_0 and U, conducted at the nominal 0.05-level. We consider the same settings as in Table 2, and the same error specifications (i.e. $I(1)$ errors with $c \geq 0$ and $I(0)$ errors). The break magnitudes are benchmarked so that the powers of S_1 for $c = 0$ in the $I(1)$ case, and S_0 in the $I(0)$ case, are equal to 0.50.

Consider first the behavior of S_1. For $I(1)$ errors, power is monotonically decreasing as c increases. Note also that for $I(0)$ errors, the power of S_1 is zero, in line with the results of Theorem 5(i)(b). The behavior of S_0 is more interesting: the power of S_0 increases significantly as the errors move from $I(0)$ to $I(1)$. The S_0 test is quite powerful, indeed power is close to 1, under the $I(1)$ environment regardless of the value of c.

	S_1	S_0	U
$I(1), c = 0$	0.500	0.987	0.997
$I(1), c = 10$	0.342	0.897	0.941
$I(1), c = 20$	0.140	0.898	0.933
$I(1), c = 40$	0.011	0.921	0.922
$I(0)$	0	0.500	0.500

Table 3 Asymptotic powers of nominal 0.05-level tests under $I(1)$ and $I(0)$ errors.

Inspection of the power performance of U shows that the conservative nature of U for $I(1)$ errors with $c > 0$, as seen in Table 2, does not translate into poor power performance for these cases. The U test essentially capitalizes on the relatively high power of the constituent tests. U generally displays power very close to the better power of the two individual tests S_1 and S_0. On the other hand, there are instances where the power of U exceeds that of either of the constituent tests S_1 and S_0, resulting from the fact that the rejections from S_1 and S_0 need not be perfectly correlated. The robust power performance of U, therefore makes a strong case for using the modified union of rejections approach in practice.

We now consider the asymptotic behavior of our feasible test under fixed alternatives in order to establish the consistency of our tests. The following theorem establishes the consistency of our tests under a fixed alternative of the form $H_1 : \gamma \neq 0$.

Theorem 6. Let the time series process $\{y_t\}$ be generated according to (4) and (5) under $H_1 : \gamma \neq 0$, and let Assumption 1 hold.

(i) If u_t is $I(0)$, then (*a*) $S_0 = O_p((T/k)^{1/2})$, and (*b*) $S_1 = O_p((lT)^{1/2})$, thus (*c*) $U = O_p((lT)^{1/2})$.

(ii) If u_t is $I(1)$, then (*a*) $S_0 = O_p(T^{1/2})$, and (*b*) $S_1 = O_p(T^{1/2})$, thus (*c*) $U = O_p(T^{1/2})$.

Under $H_1 : \gamma \neq 0$, it is seen from Theorem 6 that our feasible test, U, is consistent at rate $O_p((lT)^{1/2})$ when u_t is $I(0)$ and at rate $O_p(T^{1/2})$ when u_t is $I(1)$.

4 Disjoint Broken Trend Model

Although trend breaks are the central concern of this paper, we might also consider extending our analysis to allow (but not test for) a break in level occurring at the same time as the break in trend. Therefore, we consider the following model:

$$y_t = \alpha + \beta t + \delta DU_t(\tau^*) + \gamma DT_t(\tau^*) + u_t, \qquad t = 1, ..., T \tag{17}$$

and

$$u_t = \rho u_{t-1} + \varepsilon_t, \quad t = 2, ..., T, \quad u_1 = \varepsilon_1. \tag{18}$$

In what follows we will refer to (17) and (18) together as the disjoint broken trend model. Notice that δ and γ capture the change, respectively, in the level and slope coefficients of the series at time $T^* = \lfloor \tau^* T \rfloor$. The slope coefficient changes from β to $\beta+\gamma$ and the level shifts from α to $\alpha+\delta$ at time T^*. The trend function is discontinuous at the break date T^* if $\delta \neq 0$.

The first-differenced form of the model is given by:

$$\Delta y_t = \beta + \delta D_t(\tau^*) + \gamma DU_t(\tau^*) + \Delta u_t, \qquad t = 2, ..., T, \tag{19}$$

where $D_t(\tau^*) = 1(t = T^* + 1)$. The null hypothesis of interest continues to be $H_0 : \gamma = 0$ against the two sided alternative: $H_1 : \gamma \neq 0$. The interest lies only on the break in the trend slope.

We propose our feasible test following the same steps as in Section 3. We first consider the appropriate test statistics for $I(0)$ and $I(1)$ cases, respectively, assuming that the true break fraction, τ^*, is known. We consequently redefine $\hat{t}_0(\tau^*)$ as follows:

$$\hat{t}_0(\tau^*) = \frac{\hat{\gamma}(\tau^*)}{\sqrt{\hat{\omega}_u^2(\tau^*) \left[\left\{ \sum_{t=1}^{T} x_{DT,t}(\tau^*) x_{DT,t}(\tau^*)' \right\}^{-1} \right]_{44}}}, \tag{20}$$

$$\hat{\gamma}(\tau^*) = \left[\left\{ \sum_{t=1}^{T} x_{DT,t}(\tau^*) x_{DT,t}(\tau^*)' \right\}^{-1} \sum_{t=1}^{T} x_{DT,t}(\tau^*) y_t \right]_4 ,$$

with $x_{DT,t}(\tau^*) = \{1, t, DU_t(\tau^*), DT_t(\tau^*)\}'$ and $\hat{\omega}_u^2(\tau^*)$ calculated as in (14) under the null $H_0 : \gamma = 0$ using the Berk-type autoregressive spectral density estimator. The residuals used in the estimation of the Berk-type long run variance are the OLS residuals $\hat{u}_t(\tau^*) = y_t - \hat{\alpha} - \hat{\beta}t - \hat{\delta}DU_t(\tau^*)$. Similarly, $\hat{t}_1(\tau^*)$ is redefined to be

$$\hat{t}_1(\tau^*) = \frac{\tilde{\gamma}(\tau^*)}{\sqrt{\hat{\omega}_\varepsilon^2(\tau^*) \left[\left\{ \sum_{t=2}^{T} x_{DU,t}(\tau^*) x_{DU,t}(\tau^*)' \right\}^{-1} \right]_{33}}}, \tag{21}$$

$$\tilde{\gamma}(\tau^*) = \left[\left\{ \sum_{t=2}^{T} x_{DU,t}(\tau^*) x_{DU,t}(\tau^*)' \right\}^{-1} \sum_{t=2}^{T} x_{DU,t}(\tau^*) \Delta y_t \right]_3 ,$$

with $x_{DU,t}(\tau^*) = \{1, D_t(\tau^*), DU_t(\tau^*)\}'$, and $\hat{\omega}_\varepsilon^2(\tau^*)$ calculated nonparametrically as in (12) with bandwidth parameter $l = O(T^{1/4})$ but using the OLS residuals $\hat{\varepsilon}_t(\tau^*) = \Delta y_t - \tilde{\beta} - \tilde{\delta} D_t(\tau^*) - \tilde{\gamma}(\tau^*) DU_t(\tau^*)$.

In order to accommodate unknown break point, we again consider statistics based on the maximum of the sequences of statistics $\{|\hat{t}_0(\tau)|, \tau \in \Lambda\}$ and $\{|\hat{t}_1(\tau)|, \tau \in \Lambda\}$. These statistics are given by:

$$S_0 = \sup_{\tau \in \Lambda} |\hat{t}_0(\tau)| \tag{22}$$

$$S_1 = \sup_{\tau \in \Lambda} |\hat{t}_1(\tau)| \tag{23}$$

As in SV, the null hypothesis H_0 must be restated as $H_0 : \gamma = \delta = 0$ in the case of unknown break fraction in order to obtain a pivotal limiting null distribution for our test statistic. The following theorem, whose proof is a straightforward generalization of those in Section 3 and is therefore omitted, details the large sample behavior of the redefined S_0 and S_1 statistics under H_0.

Theorem 7. Let the time series process $\{y_t\}$ be generated according to (17) and (18) under $H_0 : \gamma = \delta = 0$, and let Assumption 1 hold.

(i) If u_t is $I(0)$, then (a) $S_0 \xrightarrow{d} \sup_{\tau\in\Lambda} |J_{U,0}(\tau)|$, (b) $S_1 = O_p\left\{(l/T)^{1/2}\right\}$. *where*

$$J_{U,0}(\tau) = \frac{\int_0^1 RT_U(r,\tau^*)dW(r)}{\{\int_0^1 RT_U(r,\tau^*)^2 dr\}^{1/2}}.$$

(ii) If u_t is $I(1)$, then (a) $S_0 \xrightarrow{d} \sup_{\tau\in\Lambda} \left|\frac{K_{U,0}(\tau,c)}{\Phi^{1/2}(c,\tau)}\right|$, (b) $S_1 \xrightarrow{d} \sup_{\tau\in\Lambda} |J_1(\tau,c)|$, *where*

$$K_{U,0}(\tau,c) = \frac{\int_0^1 RT_U(r,\tau)W_c(r)dr}{\{\int_0^1 RT_U(r,\tau)^2 dr\}^{1/2}}, \quad \textit{and}$$

$$\Phi(c,\tau) = \frac{\left\{\int_0^1 Q(r,c,\tau)^2 dr\right\}^2}{\left\{\int_0^1 Q(r,c,\tau)^2 dW_c(r)\right\}^2}$$

and $Q(r,c,\tau)$ is a continuous-time residual from the projection of $W_c(r)$ onto the space spanned by $\{1, r, 1(r > \tau)\}$, and $RT_U(r,\tau)$ is a continuous-time residual from the projection of $(r-\tau)1(r > \tau)$ onto the space spanned by $\{1, r, 1(r > \tau)\}$.

Remark 5. Observe from the result given in part $(ii)(b)$ of Theorem 7 that the limiting distribution of S_1 from the disjoint broken trend model is identical to that for joint broken trend model given in Lemma 1 $(ii)(b)$. This is because the regressor $D_t(\tau)$ has an asymptotically negligible effect on S_1.

Critical Values			
	S_1	S_0	κ_ξ
$\xi = 0.10$	2.741	2.901	1.0288
$\xi = 0.05$	3.024	3.163	1.0114
$\xi = 0.01$	3.565	3.655	1.00

Note: The critical values for S_0 and S_1 are for the $I(0)$ and $I(1)(c = 0)$ cases, respectively.

Table 4 Asymptotic critical values for nominal ξ-level S_0 and S_1 tests, and asymptotic κ_ξ values for U.

Asymptotic null critical values for S_1 under $I(1)$ errors with $c = 0$, and S_0 under $I(0)$ errors, are reported in Table 4, for the settings $\tau_L = 0.1$ and $\tau_U = 0.9$, and for the significance levels $\xi = 0.10, 0.05$ and 0.01. Table 5 presents, similar to Table 2, the asymptotic size properties of S_1 when $c > 0$, and also S_0 under both $c = 0$ and $c > 0$. From Table 5 we see that the S_1 test becomes increasingly under-sized as c increases. Similarly, the asymptotic size of S_0 remains well below the nominal size across all c. This under-sizing phenomenon seen in the S_0 test under the (local to)$I(1)$ errors and in the S_1 test when $c > 0$ or errors are $I(0)$ renders the *union of rejections* principle viable. Specifically, our feasible test which is based on the union of rejections decision rule can be stated as follows:

U : Reject H_0 if $\{S_1 > \kappa_\xi cv_\xi^1 \text{ or } S_0 > \kappa_\xi cv_\xi^0\}$

where cv_ξ^1 and cv_ξ^0 denote the ξ significance level asymptotic critical values of S_1 under $I(1)$ $(c = 0)$ errors and S_0 under $I(0)$ errors, respectively, and κ_ξ is as explained in Section 3.3.2.

	S_1	S_0	U
Panel A,$\xi = 0.10$			
$I(1), c = 0$	0.1000	0.0281	0.100
$I(1), c = 10$	0.0054	0.0037	0.008
$I(1), c = 20$	0.0000	0.0033	0.003
$I(1), c = 40$	0.0000	0.0022	0.002
$I(0)$	0.0000	0.1000	0.082
Panel B,$\xi = 0.05$			
$I(1), c = 0$	0.050	0.0083	0.050
$I(1), c = 10$	0.001	0.0014	0.002
$I(1), c = 20$	0.000	0.0013	0.001
$I(1), c = 40$	0.000	0.0011	0.001
$I(0)$	0.000	0.0500	0.044
Panel C,$\xi = 0.01$			
$I(1), c = 0$	0.01	0.0005	0.010
$I(1), c = 10$	0.00	0.0001	0.000
$I(1), c = 20$	0.00	0.0001	0.000
$I(1), c = 40$	0.00	0.0001	0.000
$I(0)$	0.00	0.0100	0.010

Note: The rejections for S_1 are computed using critical values for S_1 under $I(1), c = 0$ errors; the rejections for S_0 are computed using critical values for S_0 under $I(0)$ errors.

Table 5 Asymptotic sizes of nominal ξ-level tests under $I(1)$ and $I(0)$ errors.

The asymptotic size results for our feasible test, U, are also presented in Table 5. As expected, the testing strategy is correctly sized for $I(1)$ errors when $c = 0$. When the errors are $I(1)$ with $c > 0$, U is conservative, in line with the size properties of the constituent tests S_1 and S_0 discussed above. It is also almost correctly sized when the errors are $I(0)$.

We now turn to consistency properties of our feasible test. We consider fixed alternatives of the form $H_1 : \gamma \neq 0$, with δ now unrestricted. In the following theorem, we show that our feasible test is consistent under a fixed alternative by establishing the divergence rates of our feasible test under both $I(0)$ and $I(1)$ errors.

Theorem 8. Let the time series process $\{y_t\}$ be generated according to (17) and (18) under $H_1 : \gamma \neq 0$, and let Assumption 1 hold.

(i) If u_t is $I(0)$, then (a) $S_0 = O_p((T/k)^{1/2})$, and (b) $S_1 = O_p((lT)^{1/2})$, thus (c) $U = O_p((lT)^{1/2})$.

(ii) If u_t is $I(1)$, then (a) $S_0 = O_p(T^{1/2})$, and (b) $S_1 = O_p(T^{1/2})$, thus (c) $U = O_p(T^{1/2})$.

The proof of theorem is very similar to that of Theorem 6 and, hence, is omitted. Under H_1, when u_t is $I(0)$, we obtain that U is consistent at rate $O_p((lT)^{1/2})$, while if u_t is $I(1)$, U is consistent at rate $O_p(T^{1/2})$.

5 Finite Sample Results

In this section we report the finite sample size and power performances of our structural break tests. We employ 10% trimming $(\tau_L = 0.1, \tau_U = 0.9)$ throughout our Monte Carlo simulations. All the results reported in this section are for two-sided tests conducted at the 0.05 nominal asymptotic significance level, and were computed over 5,000 replications.

We employ the following data generating process (DGP), which is a simplified version of (17) and (18), to carry out our simulations

$$y_t = \delta DU_t(\tau^*) + \gamma DT_t(\tau^*) + u_t, \qquad t = 1, ..., T \tag{24}$$

$$u_t = \rho u_{t-1} + \varepsilon_t, \quad t = 2, ..., T, \quad u_1 = \varepsilon_1. \tag{25}$$

where $\varepsilon_t \sim NIID(0, 1)$. In Section 5.1, we provide the finite sample size properties of our proposed tests and compare the results to those of the tests proposed in HLT. In Section 5.2, we subsequently investigate the finite sample power properties of our test, again relative to the tests of HLT. We form our union of rejections decision rule as detailed in Section 3.3 for the joint broken trend model, and as detailed in Section 4 for the disjoint case.

Here we briefly outline the tests proposed by HLT before we report the finite sample results. As in this paper, HLT also utilize the two test statistics S_0 and S_1, the supremum of the t-ratios for the levels and the differenced data, respectively. The first difference is that HLT normalize the t-statistics for the levels data using a nonparametric estimator of the long run variance ω_u^2, with bandwidth parameter $l = O(T^{1/4})$. The S_0 statistic, thus, diverges to infinity under $I(1)$ errors with this choice of the long run variance estimator. Second, instead of using a union of rejections principle, which is infeasible due to their choice of the long run variance estimator, they take a weighted average of the S_0 and S_1 statistics.They adopt the stationarity test statistics of KPSS, Q_0 and Q_1, calculated from the residuals of the levels data $\{\hat{u}_t(\tau^*)\}_{t=1}^T$ and the differenced data $\{\hat{\varepsilon}_t(\tau^*)\}_{t=2}^T$ respectively to form their weight function, $\lambda(.,.)$, of the form

$$\lambda(Q_0(\tau^*), Q_1(\tau^*)) = exp[-\{gQ_0(\tau^*)Q_1(\tau^*)\}^2], \tag{26}$$

where

$$Q_0(\tau^*) = \frac{\sum_{t=1}^T \left(\sum_{i=1}^t \hat{u}_i(\tau^*)\right)^2}{T^2\hat{\omega}_u^2(\tau^*)}, \qquad Q_1(\tau^*) = \frac{\sum_{t=2}^T \left(\sum_{i=2}^t \hat{\varepsilon}_i(\tau^*)\right)^2}{(T-1)^2\hat{\omega}_\varepsilon^2(\tau^*)}$$

and g is a positive constant. Finally, the constant m_ξ is chosen such that for a given significance level, ξ, the critical values for the test coincide under $I(0)$ and $I(1)$ errors. The robust structural break test statistic of HLT, t_λ, can then be written as

$$t_\lambda = \{\lambda(Q_0(\tau^*), Q_1(\tau^*)) \times S_0\} + m_\xi \{[1 - \lambda(Q_0(\tau^*), Q_1(\tau^*))] \times S_1\}. \tag{27}$$

HLT show that under $I(0)$ errors $t_\lambda = S_0 + o_p(1)$ and under $I(1)$ errors $t_\lambda = m_\xi S_1 + o_p(1)$.

5.1 Size Properties

Table 6 reports the empirical size performance of our U test and t_λ test of HLT for each of joint and disjoint broken trend models. These we obtained by setting $\delta = \gamma = 0$ in (24). The AR parameter in (25) were varied over $\rho = 1 - (c/T)$ for $c \in \{0, 10, 20, 50, T\}$. We consider three different sample sizes; $T = 100$, $T = 150$, and $T = 300$. In the computation of our U test statistic, the long run variance estimator $\hat{\omega}_u^2$ uses values of k (in (14)) determined according to the BIC criterion with $k_{max} = \lfloor 4(T/100)^{1/4} \rfloor$. We set the bandwidth parameter $l = \lfloor 4(T/100)^{1/4} \rfloor$ in the computation of all other long variance estimators. Finally, we set $g = 250$ in (25), since it gives rise to the most accurate weights for both $I(1)$ and $I(0)$ cases.

Joint	T=100		T=150		T=300	
	U	t_λ	U	t_λ	U	t_λ
c=0	9.0%	21.3%	8.6%	16.2%	6.7%	10.2%
c=10	2.8%	15.4%	2.6%	13.0%	2.0%	8.8%
c=20	2.6%	14.9%	2.5%	16.3%	1.7%	14.5%
c=50	3.6%	10.7%	3.4%	16.7%	3.0%	20.8%
c=T	5.2%	5.4%	5.0%	5.7%	5.1%	6.1%
Disjoint	T=100		T=150		T=300	
	U	t_λ	U	t_λ	U	t_λ
c=0	8.4%	28.1%	7.8%	22.9%	5.7%	15.4%
c=10	2.2%	19.9%	2.1%	18.0%	1.2%	13.9%
c=20	2.3%	17.6%	2.0%	20.9%	1.2%	22.1%
c=50	3.2%	11.7%	2.4%	18.2%	1.3%	24.8%
c=T	4.6%	6.6%	3.8%	6.2%	4.4%	5.3%

Table 6 Empirical sizes of nominal 0.05 level tests

In the case of $I(1)$ errors ($c = 0$), Table 6 shows that our U test is slightly oversized in finite samples. However, we see that the size distortions get smaller as the sample size increases. On the other hand, when the errors are $I(0)$ ($c > 0$) the U test seems to be undersized, except for the white noise case ($c = T$). The undersizing effect is more pronounced as the persistence and the sample size increases. Also, the undersizing phenomenon is more obvious in the disjoint broken trend model. However, keep in mind that all these finite sample results are consistent with the asymptotic properties of the U test given in Tables 2 and 5.

The finite sample size properties of the t_λ test can also be seen in Table 6. The t_λ test is oversized for persistent data regardless of the sample size and the model. The size distortion can be as high as 25% at a 5% nominal level. The main source of the size distortions is the way the t_λ test is constructed. In finite samples, the weight function does not pick the appropriate statistic quite accurately[3] and since the S_0 test statistic is large (asymptotically unbounded) for persistent data, the t_λ test rejects too often.

A simple comparison of the results from Table 6 favors our U test over the t_λ test of HLT in terms of the finite sample size. In the next section, we compare the finite sample power properties of these two tests.

[3]For highly persistent data, the appropriate test statistic is S_1

5.2 Power Properties

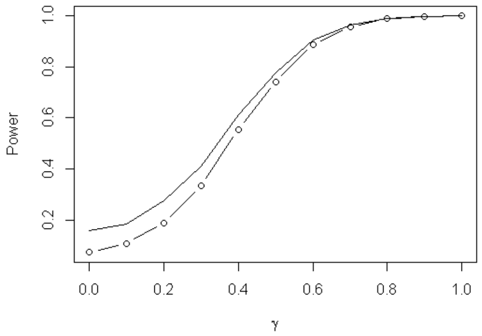
Figures 1-2 and 3 present the empirical power performance of the tests. The data were generated according to (24) and (25) for a grid of γ values, covering the range $[0, 1]$ in steps of 0.1. In order to save some space, we only present results for the joint broken trend model, where we set $\delta = 0$. The results for the disjoint broken trend model are qualitatively similar and are available upon request. We consider two break fractions $\tau^* \in \{0.25, 0.5\}$ and again let the AR parameter vary over $\rho = 1-(c/T)$ for $c \in \{0, 10, 20, 50, T\}$ in (25). The parameters values used in the estimation of the long run variances and weight function are the same as in the size simulations.

Consider first the left panels of Figures 1-2 and 3. The power of t_λ is slightly higher than U for small values of γ and the two tests are equivalent in terms of power for medium to large values of γ. However, these power advantages of t_λ over U for small γ can be attributed to the size distortions of t_λ. Because the t_λ tests are oversized (see Table 6), we also report size-adjusted powers. The size-adjusted power curves are given on the right panels of Figures 1-2 and 3. The main conclusion drawn from these results is that any power gain of t_λ over U is only due to the oversizing property of t_λ. This is not surprising; since both U and t_λ utilize the same statistics (S_0 and S_1) in their construction, we don't expect one test to be superior in extracting more information from the data regarding a potential break.

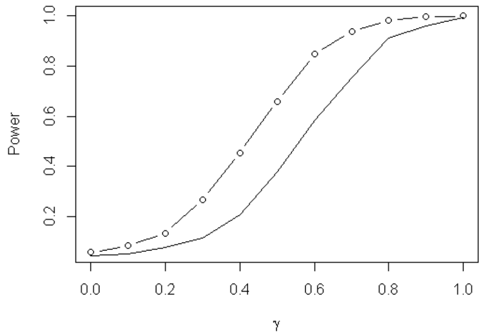
The power functions for U do not appear to depend to any noticeable degree on the location of the break. Comparing results between $\tau^* = 0.5$ and $\tau^* = 0.25$ we see that there is no power loss as the break is located away from the middle of the sample. This is also supported by an unreported simulation where we set $\tau^* = 0.75$. Consequently, our U test is as powerful for early and late breaks as for the mid-point breaks.

6 Conclusion

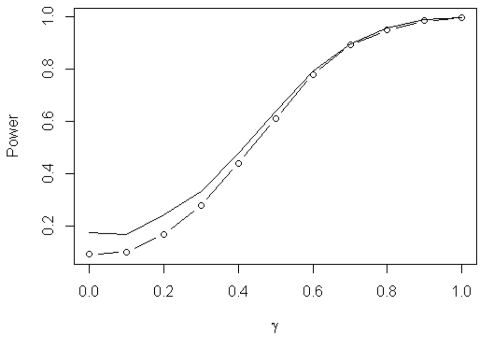
In this paper we present tests for the presence of a structural break in the trend slope of a univariate time series which do not require knowledge of the form of serial correlation in the data and are valid regardless of the errors being $I(0)$ or $I(1)$. We consider two models: a joint and a disjoint broken trend model. We propose a union of rejections based procedure using two statistics; one appropriate for stationary errors and the other for integrated errors. We provide representations for and critical values from the asymptotic distributions of our proposed statistics (also for the union of rejections approach) under the null hypothesis of no trend break, together with



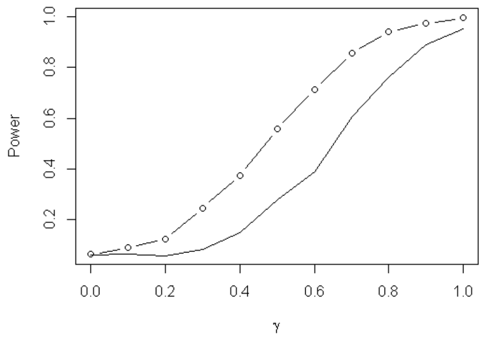
(a) $c = 0, \tau^* = 0.5$



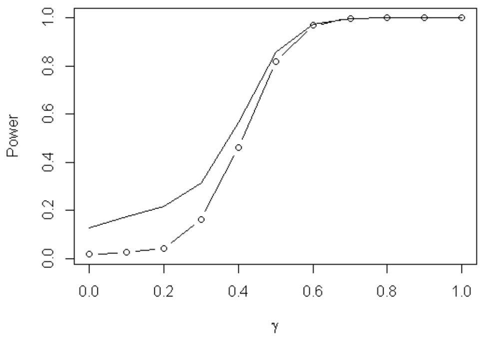
(b) $c = 0, \tau^* = 0.5$, size-adjusted



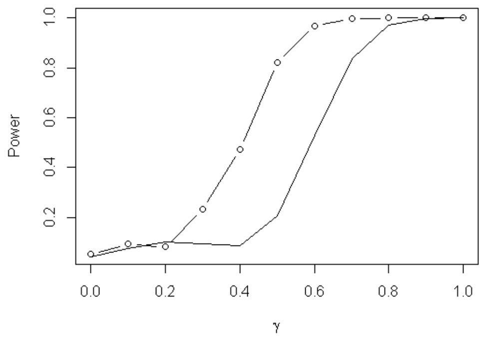
(c) $c = 0, \tau^* = 0.25$



(d) $c = 0, \tau^* = 0.25$, size adjusted

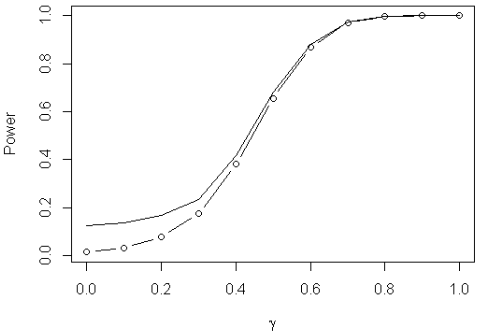


(e) $c = 10, \tau^* = 0.5$

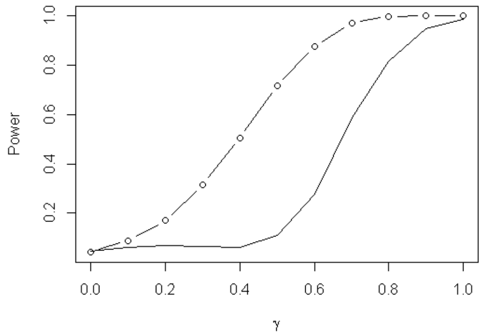


(f) $c = 10, \tau^* = 0.5$, size-adjusted

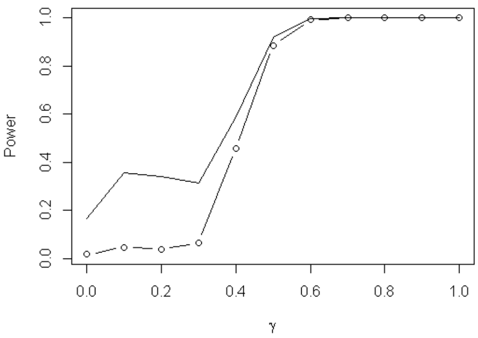
Figure 1: Power: Joint Broken Trend Model, $T = 150$, U:—o—, t_λ:—



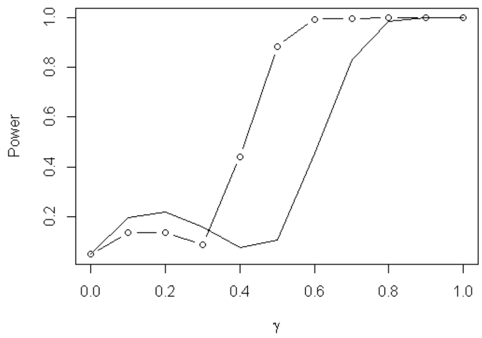
(a) $c = 10,\tau^* = 0.25$



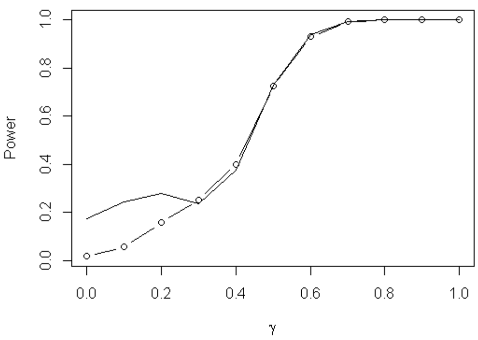
(b) $c = 10,\tau^* = 0.25$, size-adjusted



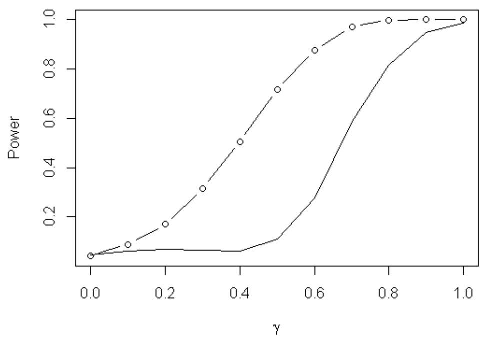
(c) $c = 20,\tau^* = 0.5$



(d) $c = 20,\tau^* = 0.5$, size adjusted

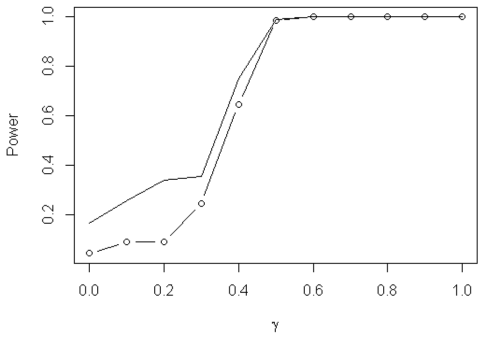


(e) $c = 20,\tau^* = 0.25$

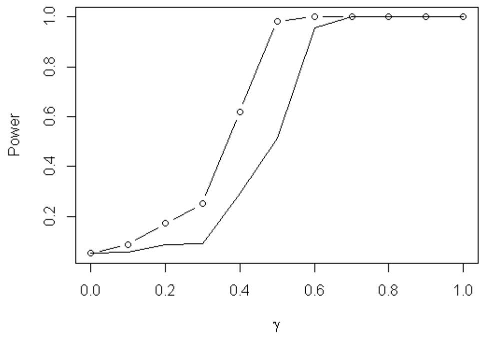


(f) $c = 20,\tau^* = 0.25$, size-adjusted

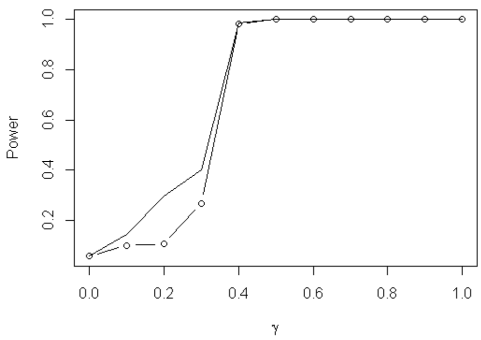
Figure 2: Power: Joint Broken Trend Model, $T = 150$, U:—o—, t_λ:—



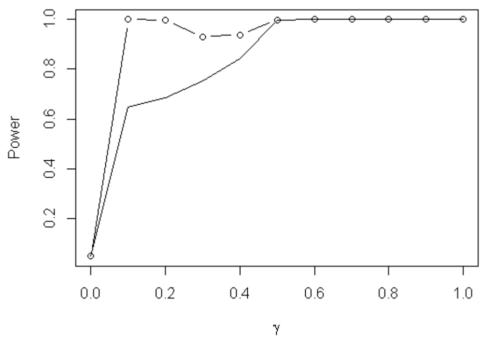
(a) $c = 50, \tau^* = 0.5$



(b) $c = 50, \tau^* = 0.5$, size-adjusted



(c) $c = T, \tau^* = 0.5$



(d) $c = T, \tau^* = 0.25$

Figure 3: Power: Joint Broken Trend Model, $T = 150$, U:—o—, t_λ:—

representations for and numerical evaluation of their asymptotic local power functions under both $I(1)$ and $I(0)$ environments. We also provide finite sample results using Monte Carlo simulations and these simulations demonstrate that our proposed method performs well in small samples, regardless of the (unknown) order of integration of the data.

It is also straightforward to extend our method to test for multiple trend breaks or to sequentially test for l versus $l+1$ breaks in the trend function. However, the main focus of this paper is on single trend break cases. Single trend break tests can be quite useful in policy analysis. For example, our robust tests can be employed to evaluate the impact of a one time policy change or a new regulation on a trending variable, without the need to know whether the data is generated by a stationary or integrated process. Overall, we believe that the robust tests we propose in this paper should prove useful in practical applications, particularly when a possible trend break in a macroeconomic or financial data is an important consideration itself, and where there is uncertainty regarding the order of integration of the data.

Appendix

In what follows, due to invariance of the statistics concerned, we can set $\alpha = \beta = 0$ without loss of generality

Proof of Theorem 1.

(i)(a) Using the Frisch-Waugh-Lovell Theorem (FWLT) we can write $t_0(\tau^*)$ in the form

$$t_0(\tau^*) = \left\{ \frac{T^{-3/2} \sum RT_t(\tau^*) u_t}{T^{-3} \sum RT_t(\tau^*)^2} \right\} \times \frac{1}{\sqrt{\omega_u^2 / T^{-3} \sum RT_t(\tau^*)^2}},$$

where $RT_t(\tau^*)$, $t = 1, ..., T$, are the OLS residuals from the regression of $DT_t(\tau^*)$ onto 1 and t. We establish the following weak convergence results using the standard time-series properties,

$$t_0(\tau^*) \xrightarrow{d} \left\{ \frac{\omega_u \int_0^1 RT(r, \tau^*) dW(r)}{\int_0^1 RT(r, \tau^*)^2 dr} \right\} \times \frac{1}{\sqrt{\omega_u^2 / \int_0^1 RT(r, \tau^*)^2 dr}},$$

where $W(r)$ is a standard Brownian motion, defined via, $\omega_u^{-1} T^{-1/2} \sum_{t=1}^{\lfloor Tr \rfloor} u_t \xrightarrow{d} W(r)$.

(b) Again appealing to the FWLT, $t_1(\tau^*)$ can be expressed as

$$T^{1/2} t_1(\tau^*) = \left\{ \frac{\sum RU_t(\tau^*) \Delta u_t}{T^{-1} \sum RU_t(\tau^*)^2} \right\} \times \frac{1}{\sqrt{\omega_\varepsilon^2 / T^{-1} \sum RU_t(\tau^*)^2}}$$

where $RU_t(\tau^*)$,$t = 1, ..., T$, are the OLS residuals from the regression of $DU_t(\tau^*)$ onto 1. $RU_t(\tau^*)$ can also be written as

$$RU_t(\tau^*) = \begin{cases} \tau^* - 1, & t \leq T^* \\ \tau^* & t > T^* \end{cases}$$

which implies that $\sum RU_t(\tau^*) \Delta u_t = \tau^* u_T + u_{T^*} + (1 - \tau^*) u_1 = O_p(1)$, since u_t is $I(0)$. Entirely standard results, all other terms on the right hand side are also $O_p(1)$. As a result,$T^{1/2} t_1(\tau^*) = O_p(1)$, which establishes the result in (b).

(ii) (a) We have that

$$T^{-1} t_0(\tau^*) = \left\{ \frac{T^{-5/2} \sum RT_t(\tau^*) u_t}{T^{-3} \sum RT_t(\tau^*)^2} \right\} \times \frac{1}{\sqrt{\omega_u^2 / T^{-3} \sum RT_t(\tau^*)^2}},$$

Consequently, since all the stochastic terms appearing above are of $O_p(1)$ with nondegenerate limiting distributions, it follows that $T^{-1} t_0(\tau^*) = O_p(1)$.

Turning to the result in (b), observe that

$$t_1(\tau^*) = \left\{ \frac{T^{-1/2}\sum RU_t(\tau^*)\Delta u_t}{T^{-1}\sum RU_t(\tau^*)^2} \right\} \times \frac{1}{\sqrt{\omega_\varepsilon^2 / T^{-1}\sum RU_t(\tau^*)^2}}$$
$$\xrightarrow{d} \left\{ \frac{\omega_\varepsilon \int_0^1 RU(r,\tau^*)dW_c(r)}{\int_0^1 RU(r,\tau^*)^2 dr} \right\} \times \frac{1}{\sqrt{\omega_\varepsilon^2 / \int_0^1 RU(r,\tau^*)^2 dr}},$$

using standard results.Rearranging delivers the stated result in (b).

Proof of Theorem 2. The proof of (i)(a) and (ii)(b) follows by applying the Continuous Mapping Theorem as in Zivot and Andrews (1992) to show the convergence in distribution from the space $D[0,1]$ to the space $C(0,1)$. t_0^* and t_1^* statistics can be written as continuous functionals of the processes $\left(T^{-3}\sum RT_t(.)^2, T^{-3/2}\sum RT_t(.)u_t\right)'$ and $\left(T^{-1}\sum RU_t(.)^2, T^{-1/2}\sum RU_t(.)\Delta u_t\right)'$, respectively. Using similar arguments from Zivot and Andrews (1992) we can show the joint weak convergence of these processes:

$$\left(T^{-3}\sum RT_t(.)^2, T^{-3/2}\sum RT_t(.)u_t\right)' \xrightarrow{d} \left(\int_0^1 RT(r,.)^2 dr, \int_0^1 RT(r,.)dW(r)\right)'$$

$$\left(T^{-1}\sum RU_t(.)^2, T^{-1/2}\sum RU_t(.)\Delta u_t\right)' \xrightarrow{d} \left(\int_0^1 RU(r,.)^2 dr, \int_0^1 RU(r,.)dW_c(r)\right)'$$

The stated results in Theorem 2 (i)(a) and (ii)(b) then follow directly from Theorem 1 (i)(a) and (ii)(b),respectively, using applications of the CMT, noting the continuity of the sup function. The result in Theorem 2 (i)(b) follows from Theorem 1 (i)(b) and the result that $t_1(\tau) = O_p(T^{-1/2})$ uniformly in τ. Finally, for the result in Theorem 2 (ii)(a) we appeal to Theorem 1 (ii)(a) and the fact that $t_0(\tau) = O_p(T)$ uniformly in τ.

Proof of Theorem 3. For the proofs of part (i) and (ii) we use the fact that $\hat{\varepsilon}_t$ is asymptotically identical to Δu_t and that $\hat{\tau}$ is consistent for τ^*.

(i) $\hat{\omega}_\varepsilon^2(\hat{\tau}) \xrightarrow{p} lim_{T\to\infty} T^{-1} E\left(\sum_{t=2}^T \Delta u_t\right)^2 = \omega_\varepsilon^2$, by the standard results for the nonparametric kernel estimators.

(ii) $\hat{\omega}_\varepsilon^2(\hat{\tau}) \xrightarrow{p} lim_{T\to\infty} T^{-1} E\left(\sum_{t=2}^T \Delta u_t\right)^2 = 0$, since Δu_t is over-differenced when u_t is $I(0)$. However, it follows from Leybourne, Taylor, and Kim (2007) that $l\hat{\omega}_\varepsilon^2 \xrightarrow{p} -2\sum_{s=1}^\infty s\psi_s'$ where $\psi_s' = E(\Delta u_t \Delta u_{t-s})$. Consequently, $l\hat{\omega}_\varepsilon^2 = O_p(1)$, which establishes the result in (ii).

Proof of Theorem 4.

(i) For $r \in \Lambda$, straightforward extensions of results in Perron and Vogelsang (1992) yield the result that $T^{-1/2}\hat{u}_{\lfloor rT \rfloor} \xrightarrow{d} \omega_\varepsilon Q(r,c)$ where $Q(r,c)$ is a continuous time residual from the projection of $W_c(r)$ onto the space spanned by $\{1, r\}$. Then, from (14) we obtain, using the CMT, that

$$T\hat{\pi} \xrightarrow{d} \frac{\sigma_\eta}{\omega_\varepsilon} \frac{\int_0^1 Q(r,c) dW_c(r)}{\int_0^1 Q(r,c)^2 dr}$$

and, since $\hat{\sigma}^2 \xrightarrow{p} \sigma_\eta^2 = 1$, we therefore obtain that

$$T^{-2}\hat{\omega}_u^2 = \frac{\hat{\sigma}^2}{(T\hat{\pi})^2} \xrightarrow{d} \omega_\varepsilon^2 \frac{\left\{\int_0^1 Q(r,c)^2 dr\right\}^2}{\left\{\int_0^1 Q(r,c)^2 dW_c(r)\right\}^2} = \omega_\varepsilon^2 \Phi(c)$$

as required.

(ii)Under the null, $\hat{u}_t$ is asymptotically identical to u_t, hence $\hat{\omega}_u^2$ behaves asymptotically as if calculated directly from u_t, and therefore $\hat{\omega}_u^2 \xrightarrow{p} \omega_u^2$.

Proof of Lemma 1.

(i)(a) The proof of Lemma 1 (i)(a) follows from the results in Theorem 1 (i)(a), Theorem 2 (i)(a) and Theorem 4 (ii) .

(b) The proof of Lemma 1 (i)(b) is a trivial extension to the results in Theorem 1 (i)(b), Theorem 2 (i)(b) and Theorem 3 (ii) combined with the result that $l\hat{\omega}_\varepsilon^2(\tau) \xrightarrow{p} -2\sum_{s=1}^{\infty} s\psi_s'$ uniformly in τ.

(ii)In order to establish the result in (a), notice first that

$$\hat{t}_0(\tau) = \left\{\frac{T^{-5/2}\sum RT_t(\tau)u_t}{T^{-3}\sum RT_t(\tau)^2}\right\} \times \frac{1}{\sqrt{T^{-2}\hat{\omega}_u^2 / T^{-3}\sum RT_t(\tau)^2}},$$

From standard weak convergence results, namely, the CMT and the Functional Limit Theorem (FCLT), we can establish that:

$$\hat{t}_0(\tau) \xrightarrow{d} \left\{\frac{\omega_\varepsilon \int_0^1 RT(r,\tau) W_c(r) dr}{\int_0^1 RT(r,\tau)^2 dr}\right\} \times \frac{1}{\sqrt{\omega_\varepsilon^2 \Phi(c) / \int_0^1 RT(r,\tau)^2 dr}}$$

Then the rest follows from the arguments given in Theorem 2.

(b)The proof of Lemma 1 (ii)(b) follows from the results in Theorem 1 (ii)(b), Theorem 2 (ii)(b) and Theorem 3 (i).

Proof of Theorem 5. In order to establish the results in Theorem 5, notice first that under $H_{1,0}$ and $H_{1,1}$ the level breaks are of order $o_p(1)$, and hence they have no asymptotic effect on $\hat{\omega}_u^2$. Then, the proof of Theorem 5 follows from trivial extensions to the results in Theorem 1, Theorem 2 and Lemma 1. The proof is therefore omitted in the interest of brevity.

Proof of Theorem 6. We omit the constant and trend regressors from (4) and the constant regressor from (7), for technical expediency, since our focus is only on establishing the orders in probability of the statistics under H_1. These particular regressors have no effect on any of the orders involved, but just introduce algebraic complexities.

(i) To establish the result in part (a), we follow the same steps outlined in Harvey et al.(2009b). We first derive the order of $\hat{t}_0(\tau^*)$ under the fixed alternative $H_1 : \gamma \neq 0$. Notice first that

$$(T/k)^{-1/2}\hat{t}_0(\tau^*) = \left\{\gamma + \frac{\sum DT_t(\tau^*)u_t}{\sum DT_t(\tau^*)^2}\right\} \times \frac{1}{\sqrt{T^{-2}k^{-1}\hat{\omega}_u^2/T^{-3}\sum DT_t(\tau^*)^2}},$$
$$= \{\gamma + o_p(1)\}O_p(1) = O_p(1)$$

which follows from Lee et.al (1997) that $\hat{\omega}_u^2 = O_p(T^2k)$ (since the long run variance is estimated ignoring the existing break) and that $T^{-3}\sum DT_t(\tau^*)^2 \to (1-\tau^*)^3/3$. Now, it is straightforward to establish that for any $\tau \in \Lambda$, we may write

$$(T/k)^{-1/2}|\hat{t}_0(\tau)| = \sqrt{\left(\frac{T^{-3}\sum y_t^2}{\hat{\sigma}^2(\tau)} - T^{-2}\right) \times \frac{\hat{\sigma}^2(\tau)}{T^{-2}k^{-1}\hat{\omega}_u^2}},$$

where $\hat{\sigma}^2(\tau) = T^{-1}\sum_{t=1}^{T}\hat{u}_t(\tau)^2$ and $\hat{u}_t(\tau) = y_t - \hat{\alpha} - \hat{\beta}t - \hat{\gamma}(\tau)DT_t(\tau)$. From above we see that the stated result will hold if $\hat{\sigma}^2(\hat{\tau}) - \hat{\sigma}^2(\tau^*)$ is asymptotically negligible. Following the results in Harvey et al. (2009b) we now demonstrate that $\hat{\sigma}^2(\hat{\tau}) - \hat{\sigma}^2(\tau^*) = O_p(T^{-1})$. It is straightforward to show that

$$\hat{\sigma}^2(\hat{\tau}) - \hat{\sigma}^2(\tau^*) = -T^{-1}\left[\frac{(\sum DT_t(\hat{\tau})y_t)^2}{\sum DT_t(\hat{\tau})^2} - \frac{(\sum DT_t(\tau^*)y_t)^2}{\sum DT_t(\tau^*)^2}\right],$$

from which it is easy to demonstrate that the dominant term of the right hand side of above equality is of the form

$$-\gamma^2 T^{-1}\left[\frac{(\sum DT_t(\hat{\tau})DT_t(\tau^*))^2}{\sum DT_t(\hat{\tau})^2} - \sum DT_t(\tau^*)^2\right].$$

After some lengthy manipulations, the dominant term of above expression can be shown to be given by

$$-\gamma^2\frac{(\hat{d}T)^2}{36}(\hat{\tau}-1)^3(4\tau^* - \hat{\tau} - 3),$$

where $\hat{d} = \tau^* - \hat{\tau}$. From Theorem 3 of Perron and Zhu (2005, p.75) we have that $\hat{d} = O_p(T^{-3/2})$ since our break fraction estimator $\hat{\tau}$ can be shown to have the same rate of consistency as the minimum sum of squares break fraction estimator of Perron and Zhu(2005). As a result, the dominant term above is $O_p(T^{-1})$, thus $\hat{\sigma}^2(\hat{\tau}) - \hat{\sigma}^2(\tau^*) = O_p(T^{-1})$. Consequently, $T^{-3/2}(|\hat{t}_0(\hat{\tau})| - |\hat{t}_0(\tau^*)|) \xrightarrow{p} 0$, which establishes the result in (a).

The proof of the result in (b) directly follows from the Proof of Theorem 3 (i)(b) in Harvey et al. (2009b).

(ii)(a) We again establish first the behavior of $\hat{t}_0(\tau^*)$ under H_1. Observe first that

$$T^{-1/2}\hat{t}_0(\tau^*) = \left\{\gamma + \frac{\sum DT_t(\tau^*)u_t}{\sum DT_t(\tau^*)^2}\right\} \times \frac{1}{\sqrt{T^{-2}\hat{\omega}_u^2/T^{-3}\sum DT_t(\tau^*)^2}},$$
$$= \{\gamma + o_p(1)\}O_p(1) = O_p(1)$$

which again follows from Lee et.al (1997) that $T^{-2}\hat{\omega}_u^2 = O_p(1)$. Again, for any $\tau \in \Lambda$, we may write

$$T^{-1/2}|\hat{t}_0(\tau)| = \sqrt{\left(\frac{T^{-3}\sum y_t^2}{T^{-1}\hat{\sigma}^2(\tau)} - T^{-1}\right) \times \frac{T^{-1}\hat{\sigma}^2(\tau)}{T^{-2}\hat{\omega}_u^2}},$$

so again we need to establish the behavior of the difference between the OLS variance estimators evaluated at τ^* and $\hat{\tau}$. Using the results from part (i)(a), we obtain that the dominant term of $T^{-1}(\hat{\sigma}^2(\hat{\tau}) - \hat{\sigma}^2(\tau^*))$ is given by

$$-\gamma^2\frac{\hat{d}^2T}{36}(\hat{\tau}-1)^3(4\tau^* - \hat{\tau} - 3),$$

Next, again utilizing the results from Theorem 3 of Perron and Zhu (2005), we may show that $\hat{d} = O_p(T^{-1/2})$ and, hence, we obtain that $T^{-1}(\hat{\sigma}^2(\hat{\tau}) - \hat{\sigma}^2(\tau^*)) = O_p(1)$. So it follows that $T^{-1/2}(|\hat{t}_0(\hat{\tau})| - |\hat{t}_0(\tau^*)|) = O_p(1)$, establishing (a).

The proof of the result in (b) again, directly follows from the Proof of Theorem 3 (ii)(b) in Harvey et al. (2009b).

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