Probabilistic stable rules and Nash Equilibrium in two-sided matching problems*

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Abstract

We study *many-to-one* matching problems when each firm has *substitutable* and *separable* preferences. We analyze the *stochastic dominance (sd) Nash equilibria* of the game induced by any *probabilistic stable* matching rule. We show that a *unique* match is obtained as the outcome of each sd-Nash equilibrium. Furthermore, *individual rationality* with respect to the *true* preferences is a necessary and sufficient condition for an equilibrium outcome. Each *stable* match for the *true* preferences is achieved as the outcome of an equilibrium in which firms behave *truthfully*. Conversely, the outcome of each such equilibrium is *stable* for the *true* preferences. Finally, we study equilibrium behavior in *many-to-many* matching problems under the same domain of preferences.

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1 Introduction

Centralized job matching procedures have received much attention in two-sided matching literature since they were introduced to address market failures (such as uncontrolled unravelling of appointment dates, and chaotic recontracting)[1]. In centralized matching, each agent submits to the clearinghouse a preference order over agents on the other side, and the clearinghouse then uses an algorithm to produce a match. These procedures are typically deterministic: matches are produced in a way that involve no uncertainty. Therefore the results do not usually reflect the case in real life situations such as labor markets where lotteries often determine outcomes. *Randomization* is the most common device to ensure procedural fairness in an environment where agents have conflicting interests. Hence, it is reasonable to allow for randomization in studies of centralized matching for purposes of achieving equity. Another motivation for studying randomization is that lotteries may be considered to represent the frictions of a decentralized matching process. Decentralized decision making often leads to uncertain outcomes in complex environments. The speed of the mail, the telephone network or the internal structure of firms determine how agents communicate. Therefore, the final match depends on the realization of random events.

We study *probabilistic matching rules* that may be used in centralized matching to achieve *procedural fairness*. When such a rule is used, agents are faced with a game in which they report a preference order to a clearinghouse which then produces a match. These rules may also appear in decentralized decision making. The game that agents face is as follows: starting from an arbitrary match, at each moment in time a pair of agents from the two sides meets at random. They are matched if this is consistent with their strategies. It may not be in an agent's best interest to behave truthfully. This implies that agents may not report their true preferences in centralized markets and may not act according to their true preferences in decentralized ones. Indeed, no stable matching rule makes it a dominant strategy for all agents to state their true preferences (Roth,1985).

Another way of stating this result is that no stable matching rule is strategy-proof. This simply says that there is room for agents to benefit from misrepresenting their true preferences when confronted with a game induced by a stable matching rule. Indeed, if all agents but one behaves truthfully, the last agent may gain from this manipulation. What we are really concerned about is not the fact that agents benefit from individually

[1] See Roth and Sotomayor (1990) for an excellent survey on two sided matching problems.

circumventing a stable rule, but rather that stable matches that the rule recommends for the true preferences may not be achieved. Therefore, it is crucial to study the equilibria of the game induced by a stable matching rule. Also, the study of incentives proved to be useful for understanding behavior in matching processes with deterministic rules. Therefore, the study of incentives facing agents is a good starting point to understand behavior in matching processes with probabilistic rules[2].

In a related paper, Roth and Vande Vate (1990) prove that starting from an arbitrary match in the marriage problem, the process of allowing randomly selected blocking pairs to create a new match leads to a stable match with probability one. It is argued that since many two-sided matching processes are not centralized and yet are not determined to encounter failures, it is reasonable to think that these markets reach stable outcomes through decentralized decision making. It can also be argued that the process of allowing randomly selected blocking pairs is a good approximation to dynamics in decentralized processes. Roth and Vande Vate (1991) study a one-period game defined by the mentioned process and show that all stable matches can be supported as equilibria in a class of undominated strategies namely *truncations*. A truncation strategy for an agent is a preference ordering that has the same order as in her true preference but may have fewer acceptable elements. However, they show that some unstable matches can arise as equilibrium outcomes in this game. They then introduce a multi-period extension of this game and show that all subgame perfect equilibrium outcomes are stable matches.

We study many-to-one matching problems such that each worker has a preference order over firms and each firm has a maximum of workers it can hire (its capacity) and a preference order over subsets of workers. We assume that each firm's preferences satisfy the following assumptions: *substitutability* and *separability*. A firm's preferences over subsets of workers are substitutable if, once a worker is chosen from a given group of workers, she is also chosen from any subset of the given group of workers. A firm's preferences over subsets of workers are separable if adding an acceptable worker to a given set of workers that does not fill the capacity makes it a preferred set and if adding an unacceptable worker makes it worse.

We analyze the equilibria of the game induced by any probabilistic stable matching rule. The equilibrium concept we study relies on *first-order stochastic dominance*. Given that preferences are ordinal, *stochastic dominance (sd) Nash equilibrium* guar-

[2]See, for example, Roth (1984a) and Roth (1991) for a study of incentives respectively in American and UK hospital-intern markets.

antees that each agent plays her best response to the others' strategies for each utility function that is compatible with the ordinal preferences. We show that a *unique* match is achieved as the outcome of an sd-Nash equilibrium of the game induced by any probabilistic stable rule. Furthermore, a match can be obtained as an equilibrium outcome if and only if it is individually-rational for the true preferences. A match is supported as the outcome of an sd-Nash equilibrium in which firms behave truthfully if and only if it is stable for the true preferences. An implication of the result is that workers can stimulate any jointly achievable outcome in the game induced by any probabilistic stable rule, while firms act truthfully.

In a related paper, Pais (2008a) proves aforementioned equilibrium results in many-to-one matching problems when each firm's preferences satisfy the following condition: *responsiveness*. A firm's preferences over subsets of workers are responsive (to its preferences over individual workers) if for any two subsets that differ in only one worker, the firm prefers the one that contains the preferred worker. Responsiveness is subsumed by substitutability and separability. Strategic issues have been the subject of papers that focus on deterministic matching rules in many-to-one matching problems. Ma (2002) assumes that firms have responsive preferences and studies a refinement of Nash equilibrium based on a class of strategies called *truncations at the match point*. A strategy in truncations at the match point for an agent is a preference ordering that is consistent with her true preferences up to her current match and that rank as unacceptable all the agents that are less preferred than her current match. He shows that a match can arise as the outcome of a strong Nash equilibrium in truncations at the match point if and only if it is stable for the true preferences.

We next turn our attention to the equilibria of the game induced by any probabilistic stable rule in the many-to-many matching problem under the same domain of preferences. In many-to-one matching problems when each firm has substitutable preferences, pairwise-stability coincides with group-stability and corewise-stability and hence is called stability (Roth and Sotomayor, 1990). Nevertheless, this result no longer holds in many-to-many matching problems when each agent has substitutable preferences. Furthermore, while the set of pairwise-stable matches is always non empty, the core and the set of group-stable matches may be empty. Therefore, we study pairwise-stability in these more general problems. We show that a unique match is achieved as the outcome of an sd-Nash equilibrium. Individual-rationality remains a neccessary and sufficient condition for a match to be an equilibrium outcome. We show that each pairwise-stable match is still obtained as the outcome of an sd-Nash equilibrium in which firms behave truthfully. Nevertheless, we establish that there are equilibrium misrepresentations that generate a match that is not pairwise-stable for the true preferences.

The paper is organized as follows. We present the model in Section 2. We impose preference restrictions in Section 3. We introduce the class of probabilistic stable matching rules and the equilibrium concept in Section 4. We conduct equilibrium analysis in Section 5. We finally analyze the equilibria in the many-to-many matching problem in Section 6. We conclude in Section 7.

2 The Model

Let $F = \{f_1, ..., f_n\}$ and $W = \{w_1, ..., w_m\}$ denote finite sets of **firms** and **workers** respectively. Generic elements of F and W are denoted by f and w respectively while a generic element of $F \cup W$ is denoted by v. "It" refers to a firm and "she" refers to a worker. Each worker can work for at most one firm and each firm f has a **capacity** c_f, i.e, each firm f has at most c_f positions to fill. Let $c \equiv (c_f)_{f \in F}$ denote the list of capacities. Each worker $w \in W$ has a **strict** preference relation P_w over the set $F \cup \{w\}$. We write $P_w : f_1, f_2, w, f_3, ..., f_n$, for example, to indicate that w's first choice is being matched to f_1, her second choice is being matched to f_2 and she prefers remaining unmatched rather than being matched to any other firm. Each firm $f \in F$ has a **strict** preference relation over the set of all subsets of workers 2^W. We write $P_f : \{w_1, w_2\}, w_2, \emptyset, \{w_2, w_3\}, ..., w_3$, for example, to indicate that f's first choice is being matched to w_1 and w_2, her second choice is being matched to w_2 only, and it prefers remaining unmatched to being matched to any other subset of workers. **Preference profiles** are $(n + m)$ tuples of preference relations denoted by $P \equiv (P_{f_1}, ..., P_{f_n}, P_{w_1}, ..., P_{w_m})$. A **problem** is a pair (P, c). Let P_{-v} denote the profile $P_{(F \cup W) \setminus \{v\}}$. We sometimes write the profile of preferences P as (P_v, P_{-v}). Let $\mathcal{P}_v$ denote the set of all possible preference relations for agent v and $\mathcal{P} \equiv \prod_{v \in (F \cup W)} \mathcal{P}_v$ be the set of all possible preference profiles. Let R_v denote the **at least as desirable as** relation associated with P_v. For each $w \in W$ and each pair $v', v'' \in F \cup \{w\}$, $v' \; R_w \; v''$ means either $v' = v''$ or $v' \; P_w \; v''$. For each $f \in F$ and each $S, S' \subseteq W$, $S \; R_f \; S'$ means either $S = S'$ or $S \; P_f \; S'$. Let $w \in W$ and $v \in F \cup \{w\}$ be given. Let $U_v(P_w)$ denote the set of agents that w finds at least as desirable as v, i.e, $U_v(P_w) \equiv \{v' \in F \cup \{w\} : v' \; R_w \; v\}$. Let $f \in F$ and $S \subseteq W$ be given. Let $U_S(P_f)$ denote the set of all subsets of workers that f finds at least as desirable as S, i.e, $U_S(P_f) \equiv \{S' \subseteq W : S' \; R_f \; S\}$.

A **match** is a mapping μ from the set $F \cup W$ to the set $2^W \cup F \cup W$ satisfying the following conditions:

1. For each $w \in W$, either $\mu(w) \in F$ or $\mu(w) = w$;
2. For each $f \in F$, $\mu(f) \in 2^W$ and $|\mu(f)| \leq c_f$;
3. For each $(f, w) \in F \times W$, $\mu(w) = f$ if and only if $w \in \mu(f)$.

Let $\mathcal{M}$ denote the set of all matches. We say that agent v is **unmatched** at μ if $\mu(v) = v$ and **matched** otherwise. We say that f and w are matched at μ if $w \in \mu(f)$. A match is **one-to-one** if each agent is matched to at most one member on the other side. A matching problem with one-to-one matches is known as a **marriage problem**. Preferences over partners are extended to preferences over matches in the conventional way: an agent's preferences over matches parallel to her preferences over her own assignments at the matches. For example, agent v prefers μ to μ' if and only if $\mu(v)\ P_v\ \mu'(v)$.

Let $w \in W$ and $P_w \in \mathcal{P}_w$ be given. Firm f is **acceptable to $\boldsymbol{w}$** if she prefers to work for f rather than remaining unmatched. The set of acceptable firms for w is given by $A(P_w) = \{f \in F : f\ P_w\ w\}$. Let $f \in F$ and $P_f \in \mathcal{P}_f$ be given. Worker w is **acceptable to $\boldsymbol{f}$** if it prefers to employ w rather than remaining unmatched. The set of acceptable workers for f is given by $A(P_f) = \{w \in W : \{w\}\ P_f\ \emptyset\}$. Let $S \subseteq W$ be given.

Let $\boldsymbol{Ch(S, P_f)}$ denote firm f's chosen set in S, i.e, its most preferred subset of S according to its preference relation P_f. Since preferences are strict, then $Ch(S, P_f)$ is the unique subset S' of S that satisfies the following: for each $S'' \subseteq S$, $S'' \neq S'$, $S'\ P_f\ S''$. Match μ is **individually-rational for $\boldsymbol{P}$** if, for each $f \in F$, $Ch(\mu(f), P_f) = \mu(f)$ and for each $w \in W$, $\mu(w)\ R_w\ w$. Let $\boldsymbol{IR(P)}$ denote the set of all individually-rational matches for preference profile P.

Let $P \in \mathcal{P}$ be given. A pair $\boldsymbol{(f, w)}$ **blocks $\boldsymbol{\mu}$ at $\boldsymbol{P}$** if they are not matched to each other at μ but would prefer to be matched to each other, i.e, $w \notin \mu(f)$, $f\ P_w\ \mu(w)$ and $w \in Ch(\mu(f) \cup \{w\}, P_f)$. Match μ is **stable for $\boldsymbol{P}$** if it is individually-rational for P and is not blocked by any firm-worker pair. Let $\boldsymbol{S(P)}$ denote the set of stable matches for P. Firm f and worker w are **achievable at $\boldsymbol{P}$** for each other if they are matched at some match in $S(P)$. The set of stable matches may be empty if no restriction is imposed on firms' preferences (see Example 2.7 in Roth and Sotomayor, 1990). The class of "substitutable" preferences is the most general preference domain so far under which the existence of stable matches has been guaranteed.

3 Preference Restrictions

We now define several restrictions on firms' preferences.

Firm f's preference relation P_f is **substitutable** if for each set of workers S, S' with $S' \subseteq S$ and each $w \in S'$; if $w \in Ch(S, P_f)$ then $w \in Ch(S', P_f)$.

Substitutability requires that a firm is willing to continue to be matched to a worker even if some other workers become unavailable.

Firm f's preference relation P_f is **separable** if (i) for each set of workers S with $|S| < c_f$ and each $w \notin S$, $S \cup \{w\}\ P_w\ S$ if and only if $w\ P_w\ \emptyset$ and (ii) for each S with $|S| > c_f$, $\emptyset\ P_f\ S$.

Separability says that adding an acceptable worker to a set that does not complete the capacity makes it a preferred set and adding an unacceptable worker to such a set makes it worse. The following ordering (from Martinez et al. 2000) over 2^W, where $W = \{w_1, w_2, w_3, w_4\}$ and $c_f = 2$

$$\begin{aligned} P_f \quad : \quad & \{w_1, w_2\}, \{w_3, w_4\}, \{w_1, w_3\}, \{w_1, w_4\}, \{w_2, w_3\}, \\ & \{w_2, w_4\}, \{w_1\}, \{w_2\}, \{w_3\}, \{w_4\} \end{aligned}$$

illustrates that separability does not imply substitutability. Notice that P_f is separable but not substitutable: $w_1 \in Ch(W, P_f)$, but $w_1 \notin Ch(W \backslash \{w_2\}, P_f)$. We now define a well studied preference restriction in many-to-one matching problems.

Firm f's preference relation P_f over subsets of workers is **responsive** to its preference relation over individual workers if it is separable and for each $S \subseteq W$, each $w' \in S$ and each $w \notin S$, we have $(S \backslash \{w'\} \cup \{w\})\ P_v\ S$ if and only if $w\ P_v\ w'$.

Firm f's preferences over subsets of workers is responsive to its preferences over individual workers if its preferences over individual workers determine the ordering of sets of workers that differ only in one worker.

The following ordering (from Martinez et al. 2000) over 2^W, where $W = \{w_1, w_2, w_3\}$ and $c_f = 2$

$$P_f : \{w_1, w_3\}, \{w_1, w_2\}, \{w_2, w_3\}, \{w_1\}, \{w_2\}, \{w_3\}$$

illustrates that the set of responsive preferences is a proper subset of the set of substitutable and separable preferences. A **many-to-one** matching problem with responsive preferences is known as a **college admissions** problem. We study many-to-one matching problems with substitutable and separable preferences.

When each firm has substitutable preferences, the set of stable matches is always nonempty (Theorem 6.5 in Roth and Sotomayor 1990). The proof of the theorem uses the deferred acceptance algorithm with firms proposing. At the first step, each firm proposes to its most preferred set of workers and each worker tentatively "holds" her most preferred acceptable firm that has proposed to her, and rejects the others. At the following step, each firm proposes to its most preferred set of workers that includes the workers to whom it proposed at the previous step and who have not rejected it. Each worker tentatively holds her most preferred firm that has proposed to her and rejects the others. The algorithm ends when there are no rejections. Each worker is matched to the firm she holds at the final step and workers who have not received any proposals at the final step remain unmatched.

The stability of the match produced by the deferred acceptance algorithm with firms proposing lies in the observation that as firms' preferences are substitutable, no firm ever regrets having to offer employment at subsequent steps of the algorithm to workers who have not rejected its earlier offers. For a preference profile P, the algorithm with firms proposing produces a stable match that is optimal for the firms in the sense that all firms find it at least as desirable as any other stable match. It is denoted by $\boldsymbol{\mu^F(P)}$. The algorithm with workers proposing produces a stable match that is optimal for the workers in the corresponding sense. It is denoted by $\boldsymbol{\mu^W(P)}$. The optimal stable match for firms is the worst stable match for workers and vice versa (Roth and Sotomayor, 1990).

4 Probabilistic matching and equilibrium notions

In practice, matching is often not centralized. Instead, mathes are reached through decentralized procedures. Many two-sided matching markets do not adopt centralized procedures. Such procedures introduce randomness into what matches are achieved (the order according to which agents communicate depends on the speed of the mail, the internal structure of firms and the telephone network). One way to model decentralized decision making is to consider a random process that develops a sequence of matches such that each match in the sequence is derived from the previous one by satisfying a randomly selected blocking agent or a blocking pair (See Roth and Vande Vate (1990) for one-to-one matching problems and Kojima and Ünver (2008) for many-to-many matching problems). As deterministic rules inherently favor some agents over others, randomness can be introduced in centralized matching to achieve procedural fairness. This is felt most strongly in two-sided matching where the polarization of interests of agents on different sides is reflected in the structure of the set of stable matches. School choice, public housing and on campus housing in American universities are examples of allocation problems that have adopted probabilistic procedures.

We reproduce the definitions and notation in Pais (2008a). A **probabilistic** (matching) **rule** $\widetilde{\varphi}$ maps preference profiles to lotteries over the set of matches: $\mathcal{P}\xrightarrow{\widetilde{\varphi}}\Delta\mathcal{M}$. A probabilistic match $\widetilde{\varphi}[P]$ is the image of a preference profile P under a rule. We consider only **probabilistic stable rules** that yield a lottery whose support (abreviated as supp) is a subset of the set of stable matches for each preference profile P. Formally, for each $P \in \mathcal{P}$, supp$\widetilde{\varphi}[P] \in S(P)$. Let $\widetilde{\varphi}_v[P]$ denote the probability distribution induced by $\widetilde{\varphi}[P]$ over agent v's achievable partners. If the distribution $\widetilde{\varphi}[P]$ is degenerate, we abuse notation and denote by $\widetilde{\varphi}[P]$ the unique outcome. Similarly, if for some agent v, the distribution $\widetilde{\varphi}_v[P]$ is degenerate, we denote by $\widetilde{\varphi}_v[P]$, v's unique partner in the probabilistic stable match $\widetilde{\varphi}[P]$. A **deterministic rule** φ maps preference profiles to the set of matches: $\mathcal{P}\xrightarrow{\widetilde{\varphi}}\mathcal{M}$. We consider only deterministic stable matching rules that yield a unique stable match for each preference profile P. We let $\varphi^F(\varphi^W)$ denote the deterministic stable rule that recommends the firm (worker) optimal stable match $\mu^F(P)$ $(\mu^W(P))$ for each preference profile P. We denote v's assignment at the match $\varphi[P]$ by $\varphi_v[P]$.

We study the game induced by any probabilistic stable rule $\widetilde{\varphi}$ in which agents are called upon to state their preferences. No stable rule makes it a dominant strategy for all workers and firms to state their true preferences (Roth,1985). This implies that an agent may reveal a different order than her true preferences. To understand what outcomes will result when all agents behave in this way, we need to study the manipulation "game" associated with the rule. Consider the following game in which the strategy space for each agent v is the set of all preferences $\mathcal{P}_v$. Each agent announces a preference list $Q_v \in \mathcal{P}_v$ over agents on the other side and then a match is randomly selected among all matches that are stable for the stated preferences Q. Formally, the set of strategy profiles $\mathcal{P}$ and a probabilistic stable matching rule $\widetilde{\varphi}$ define a mechanism $(\mathcal{P}, \widetilde{\varphi})$. The mechanism together with the true preference profile defines the game $(\mathcal{P}, \widetilde{\varphi}, P)$. Similarly, $(\mathcal{P}, \varphi)$ is a deterministic stable mechanism that induces the game

$(\mathcal{P}, \varphi, P)$.

We need to introduce more notation to define our equilibrium concept. Let $\pi \in \Delta\mathcal{M}$ and $v \in F \cup W$. Let π_v denote the lottery induced by π over v's set of assignments, i.e, over $F \cup \{w\}$ if $v = w$, and over 2^W if $v = f$. Let $w \in W$ and $v \in F \cup \{w\}$. Let $\pi_w(U_v(P_w))$ denote the probability that w obtains a partner at least as desirable as v. Let $f \in F$ and $S \subseteq W$. Let $\pi_f(U_S(P_f))$ denote the probability that f obtains a set of partners that is at least as desirable as S. For each pair $\pi, \pi' \in \Delta\mathcal{M}$ and each $w \in W$, **π stochastically P_w-dominates π'**, denoted as $\pi \; P_w^{sd} \; \pi'$, if for each $v \in F \cup \{w\}$, $\pi_w(U_v(P_w)) \geq \pi'_w(U_v(P_w))$. For each pair $\pi, \pi' \in \Delta\mathcal{M}$ and each $f \in F$, **π stochastically P_f-dominates π'**, denoted as $\pi \; P_f^{sd} \; \pi'$, if for each $S \subseteq W$, $\pi_f(U_S(P_f)) \geq \pi'_f(U_S(P_f))$.

Next, we define what constitutes a best strategy for an agent. Let $Q \in \mathcal{P}$ and $v \in F \cup W$. Given Q_{-v}, we say that strategy Q_v **stochastically P_v-dominates** an alternative strategy Q'_v, if $\widetilde{\varphi}_v[Q_v, Q_{-v}] \; P_v^{sd} \; \widetilde{\varphi}_v[Q'_v, Q_{-v}]$. This means that no worker w can increase the probability of obtaining $v \in F \cup \{w\}$ or a higher ranked agent in her list P_w by using Q'_w rather than using Q_w and that no firm f can increase the probability of obtaining a set of partners S or a higher ranked set of partners in its list P_f by using Q'_f rather than using Q_f. The following equilibrium notion relies on the criterion of stochastic dominance.

The profile of strategies Q is an **stochastic-dominance (sd) Nash equilibrium** of the game $(\mathcal{P}, \widetilde{\varphi}, P)$ if for each $v \in F \cup W$, Q_v stochastically P_v-dominates each alternative strategy, given Q_{-v}[3].

The profile of strategies Q is an sd-Nash equilibrium of the game $(\mathcal{P}, \widetilde{\varphi}, P)$ if, once adopted by the agents, no agent finds unilateral deviation profitable for each utility function compatible with the ordinal preferences. It is one of our objectives to show the existence of an sd-Nash equilibrium.

5 Equilibrium Analysis:

We now analyze the sd-Nash equilibria of the game induced by any probabilistic stable rule. The following lemmas will be used repeatedly in the proofs of the propositions.

[3]This name is taken from Thomson (2011). It is referred to as ordinal Nash equilibrium in the literature.

Lemma 1 below shows that each partner of a firm at a stable match is acceptable to the firm.

Lemma 1 *Let $Q \in \mathcal{P}$ and $\mu \in S(Q)$. Then for each $f \in F$ and each $w \in \mu(f)$, $w\ Q_f\ \emptyset$.*

Proof. Let $f \in F$ and $w \in \mu(f)$. By stability of μ, $Ch(\mu(f), Q_f) = \mu(f)$. Thus, $w \in Ch(\mu(f), Q_f)$. By substitutability, $w \in Ch(\{w\}, Q_f)$. Hence, $w\ Q_f\ \emptyset$. ■

Let S be a subset of workers that does not fill firm f's capacity. Let each of its elements be acceptable to firm f. Lemma 2 says that f's chosen set in S is S itself.

Lemma 2 *Let $Q \in \mathcal{P}$, $f \in F$ and $S \subseteq W$ with $|S| \leq c_f$ be given. If $S \subseteq A(Q_f)$, then $Ch(S, Q_f) = S$.*

Proof. Assume, by contradiction that $Ch(S, Q_f) \neq S$. Then, $Ch(S, Q_f) \subsetneq S$ and there is $w \in S \backslash Ch(S, Q_f)$. Since $|S| \leq c_f$, then $|Ch(S, Q_f)| < c_f$. Since $(Ch(S, Q_f) \cup \{w\}) \subseteq S$, then by definition of the chosen set, $Ch(S, Q_f)\ Q_f\ (Ch(S, Q_f) \cup \{w\})$. By separability, $\emptyset\ Q_f\ w$, contradicting the definition of Q_f. ■

Let $Q \in \mathcal{P}$ and $\mu \in S(Q)$. Let $Q' \in \mathcal{P}$ be such that each agent announces her partner(s) at μ as the only acceptable partner(s) at Q'. Lemma 3 says that $\mu \in S(Q')$.

Lemma 3 *Let $Q \in \mathcal{P}, \mu \in S(Q)$ and $v \in F \cup W$. Let $Q' \in \mathcal{P}$ be such that $A(Q'_v) = \mu(v)$ and for each $v' \neq v$, $Q'_{v'} = Q_{v'}$. Then, $\mu \in S(Q')$.*

Proof. Let $Q \in \mathcal{P}, \mu \in S(Q)$ and $v \in F \cup W$.
Case 1: $v = w$.
Let $Q' \in \mathcal{P}$ be such that $A(Q'_w) = \{\mu(w)\}$ and for each $v' \in F \cup W \backslash \{w\}$, $Q'_{v'} = Q_{v'}$. We show that $\mu \in S(Q')$. By definition of Q', for each $\widehat{w} \in W$, $\mu(\widehat{w})$ is acceptable to $\widehat{w}$ at Q' and for each $\widehat{f} \in F$, $Ch(\mu(\widehat{f}), Q'_{\widehat{f}}) = \mu(\widehat{f})$. Hence, $\mu \in IR(Q')$. Suppose that the pair (f', w') blocks μ at Q', i.e,

$$w' \notin\ \mu(f'),\ f'Q'_{w'}\mu(w') \text{ and } w' \in Ch(\mu(f') \cup \{w'\}, Q'_{f'}). \tag{1}$$

As only $\mu(w)$ is acceptable to w at Q'_w, we have $w' \neq w$. Hence, $Q'_{w'} = Q_{w'}$. Then, (1) becomes $w' \notin \mu(f')$, $f'\ Q_{w'}\ \mu(w')$ and $w' \in Ch(\mu(f') \cup \{w'\}, Q_{f'})$, contradicting $\mu \in S(Q)$.
Case 2: $v = f$.
Let $Q' \in \mathcal{P}$ be such that $A(Q'_f) = \mu(f)$ and for each $v' \in F \backslash \{f\} \cup W$, $Q'_{v'} = Q_{v'}$. We

show that $\mu \in S(Q')$. By definition of Q', for each $\widehat{w} \in W$, $\mu(\widehat{w})$ is acceptable to $\widehat{w}$ at Q' and for each $\widehat{f} \in F \backslash \{f\}$, $Ch(\mu(\widehat{f}), Q'_{\widehat{f}}) = \mu(\widehat{f})$. As $A(Q'_f) = \mu(f)$ and $|\mu(f)| \leq c_f$, by Lemma 2, $Ch(\mu(f), Q'_f) = \mu(f)$. Hence, $\mu \in IR(Q')$. Suppose that the pair (f', w') blocks μ at Q', i.e,

$$w' \notin \mu(f'),\ f' Q'_{w'} \mu(w') \text{ and } w' \in Ch(\mu(f') \cup \{w'\}, Q'_{f'}). \tag{2}$$

We show that $f' \neq f$. Assume, by contradiction that $f' = f$. By substitutability, $w' \in Ch(\{w'\}, Q'_f)$. Thus, $w'\ Q'_f\ \emptyset$, contradicting the definition of Q'_f. Hence, $f' \neq f$. Then, (2) becomes $w' \notin \mu(f')$, $f'\ Q_{w'}\ \mu(w')$ and $w' \in Ch(\mu(f') \cup \{w'\}, Q_{f'})$, contradicting $\mu \in S(Q)$. ■

Proposition 1 states that one and only one stable match is achieved as the outcome of each sd-Nash equilibrium of $(\mathcal{P}, \widetilde{\varphi}, P)$.

Proposition 1 *Let Q be an sd-Nash equilibrium of the game $(\mathcal{P}, \widetilde{\varphi}, P)$. Then, a single match is obtained with probability one.*

Proof. Given that the set of matched workers is the same at each stable match (Martinez et al. 2000), the proof mimics that of Proposition 1 in Pais (2008a). ■

Propositions 2 and 3 establish that a match can be supported as an equilibrium outcome if and only if it is individually-rational for the true preferences. We provide a complete characterization of the sd-Nash equilibria of the game induced by any probabilistic stable rule.

Proposition 2 *Let $\mu \in IR(P)$ and let $\widetilde{\varphi}$ be a probabilistic stable rule. Then, there is an sd-Nash equilibrium Q of the game $(\mathcal{P}, \widetilde{\varphi}, P)$ that supports μ.*

Proof. Let $Q \in \mathcal{P}$ be such that for each $w \in W$, $A(Q_w) = \{\mu(w)\}$ and for each $f \in F$, let $A(Q_f) = \mu(f)$. We show that $\mu \in S(Q)$. Clearly, for each $w \in W$, $\mu(w)$ is acceptable to w at Q. Since for each $f \in F$, $|\mu(f)| \leq c_f$ and $A(Q_f) = \mu(f)$, then by Lemma 2, for each $f \in F$, $Ch(\mu(f), Q_f) = \mu(f)$. Since no worker finds any firm but her partner at μ acceptable at Q, then no worker can be part of a blocking pair. Hence, $\mu \in S(Q)$. The set of matched workers is the same at each stable match for Q (Martinez et al. 2000). As for each $w \in W$, $A(Q_w) = \{\mu(w)\}$, each worker is matched to her partner at μ at each stable match for Q. Hence, $S(Q) = \{\mu\}$ and μ is reached with probability one. The rest of the proof mimics the proof of Proposition 2 in Pais (2008a). ■

Proposition 3 *Let Q be an sd-Nash equilibrium of the game $(\mathcal{P}, \widetilde{\varphi}, P)$. Then, the unique equilibrium outcome $\widetilde{\varphi}[Q]$ is individually-rational for the true preferences.*

Proof. By Proposition 1, a unique match is obtained from each sd-Nash equilibrium of $(\mathcal{P}, \widetilde{\varphi}, P)$. Let $\mu \equiv \widetilde{\varphi}[Q]$. We prove that $\mu \in IR(P)$. Assume, by contradiction that there is $w \in W$ such that $w \; P_w \; \mu(w)$. Let Q'_w be an alternative strategy for w such that $A(Q'_w) = \emptyset$. Let $Q' \equiv (Q'_w, Q_{-w})$. By announcing each firm unacceptable, w is unmatched at each match in $S(Q')$. Hence, $1 = \widetilde{\varphi}[Q'](U_w(P_w)) > \widetilde{\varphi}[Q](U_w(P_w))$ and Q_w does not stochastically P_w-dominate Q'_w. Thus, Q is not an sd-Nash equilibrium of the game $(\mathcal{P}, \widetilde{\varphi}, P)$.

Suppose now that there is $f \in F$ such that $Ch(\mu(f), P_f) \neq \mu(f)$. Then there is a set of workers $S^G \subsetneq \mu(f)$ such that $S^G \; P_f \; \mu(f)$. Let Q'_f be an alternative strategy for f such that $A(Q'_f) = S^G$. We show that Q_f does not stochastically P_f-dominate Q'_f. Let $Q' \equiv (Q'_f, Q_{-f})$. Let $S^B \equiv \mu(f) \backslash S^G$. Let Q'^R be a strategy profile restricted to $F \cup W \backslash S^B$ such that for each $\widehat{w} \in W \backslash S^B$, $Q'^R_{\widehat{w}} = Q'_{\widehat{w}}$, and for each $\widehat{f} \in F$, $Q'^R_{\widehat{f}}$ is the same strategy as $Q'_{\widehat{f}}$ but restricted to subsets of $W \backslash S^B$, i.e, $Q'^R_{\widehat{f}} = Q'_{\widehat{f}} \mid_{W \backslash S^B}$. Let $(F, W \backslash S^B, Q'^R)$ be a restricted matching problem. Let μ'^R be the match for the set $F \cup W \backslash S^B$ such that $\mu'^R(f) = S^G$ and for each $\widehat{f} \neq f$, $\mu'^R(\widehat{f}) = \mu(\widehat{f})$, and for each $\widehat{w} \in W \backslash S^B$ $\mu'^R(\widehat{w}) = \mu(\widehat{w})$. We prove that $\mu'^R \in S(Q'^R)$.

For each $\widehat{w} \in W \backslash S^B$, $\mu(\widehat{w})$ is acceptable to $\widehat{w}$ at Q. Since $Q'^R_{\widehat{w}} = Q'_{\widehat{w}} = Q_{\widehat{w}}$ and $\mu'^R(\widehat{w}) = \mu(\widehat{w})$, then $\mu'^R(\widehat{w})$ is acceptable to $\widehat{w}$ at $Q'^R_{\widehat{w}}$. For each $\widehat{f} \neq f$, $Ch(\mu(\widehat{f}), Q_{\widehat{f}}) = \mu(\widehat{f})$. Since $Q'^R_{\widehat{f}} = Q'_{\widehat{f}} \mid_{W \backslash S^B} = Q_{\widehat{f}} \mid_{W \backslash S^B}$ and $\mu'^R(\widehat{f}) = \mu(\widehat{f})$, then $Ch(\mu'^R(\widehat{f}), Q'^R_{\widehat{f}}) = \mu'^R(\widehat{f})$. We now show that $Ch(\mu'^R(f), Q'^R_f) = \mu'^R(f)$. Since $|\mu(f)| \leq c_f$ and $S^B \neq \emptyset$, then $\left|S^G\right| < c_f$. As $A(Q'_f) = S^G$, by Lemma 2, $Ch(S^G, Q'_f) = S^G$. Recalling that $\mu'^R(f) = S^G$ and Q'^R_f is the same strategy as Q'_f but restricted to subsets of $W \backslash S^B$, we have $Ch(\mu'^R(f), Q'^R_f) = \mu'^R(f)$. It follows that $\mu'^R \in IR(Q'^R)$. Suppose now that the pair (f', w') blocks μ'^R at Q'^R, i.e,

$$w' \notin \mu'^R(f') \text{ but } f' Q'^R_{w'} \mu'^R(w') \text{ and } w' \in Ch(\mu'^R(f') \cup \{w'\}, Q'^R_{f'}). \tag{3}$$

We argue that $f' \neq f$. Assume, by contradiction that $f' = f$. Recalling again that $\mu'^R(f) = S^G$ and that Q'^R_f is the same strategy as Q'_f but restricted to subsets of $W \backslash S^B$, we have $w' \notin S^G$ and $w' \in Ch(S^G \cup \{w'\}, Q'_f)$. By substitutability, $w' \in Ch(\{w'\}, Q'_f)$. Thus, $w' \; Q'_f \; \emptyset$, contradicting the definition of Q'_f. Hence, $f' \neq f$. Hence, $Q'^R_{f'} = Q'_{f'} \mid_{W \backslash S^B} = Q_{f'} \mid_{W \backslash S^B} = Q^R_{f'}$. Also, $Q'^R_{w'} = Q'_{w'} = Q_{w'}$. By the definition of μ'^R, for each $\widehat{f} \neq f$, $\mu'^R(\widehat{f}) = \mu(\widehat{f})$, and for each $\widehat{w} \in W \backslash S^B$ $\mu'^R(\widehat{w}) = \mu(\widehat{w})$. Then (3) becomes

$w' \notin \mu(f')$ but $f' \; Q_{w'} \; \mu(w')$ and $w' \in Ch(\mu(f') \cup \{w'\}, Q_{f'})$, contradicting $\mu \in S(Q)$. Hence, $\mu'^R \in S(Q'^R)$. Since f is matched to S^G at a stable match for $(F, W \backslash S^B, Q'^R)$ and $|S^G| < c_f$, then by Proposition 2 in Martinez et al (2000), f is matched to S^G at each stable match for $(F, W \backslash S^B, Q'^R)$, in particular at the firm-optimal stable match for $(F, W \backslash S^B, Q'^R)$,

Suppose S^B join in. By Theorem 5 in Kelso and Crawford (1982), each firm is at least as well off as in the new firm-optimal stable match[4]. Let μ_F denote the firm-optimal stable match for (F, W, Q'). Since $A(Q'_f) = S^G$, then by Lemma 1, $\mu_F(f) \subseteq S^G$. Suppose $\mu_F(f) \subsetneq S^G$. By the definition of the firm-optimal stable match, $\mu_F(f) \; Q'_f \; S^G$. Then, $Ch(S^G, Q'_f) \neq S^G$. This implies that $Ch(S^G, Q'^R_f) \neq S^G$. Noting that $\mu'^R(f) = S^G$, the previous statement becomes $Ch(\mu'^R(f), Q'^R_f) \neq \mu'^R(f)$, contradicting $\mu'^R \in S(Q'^R)$. Hence, $\mu_F(f) = S^G$.

We now complete the proof. Note that $|S^G| < c_f$. By Proposition 2 in Martinez et al. (2000), f is matched to S^G at each stable match for (F, W, Q'). Therefore, by deviating and acting according to Q'_f, f gets S^G with probability one instead of $\mu(f)$. Thus, Q_f does not stochastically P_f-dominate Q'_f and hence, Q is not an sd-Nash equilibrium of $(\mathcal{P}, \widetilde{\varphi}, P)$. ■

We provide a sufficient condition for stability of the outcome of each sd-Nash equilibrium in the game induced by any probabilistic stable rule. To this end, we turn our attention to sd-Nash equilibria where firms behave truthfully. Each sd-Nash equilibrium where firms behave truthfully generates a stable match for the true preferences.

Proposition 4 *Let $Q \equiv (P_F, Q_W)$ be an sd-Nash equilibrium of the game $(\mathcal{P}, \widetilde{\varphi}, P)$. Then, the unique equilibrium outcome $\widetilde{\varphi}[Q]$ is stable for the true preferences.*

Proof. By Proposition 1, a unique match is obtained from each sd-Nash equilibrium of $(\mathcal{P}, \widetilde{\varphi}, P)$. Let $\mu \equiv \widetilde{\varphi}[Q]$. By Proposition 3, $\mu \in IR(P)$. We prove that $\mu \in S(P)$. Assume, by contradiction that $\mu \notin S(P)$. Suppose (f, w) blocks μ at P, i.e, $w \notin \mu(f)$, $f \; P_w \; \mu(w)$ and $w \in Ch(\mu(f) \cup \{w\}, P_f)$.

Let Q'_w be an alternative strategy for w such that for each $v, v' \in F \backslash \{f\} \cup W$, $f \; Q'_w \; v'$ and $v \; Q'_w \; v'$. Let $Q' \equiv (Q'_w, Q_{-w})$. If w is matched to f with positive probability at $\widetilde{\varphi}[Q']$, then Q_w does not stochastically P_w-dominate Q'_w and hence, Q is not an sd-Nash equilibrium of $(\mathcal{P}, \widetilde{\varphi}, P)$. Suppose w is not matched to f with positive probability at $\widetilde{\varphi}[Q']$.

[4] The result first proved by Kelso and Crawford (1982) is later shown by Crawford (1990) for the wide class of matching problems including many-to-one and many-to-many matching problems.

Let $\mu' \in \widetilde{\varphi}[Q']$. We show that $\mu' \in S(Q)$. Assume, by contradiction that $\mu' \notin S(Q)$. Since the definitions of Q' and of Q and $\mu'(w) \neq f$ ensure $\mu' \in IR(Q)$, then there is a blocking pair (f', w') for μ' at Q, i.e, $w' \notin \mu'(f')$, $f'\ Q_{w'}\ \mu'(w')$ and $w' \in Ch(\mu'(f') \cup \{w'\}, Q_{f'})$. This implies that (f', w') blocks μ' for Q' unless $w' = w$ and $\mu'(w') = f$, contradicting the assumption that w is not matched to f with positive probability at $\widetilde{\varphi}[Q']$. Hence, $\mu' \in S(Q)$.

We next show that μ is the firm-optimal stable match for Q. Suppose not. Then, there are $\widetilde{\mu} \in S(Q)$ and $\widetilde{f} \in F$ such that $\widetilde{\mu}(\widetilde{f})\ Q_{\widetilde{f}}\ \mu(\widetilde{f})$. Since $P_{\widetilde{f}} = Q_{\widetilde{f}}$, then $\widetilde{\mu}(\widetilde{f})\ P_{\widetilde{f}}\ \mu(\widetilde{f})$. Let $\widetilde{Q}_{\widetilde{f}}$ be an alternative strategy for $\widetilde{f}$ such that $A(\widetilde{Q}_{\widetilde{f}}) = \widetilde{\mu}(\widetilde{f})$. Let $\widetilde{Q} \equiv (\widetilde{Q}_{\widetilde{f}}, Q_{-\widetilde{f}})$. By Lemma 3, $\widetilde{\mu} \in S(\widetilde{Q})$. By Proposition 2 in Martinez et al. (2000), $\widetilde{f}$ has the same number of positions filled at each match in $S(\widetilde{Q})$. As the set of agents $\widetilde{f}$ finds acceptable in $\widetilde{Q}_{\widetilde{f}}$ is exactly $\widetilde{\mu}(\widetilde{f})$, by Lemma 1, $\widetilde{f}$ is matched to $\widetilde{\mu}(\widetilde{f})$ at each match in $S(\widetilde{Q})$ and in particular at each match in $\widetilde{\varphi}[\widetilde{Q}]$. Since $\widetilde{\mu}(\widetilde{f})\ P_{\widetilde{f}}\ \mu(\widetilde{f})$, then $\widetilde{f}$ has a profitable deviation when the other agents use $Q_{-\widetilde{f}}$. Hence, μ is the firm-optimal stable match for Q.

We next show that $w \notin Ch(\mu'(f) \cup \{w\}, P_f)$. Assume, by contradiction that $w \in Ch(\mu'(f) \cup \{w\}, P_f)$. Since $Q'_f = Q_f = P_f$, then $w \in Ch(\mu'(f) \cup \{w\}, Q'_f)$. Since w is not matched to f with positive probability at $\widetilde{\varphi}[Q']$, then by the definition of Q'_w, $f\ Q'_w\ \mu'(w)$, contradicting $\mu' \in S(Q')$. Hence, $w \notin Ch(\mu'(f) \cup \{w\}, P_f)$.

We now complete the proof. By substitutability, $w \notin Ch(\mu(f) \cup \mu'(f) \cup \{w\}, P_f)$. Let $K \equiv Ch(\mu(f) \cup \mu'(f) \cup \{w\}, P_f)$. Thus, $w \notin K$. We show that $K \nsubseteq \mu(f)$. Assume, by contradiction that $K \subseteq \mu(f)$. Then, by definition of the chosen set, $Ch(\mu(f) \cup \{w\}, P_f) = K$. Thus, $w \notin Ch(\mu(f) \cup \{w\}, P_f)$. This is a contradiction. Hence, $K \nsubseteq \mu(f)$. This, together with $w \notin K$ implies that $(K \cap \mu'(f)) \backslash \mu(f) \neq \emptyset$. Let $\widetilde{w} \in (K \cap \mu'(f)) \backslash \mu(f)$. By substitutability, $\widetilde{w} \in Ch(\mu(f) \cup \{\widetilde{w}\}, P_f)$. Since $Q_f = P_f$, then $\widetilde{w} \in Ch(\mu(f) \cup \{\widetilde{w}\}, Q_f)$. Since μ is the firm-optimal stable match for Q, then workers unanimously find μ the worst among all stable matches for Q. This, together with $\mu' \in S(Q)$ implies that $\mu'(\widetilde{w}) = f\ Q_{\widetilde{w}}\ \mu(\widetilde{w})$, Hence, $(f, \widetilde{w})$ blocks μ at Q, contradicting $\mu \in S(Q)$. ■

Proposition 5 states the converse result that each stable match for the true preferences can be achieved as the outcome of an sd-Nash equilibrium in which firms behave truthfully. An immediate implication of the result is that workers can obtain any jointly achievable match as the outcome of the game induced by any probabilistic stable rule.

Proposition 5 *Let $\mu \in S(P)$ and $\widetilde{\varphi}$ be a probabilistic stable rule. Then, there is an* *sd-Nash equilibrium $Q = (P_F, Q_W)$ of the game $(\mathcal{P}, \widetilde{\varphi}, P)$ that supports μ.*

Proof. Let $Q \in \mathcal{P}$ be such that for each $w \in W$, $A(Q_w) = \{\mu(w)\}$ and for each $f \in F$, $Q_f = P_f$. By repeatedly applying Lemma 3, we obtain $\mu \in S(Q)$. By Proposition 1 in Martinez et al. (2000), the set of matched workers is the same at each stable match for Q. As for each $w \in W$, $A(Q_w) = \{\mu(w)\}$, each worker is matched to her partner at μ at each stable match for Q. Hence, $S(Q) = \{\mu\}$ and μ is reached with probability one.

We now prove that Q is an sd-Nash equilibrium of $(\mathcal{P}, \widetilde{\varphi}, P)$. Let $w \in W$ and Q'_w be an alternative strategy for w. Let $f \in F$ be such that $f \; P_w \; \mu(w)$. Let $Q' \equiv (Q'_w, Q_{-w})$. We show that w can not get matched to f at any stable match for Q'. Assume on the contrary that there is $\mu' \in S(Q')$ such that $\mu'(w) = f$. By Lemma 1, $w \; Q'_f \; \emptyset$. Since $\mu \in S(P)$ and $f \; P_w \; \mu(w)$, then $w \notin Ch(\mu(f) \cup \{w\}, P_f)$. This, together with the definition of the chosen set implies that $Ch(\mu(f), P_f) = Ch(\mu(f) \cup \{w\}, P_f)$. By stability of μ, $Ch(\mu(f), P_f) = \mu(f)$. Thus, $Ch(\mu(f) \cup \{w\}, P_f) = \mu(f)$. Since $Q'_f = Q_f = P_f$, then $Ch(\mu(f) \cup \{w\}, Q'_f) = \mu(f)$. Thus, $\mu(f) \; Q'_f \; (\mu(f) \cup \{w\})$. We show that $|\mu(f)| = c_f$. Assume on the contrary $|\mu(f)| < c_f$. By separability, $\emptyset \; Q'_f \; w$. This is a contradiction. Hence, $|\mu(f)| = c_f$. This, together with $\mu'(f) \neq \mu(f)$ implies that there is $w' \in \mu(f) \backslash \mu'(f)$. As $A(Q'_{w'}) = \{f\}$, we have $\mu'(w') = w'$. Thus, $f \; Q'_{w'} \; \mu'(w')$. Since only workers in $\mu(f) \cup \{w\}$ find f acceptable at Q', then $\mu'(f) \subseteq (\mu(f) \cup \{w\})$. Thus, $(\mu'(f) \cup \{w'\}) \subseteq (\mu(f) \cup \{w\})$. As $Ch(\mu(f) \cup \{w\}, Q'_f) = \mu(f)$, by substitutability, $w' \in Ch(\mu'(f) \cup \{w'\}, Q'_f)$, contradicting $\mu' \in S(Q')$. Hence, w cannot get matched to f at any stable match for Q'. This implies that w cannot improve upon $\mu(w)$ by deviating.

Now let $f \in F$. The only workers willing to get matched to f are those in $\mu(f)$. Moreover, by individual-rationality of μ, for each $S \subseteq \mu(f)$, $\mu(f) \; R_f \; S$. Hence, f cannot improve upon $\mu(f)$ by deviating. ■

6 A general model: Many-to-many matching problems

An extension of the problem we have considered in the previous section is when each agent is allowed to be matched to multiple agents on the other side of the problem. This extension has also attracted much attention in the recent literature. In many-to-many matching problems, Kojima and Ünver (2008) analyze a random procedure of the sort studied by Roth and Vande Vate (1990) in one-to-one matching problems.

When agents on one side have substitutable preferences and those on the other have responsive preferences, they prove that the decentralized process of satisfying randomly chosen blocking pairs converges to a pairwise-stable match. Thus, the decentralized interpretation of the model remains valid in the extended setup.

We first provide the formal extension. Each firm f has a **strict** preference relation over the set of all subsets of workers 2^W and each worker has a **strict** preference relation over the set of all subsets of firms 2^F. Each agent v has a capacity, i.e, each firm f can fill at most c_f positions and each worker w can work for at most c_w firms. Let $c \equiv (c)_{v\in(F\cup W)}$ denote the list of capacities. Let $\mathcal{P}_v$ denote the set of all possible preference relations for agent v and let $\mathcal{P} = \prod_{v\in(F\cup W)} \mathcal{P}_v$ be the set of all possible preference profiles. Let R_v denote the **at least as desirable as relation** associated with P_v. For each $w \in W$ and each $S, S' \subseteq F$, $S\ R_w\ S'$ indicates either $S = S'$ or $S\ P_w\ S'$. For each $f \in F$ and each $S, S' \subseteq W$, $S\ R_f\ S'$ indicates either $S = S'$ or $S\ P_f\ S'$.

A **match** is a mapping μ from the set $F \cup W$ to the set of all subsets of $F \cup W$ satisfying the following conditions:

1. For each $w \in W$, $\mu(w) \in 2^F$ and $|\mu(w)| \leq c_w$;
2. For each $f \in F$, $\mu(f) \in 2^W$ and $|\mu(f)| \leq c_f$;
3. For each $(f, w) \in F \times W$, $f \in \mu(w)$ if and only if $w \in \mu(f)$.

Let $P \in \mathcal{P}$ be given. Agent v **blocks $\boldsymbol{\mu}$ at $\boldsymbol{P}$** if $Ch(\mu(v), P_v) \neq \mu(v)$. Match μ is **individually-rational for $\boldsymbol{P}$** if it is not blocked by an agent. Let $\boldsymbol{IR(P)}$ denote the set of all individually-rational matches for preference profile P. A pair **$\boldsymbol{(f, w)}$ blocks $\boldsymbol{\mu}$ at $\boldsymbol{P}$** if they are originally not matched to each other and prefer to be matched to each other possibly instead of some of their current partnerships, i.e, $w \in Ch(\mu(f)\cup\{w\}, P_f)$ and $f \in Ch(\mu(w) \cup \{f\}, P_w)$. Match μ is **pairwise-stable for $\boldsymbol{P}$** if it is individually-rational for P and is not blocked by any firm-worker pair. Let $\boldsymbol{S(P)}$ denote the set of pairwise-stable matches for preference profile P. The class of preferences that satisfy substitutability is the most general preference domain so far under which the existence of pairwise-stable matches has been guaranteed (Roth 1984b, 1991).

Lemmas 1 and 2 easily extend to many-to-many matching problems.

Lemma 4 *Let $Q \in \mathcal{P}$ and $\mu \in S(Q)$. Then for each $v \in F \cup W$ and each $v' \in \mu(v)$, v' Q_v $\emptyset$.*

Lemma 5 *Let $Q \in \mathcal{P}$, $v \in F \cup W$ and a subset S of partners with $|S| \leq c_v$ be given. If $S \subseteq A(Q_v)$, then $Ch(S, Q_v) = S$.*

Proposition 6 is an extension of Proposition 1 to many-to-many matching problems: a unique match arises as the outcome of each sd-Nash equilibrium of the game $(\mathcal{P}, \widetilde{\varphi}, P)$

Proposition 6 *Let Q be an sd-Nash equilibrium of the game $(\mathcal{P}, \widetilde{\varphi}, P)$. Then, a single match is obtained with probability one.*

Proof. Let Q be an sd-Nash equilibrium of the game $(\mathcal{P}, \widetilde{\varphi}, P)$. Assume, by contradiction that $|\text{supp } \widetilde{\varphi}[Q]| \geq 2$. Then there are $w \in W$ and $\mu, \widehat{\mu} \in \text{supp } \widetilde{\varphi}[Q]$ such that $\mu(w) \neq \widehat{\mu}(w)$. Let $\mu' \in \text{supp } \widetilde{\varphi}[Q]$ be such that for each $\mu \in \text{supp } \widetilde{\varphi}[Q]$, $\mu'(w) \; R_w \; \mu(w)$. Let Q'_w be such that $A(Q'_w) = \mu'(w)$ and let $Q' \equiv (Q'_w, Q_{-w})$. We prove that $\mu' \in S(Q')$. Assume, by contradiction that $\mu' \notin S(Q')$. By Lemma 5, $Ch(\mu'(w), Q'_w) = \mu'(w)$. Since for each $v \neq w$, $Q'_v = Q_v$ and $\mu' \in S(Q)$, then $Ch(\mu'(v), Q'_v) = \mu'(v)$. It follows that $\mu' \in IR(Q')$. This implies that there is a blocking pair (f', w') for μ' at Q', i.e,

$$w' \notin \mu'(f'),\ f' \in Ch(\mu'(w') \cup \{f'\}, Q'_{w'}) \text{ and } w' \in Ch(\mu'(f') \cup \{w'\}, Q'_{f'}). \tag{4}$$

We prove that $w' \neq w$. Suppose by contradiction that $w' = w$. Then, $w \notin \mu'(f')$ and $f' \in Ch(\mu'(w) \cup \{f'\}, Q'_w)$. By substitutability, $f' \in Ch(\{f'\}, Q'_w)$. Thus, $f' \; Q'_w \; \emptyset$, contradicting the definition of Q'_w. Hence, $w' \neq w$. Then (4) becomes $w' \notin \mu'(f')$, $f' \in Ch(\mu'(w') \cup \{f'\}, Q_{w'})$ and $w' \in Ch(\mu'(f') \cup \{w'\}, Q_{f'})$, contradicting $\mu' \in S(Q)$. Hence, $\mu' \in S(Q')$. Each agent is assigned to the same number of partners at each pairwise-stable match for Q (Klijn and Yazıcı, 2011). As $A(Q'_w) = \mu'(w)$, w is matched to $\mu'(w)$ at each stable match for Q'. Hence, Q_w does not stochastically P_w-dominate Q'_w. ■

As in many-to-one matching problems we identify individual-rationality for the true preferences as a necessary and sufficient condition for an equilibrium outcome.

Proposition 7 *Let $\mu \in IR(P)$ and let $\widetilde{\varphi}$ be a probabilistic stable matching rule. Then, there is an sd-Nash equilibrium Q of the game $(\mathcal{P}, \widetilde{\varphi}, P)$ that supports μ.*

Proof. Let $Q \in \mathcal{P}$ be such that for each $v \in F \cup W$, $A(Q_v) = \mu(v)$. We first show that $\mu \in S(Q)$. Since for each $v \in F \cup W$, $|\mu(v)| \leq c_v$ and $A(Q_v) = \mu(v)$, then by Lemma 5, for each $v \in F \cup W$, $Ch(\mu(v), Q_v) = \mu(v)$. It follows that $\mu \in IR(Q)$. We next show that there is no blocking pair for μ at Q. Assume, by contradiction that there is a blocking pair (f, w) for μ at Q, i.e, $w \notin \mu(f)$, $f \in Ch(\mu(w) \cup \{f\}, Q_w)$ and $w \in Ch(\mu(f) \cup \{w\}, Q_f)$. By substitutability, $w \in Ch(\{w\}, Q_f)$. Thus, $w \; Q_f$

$\emptyset$, contradicting the definition of Q_f. Hence, $\mu \in S(Q)$. Each agent is assigned to the same number of partners at each pairwise-stable match for Q (Klijn and Yazıcı, 2011). As for each $v \in F \cup W$, $A(Q_v) = \mu(v)$, each agent v is matched to $\mu(v)$ at each stable match for Q. Hence, $S(Q) = \{\mu\}$ and μ is reached with probability one. We next show that no agent can profitably deviate. Let $v \in F \cup W$ and Q'_v be an alternative strategy. Let $Q' \equiv (Q'_v, Q_{-v})$. By Lemma 4, each partner of v at each stable match for Q' should find v acceptable at Q'. If $\mu(v) \neq \emptyset$, then by individual-rationality of μ, for each $S \subseteq \mu(f)$, $\mu(f)\ R_f\ S$. Therefore, agent v can only be matched to $\mu(v)$ at each stable match for Q'. If $\mu(v) = \emptyset$, then agent v should be unmatched at each stable match for Q'. In either case, agent v cannot benefit from deviating. Hence, Q is an sd-Nash equilibrium. ■

Proposition 8 *Let Q be an sd-Nash equilibrium of the game $(\mathcal{P}, \widetilde{\varphi}, P)$. Then, the unique equilibrium outcome $\widetilde{\varphi}[Q]$ is individually-rational for the true preferences.*

The proof of many-to-one version of Proposition 8 relies on the rural hospital theorem (Martinez et al. 2000). It says that each agent is matched to the same number of partners at each stable match and that if a firm does not fill its capacity at a stable match then it is matched to the same set of partners at each stable match. The rural hospital theorem holds in many-to-many matching problems with substitutable and separable preferences (Klijn and Yazıcı, 2011). Moreover, Proposition 1 and Lemma 2 extend to this problem, too. As mentioned in footnote 6, the comparative statics of adding agents to the problem is first studied by Kelso and Crawford (1982) in many-to-one matching problems and later shown by Crawford (1990) for the wide class of many-to-many matching problems. He shows that adding an agent to one side of the problem strengthens the competitive positions of the agents on the other side of the problem. Having mentioned all these results, the argument for an arbitrary firm in the proof of Proposition 3 is valid for an arbitrary agent v.

The following example shows that truthtelling by firms in equilibrium is not sufficient for the stability of the equilibrium outcome.

Example 1: Let $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Let for each $v \in F \cup W \backslash \{w_4\}$, $c_v = 2$ and $c_{w_4} = 1$. Let the preference profile P be as follows.

$$P_{f_1}: \quad \{w_1, w_3\}, \{w_1, w_2\}, \{w_2, w_3\}, \{w_1\}, \{w_3\}, \{w_2\},$$
$$P_{f_2}: \quad \{w_2, w_4\}, \{w_3, w_4\}, \{w_2, w_3\}, \{w_4\}, \{w_2\}, \{w_3\},$$
$$P_{f_3}: \quad \{w_2, w_3\}, \{w_1, w_2\}, \{w_1, w_3\}, \{w_2\}, \{w_3\}, \{w_1\},$$

$P_{w_1}: \quad \{f_1, f_3\}, \{f_1\}, \{f_3\},$
$P_{w_2}: \quad \{f_1, f_3\}, \{f_2, f_3\}, \{f_1, f_2\}, \{f_3\}, \{f_1\}, \{f_2\},$
$P_{w_3}: \quad \{f_1, f_2\}, \{f_2, f_3\}, \{f_1, f_3\}, \{f_2\}, \{f_1\}, \{f_3\},$
$P_{w_4}: \quad \{f_2\},$

Consider the game $(\mathcal{P}, \varphi^F, P)$ induced by the probabilistic matching rule that assigns probability one to the firm-optimal pairwise-stable match for each profile.
Let Q be a strategy profile such that for each $v \in F \cup W \backslash \{w_3\}$ $Q_v = P_v$ and Q_{w_3} : $\{f_2, f_3\}, \{f_2\}, \{f_3\}$. The match $\mu \equiv \varphi^F[Q]$ is the firm-optimal pairwise-stable match for Q shown below.

$$\mu = \begin{pmatrix} f_1 & f_2 & f_3 \\ \{w_1, w_2\} & \{w_3, w_4\} & \{w_2, w_3\} \end{pmatrix}.$$

Notice that since (f_1, w_3) is a blocking pair for μ at P, then μ is not pairwise-stable for P . We now prove that Q is an sd-Nash equilibrium of the game $(\mathcal{P}, \varphi^F, P)$. Since firm f_3 and workers w_2 and w_4 are assigned at μ to their most preferred partners according to their true preferences, we only need to consider deviations for firms f_1 and f_2 and workers w_1 and w_3. Since w_3 finds any subsets of partners that include f_1 unacceptable at Q_{w_3}, firm f_1 cannot deviate to obtain $\{w_1, w_3\}$. Thus, firm f_1 cannot improve upon μ by deviating.

We next consider deviations for f_2. Firm f_2 can benefit from deviation only if it obtains $\{w_2, w_4\}$. Let Q'_{f_2} be an alternative strategy for f_2. We show that firm f_2 can not be assigned to $\{w_2, w_4\}$ at $\varphi^F(Q'_{f_2}, Q_{-f_2})$. Assume, by contradiction that f_2 is assigned to $\{w_2, w_4\}$ at $\varphi^F(Q'_{f_2}, Q_{-f_2})$. Then there are three possibilities for w_2's assignment at $\varphi^F(Q'_{f_2}, Q_{-f_2})$. 1) Worker w_2 is assigned to $\{f_2, f_3\}$ at $\varphi^F(Q'_{f_2}, Q_{-f_2})$. Since w_3 finds any subsets of partners that include f_1 unacceptable at Q_{w_3}, f_1 is either assigned to w_1 or unmatched at $\varphi^F(Q'_{f_2}, Q_{-f_2})$. In either case (f_1, w_2) blocks $\varphi^F(Q'_{f_2}, Q_{-f_2})$ at (Q'_{f_2}, Q_{-f_2}), contradicting pairwise-stability of φ^F. 2) Worker w_2 is assigned to $\{f_1, f_2\}$ at $\varphi^F(Q'_{f_2}, Q_{-f_2})$. Then f_3 is assigned to $\{w_1, w_3\}$ or $\{w_1\}$ only or $\{w_3\}$ only at $\varphi^F(Q'_{f_2}, Q_{-f_2})$. In all cases (f_3, w_2) blocks $\varphi^F(Q'_{f_2}, Q_{-f_2})$ at (Q'_{f_2}, Q_{-f_2}), contradicting pairwise-stability of φ^F. 3) Worker w_2 is assigned to $\{f_2\}$ only at $\varphi^F(Q'_{f_2}, Q_{-f_2})$. Then as before f_3 is assigned to $\{w_1, w_3\}$ or $\{w_1\}$ only or $\{w_3\}$ only at $\varphi^F(Q'_{f_2}, Q_{-f_2})$. In all cases (f_3, w_2) blocks $\varphi^F(Q'_{f_2}, Q_{-f_2})$ at (Q'_{f_2}, Q_{-f_2}), contradicting pairwise-stability of φ^F. Hence, firm f_2 can not be assigned to $\{w_2, w_4\}$ at $\varphi^F(Q'_{f_2}, Q_{-f_2})$. Since Q'_{f_2} is arbitrary, then firm f_2 cannot improve upon μ by deviating.

We now consider deviations for w_1. Worker w_1 can benefit from deviation only if she obtains $\{f_1, f_3\}$. Let Q'_{w_1} be an alternative strategy for w_1. We show that w_1 can not be assigned to $\{f_1, f_3\}$ at $\varphi^F(Q'_{w_1}, Q_{-w_1})$. Assume, by contradiction that w_1 is assigned to $\{f_1, f_3\}$ at $\varphi^F(Q'_{w_1}, Q_{-w_1})$. Then there are three possibilities for f_3's assignment at $\varphi^F(Q'_{w_1}, Q_{-w_1})$. 1) Firm f_3 is assigned to $\{w_1, w_2\}$ at $\varphi^F(Q'_{w_1}, Q_{-w_1})$. Then w_3 is either assigned to f_2 or unmatched at $\varphi^F(Q'_{w_1}, Q_{-w_1})$. In either case (f_3, w_3) blocks $\varphi^F(Q'_{w_1}, Q_{-w_1})$ at (Q'_{w_1}, Q_{-w_1}), contradicting pairwise-stability of φ^F. 2) Firm f_3 is assigned to $\{w_1, w_3\}$ at $\varphi^F(Q'_{w_1}, Q_{-w_1})$. Then w_2 is assigned to $\{f_1, f_2\}$ or $\{f_1\}$ only or $\{f_2\}$ only at $\varphi^F(Q'_{w_1}, Q_{-w_1})$. In all cases (f_3, w_2) blocks $\varphi^F(Q'_{w_1}, Q_{-w_1})$ at (Q'_{w_1}, Q_{-w_1}), contradicting pairwise-stability of φ^F. 3) Firm f_3 is assigned to $\{w_1\}$ only at $\varphi^F(Q'_{w_1}, Q_{-w_1})$. Then w_3 is either assigned to f_2 or unmatched at $\varphi^F(Q'_{w_1}, Q_{-w_1})$. In either case (f_3, w_3) blocks $\varphi^F(Q'_{w_1}, Q_{-w_1})$ at (Q'_{w_1}, Q_{-w_1}), contradicting pairwise-stability of φ^F. Hence, worker w_1 can not be assigned to $\{f_1, f_3\}$ at $\varphi^F(Q'_{w_1}, Q_{-w_1})$. Since Q'_{w_1} is arbitrary, then worker w_1 cannot benefit from deviating at Q.

We finally consider deviations for w_3. Worker w_3 can benefit from deviation only if she obtains $\{f_1, f_2\}$. Let Q'_{w_3} be an alternative strategy for w_3. We show that w_3 can not be assigned to $\{f_1, f_2\}$ at $\varphi^F(Q'_{w_3}, Q_{-w_3})$. Assume, by contradiction that w_3 is assigned to $\{f_1, f_2\}$ at $\varphi^F(Q'_{w_3}, Q_{-w_3})$. Then there are three possibilities for f_2's assignment at $\varphi^F(Q'_{w_3}, Q_{-w_3})$. 1) Firm f_2 is assigned to $\{w_3, w_4\}$ at $\varphi^F(Q'_{w_3}, Q_{-w_3})$. If w_2 is assigned to $\{f_1\}$ or $\{f_3\}$ only, then (f_2, w_2) blocks $\varphi^F(Q'_{w_3}, Q_{-w_3})$ at (Q'_{w_3}, Q_{-w_3}), contradicting pairwise-stability of φ^F. Thus, w_2 is assigned to $\{f_1, f_3\}$ at $\varphi^F(Q'_{w_3}, Q_{-w_3})$. This implies that f_1 is assigned to $\{w_2, w_3\}$ at $\varphi^F(Q'_{w_3}, Q_{-w_3})$. Then, w_1 is either assigned to $\{f_3\}$ or unmatched at $\varphi^F(Q'_{w_3}, Q_{-w_3})$. In either case (f_1, w_1) blocks $\varphi^F(Q'_{w_3}, Q_{-w_3})$ at (Q'_{w_3}, Q_{-w_3}), contradicting pairwise-stability of φ^F. 2) Firm f_2 is assigned to $\{w_2, w_3\}$ at $\varphi^F(Q'_{w_3}, Q_{-w_3})$. Then, w_4 is unmatched at $\varphi^F(Q'_{w_3}, Q_{-w_3})$. Thus, (f_2, w_4) blocks $\varphi^F(Q'_{w_3}, Q_{-w_3})$ at (Q'_{w_3}, Q_{-w_3}), contradicting pairwise-stability of φ^F. 3) Firm f_2 is assigned to $\{w_3\}$ at $\varphi^F(Q'_{w_3}, Q_{-w_3})$. Again w_4 is unmatched at $\varphi^F(Q'_{w_3}, Q_{-w_3})$. Thus, (f_2, w_4) blocks $\varphi^F(Q'_{w_3}, Q_{-w_3})$ at (Q'_{w_3}, Q_{-w_3}), contradicting pairwise-stability of φ^F.$\Diamond$

The example illustrates that when each agent has responsive preferences, the result does not hold either. We next question the existence of an sd-Nash equilibrium where firms behave truthfully.

Proposition 9 *Let $\mu \in S(P)$ and $\widetilde{\varphi}$ be a probabilistic stable matching rule. Then, there is an sd-Nash equilibrium $Q = (P_F, Q_W)$ of the game $(\mathcal{P}, \widetilde{\varphi}, P)$ that supports μ.*

Proof. Let $Q \in \mathcal{P}$ be such that for each $w \in W$, $A(Q_w) = \mu(w)$ and for each $f \in F$, $Q_f = P_f$. Since $\mu \in S(P)$ and $Q_F = P_F$, then for each $f \in F$, $Ch(\mu(f), Q_f) = \mu(f)$.

We show that $\mu \in S(Q)$. By Lemma 5, for each $w \in W$, $Ch(\mu(w), Q_w) = \mu(w)$. Thus, $\mu \in IR(Q)$. Suppose that the pair (f', w') blocks μ at Q, i.e,

$$w' \in \mu(f'),\ f' \in Ch(\mu(w') \cup \{f'\}, Q_{w'}) \text{ and } w' \in Ch(\mu(f') \cup \{w'\}, Q_{f'}).$$

By substitutability, $f' \in Ch(\{f'\}, Q_{w'})$. Thus, $f'\ Q_{w'}\ \emptyset$. Then, $f' \in A(Q_{w'})$ which implies that $f' \in \mu(w')$. This is a contradiction. Hence, $\mu \in S(Q)$. Each agent is matched to the same number of partners at each pairwise-stable match for Q (Klijn and Yazıcı, 2011). As for each $w \in W$, $A(Q_w) = \mu(w)$, each worker is matched to her μ-partners at each pairwise-stable match for Q. Hence, $S(Q) = \{\mu\}$ and μ is reached with probability one.

We now prove that Q is an sd-Nash equilibrium of $(\mathcal{P}, \widetilde{\varphi}, P)$. Let $w \in W$ and Q'_w be an alternative strategy for w. Let $S \subseteq F$ be such that $S\ P_w\ \mu(w)$. Let $Q' \equiv (Q'_w, Q_{-w})$. We show that w can not be matched to S at any pairwise-stable match for Q'. Assume, by contradiction that there is $\mu' \in S(Q')$ such that $\mu'(w) = S$.

We first show that $Ch(\mu'(w) \cup \mu(w), P_w) \backslash \mu(w) \neq \emptyset$. Suppose not. Then, $Ch(\mu'(w) \cup \mu(w), P_w) \subseteq \mu(w)$. By pairwise-stability of μ for P, $Ch(\mu'(w) \cup \mu(w), P_w) = \mu(w)$, contradicting $\mu'(w) = S\ P_w\ \mu(w)$.

Now let $f \in Ch(\mu'(w) \cup \mu(w), P_w) \backslash \mu(w)$. Then, $f \in \mu'(w) \backslash \mu(w)$. Also, $\mu'(f) \neq \mu(f)$. By Lemma 4, $w\ Q'_f\ \emptyset$. By substitutability, $f \in Ch(\mu(w) \cup \{f\}, P_w)$. By pairwise-stability of μ for P, $w \notin Ch(\mu(f) \cup \{w\}, P_f)$. This, together with the definition of the chosen set implies that $Ch(\mu(f), P_f) = Ch(\mu(f) \cup \{w\}, P_f)$. By pairwise-stability of μ for P, $Ch(\mu(f), P_f) = \mu(f)$. Thus, $Ch(\mu(f) \cup \{w\}, P_f) = \mu(f)$. Since $Q'_f = Q_f = P_f$, then $Ch(\mu(f) \cup \{w\}, Q'_f) = \mu(f)$. Thus, $\mu(f)\ Q'_f\ (\mu(f) \cup \{w\})$. We show that $|\mu(f)| = c_f$. Assume on the contrary that $|\mu(f)| < c_f$. By separability, $\emptyset\ Q'_f\ w$. This is a contradiction. Hence, $|\mu(f)| = c_f$. This, together with $\mu'(f) \neq \mu(f)$ implies that $\mu(f) \backslash \mu'(f) \neq \emptyset$.

Let $\overline{w} \in \mu(f) \backslash \mu'(f)$. Notice that $\overline{w} \neq w$. Since $A(Q'_{\overline{w}}) = \mu(\overline{w})$, $|\mu(\overline{w})| \leq c_{\overline{w}}$, $\overline{w} \notin \mu'(f)$ and $\mu' \in S(Q')$, by Lemma 4, $|\mu'(\overline{w})| < c_{\overline{w}}$. We show that $f \in Ch(\mu'(\overline{w}) \cup \{f\}, Q'_{\overline{w}})$. Suppose not. Then $Ch(\mu'(\overline{w}) \cup \{f\}, Q'_{\overline{w}}) \subseteq \mu'(\overline{w})$. By pairwise-stability of μ' for $Q'_{\overline{w}}$, $Ch(\mu'(\overline{w}) \cup \{f\}, Q'_{\overline{w}}) = \mu'(\overline{w})$ which implies that $\mu'(\overline{w})\ Q'_{\overline{w}}\ \mu'(\overline{w}) \cup \{f\}$. By separability, $\emptyset\ Q'_{\overline{w}}\ f$, contradicting $A(Q'_{\overline{w}}) = \mu(\overline{w})$. Thus,

$$f \in Ch(\mu'(\overline{w}) \cup \{f\}, Q'_{\overline{w}}). \tag{5}$$

Since only workers in $\mu(f) \cup \{w\}$ find f acceptable at Q', then $\mu'(f) \subseteq (\mu(f) \cup \{w\})$.

Thus, $(\mu'(f) \cup \{\overline{w}\}) \subseteq (\mu(f) \cup \{w\})$. As $Ch(\mu(f) \cup \{w\}, Q'_f) = \mu(f)$, by substitutability

$$\overline{w} \in Ch(\mu'(f) \cup \{\overline{w}\}, Q'_f). \tag{6}$$

Conditions 5 and 6 imply that $(f, \overline{w})$ blocks μ' at Q', contradicting $\mu' \in S(Q')$. Hence, w cannot get matched to f at any pairwise-stable match for Q'. This implies that w cannot improve upon $\mu(w)$ by deviating.

Now let $f \in F$. The only workers willing to get matched to f are those in $\mu(f)$. Moreover, by individual-rationality of μ, for each $S \subseteq \mu(f)$, $\mu(f)\ R_f\ S$. Hence, f cannot improve upon $\mu(f)$ by deviating. ■

7 Conclusion

We analyzed the sd-Nash equilibria of the game induced by any probabilistic stable matching rule in many-to-one matching problems when each firm has substitutable and separable preferences. We first proved that a unique match is achieved as the outcome of each sd-Nash equilibrium whereas multiple matches may arise with positive probability in the game as the outcome of truthful behavior. We next showed that individual-rationality with respect to the true preferences is a necessary and sufficient condition for a match to be achieved as the outcome of an sd-Nash equilibrium. Stochastically dominant Nash equilibria where firms behave truthfully always lead to stable matches for the true preferences. Conversely, each stable match for the true preferences is supported as the outcome of an sd-Nash equilibrium where firms behave truthfully. We also studied equilibria in many-to-many matching problems and extended the above results but the one concerning the stability of equilibrium outcomes for the true preferences. We identified an sd-Nash equilibrium of the game induced by the firm-optimal stable rule where firms behave straightforwardly and yet the equilibrium outcome is not pairwise-stable for the true preferences. Nevertheless, for each pairwise-stable match μ for the true preferences, we can still find an equilibrium that leads to μ with probability one.

Pais (2008a) examines the connection between the equilibria of the game induced by a probabilistic stable rule with those induced by a deterministic stable rule in the college admissions problem. In particular, each sd-Nash equilibrium of the game induced by any probabilistic stable rule is a Nash equilibrium of the game induced by some deterministic stable rule. A partially converse result is the following. Let Q be an sd-Nash equilibrium of the game induced by the firm-optimal stable rule and of the game induced by the worker-optimal stable rule. Then Q is an sd-Nash equilibrium of the game induced by any probabilistic stable rule. With obvious modifications, the proofs of all these results remain valid in many-to-many matching problems when each agent has substitutable and separable preferences.

Pais (2008a) studies the sd-Nash equilibria of the sequential game of the sort studied by Roth and Vande Vate (1990). She allows for a broader set of strategies that are not required to be consistent with a unique preference ordering. However, when concerned with strategies that are compatible with a distinct preference ordering for each play of the game (that corresponds to a sequence of randomly selected pairs of agents), given a profile of stated preferences for agents other than an arbitrary agent v, agent v has a best response that is compatible with a preference ordering. This result extends to many-to-many matching problems when agents on one side have substitutable and separable preferences and agents on the other side have responsive preferences.

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