Robust portfolios that do not tilt factor exposure

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\begin{abstract}
Robust portfolios reduce the uncertainty in portfolio performance. In particular, the worst-case optimization approach is based on the Markowitz model and form portfolios that are more robust compared to mean–variance portfolios. However, since the robust formulation finds a different portfolio from the optimal mean–variance portfolio, the two portfolios may have dissimilar levels of factor exposure. In most cases, investors need a portfolio that is not only robust but also has a desired level of dependency on factor movement for managing the total portfolio risk. Therefore, we introduce new robust formulations that allow investors to control the factor exposure of portfolios. Empirical analysis shows that the robust portfolios from the proposed formulations are more robust than the classical mean–variance approach with comparable levels of exposure on fundamental factors.
\end{abstract}

\section{Introduction}

Robust optimization was developed to solve problems where there is uncertainty in the decision environment, and therefore is sometimes referred to as uncertain optimization (Ben-Tal and Nemirovski, 1998). Robust models were adapted in portfolio optimization to resolve the sensitivity issue of the mean–variance model (Markowitz, 1952) to its inputs. Even though the Markowitz model is still the basis for portfolio optimization and one of the most significant contributions to portfolio selection, its drawbacks are well documented. For example, Best and Grauer (1991) show that mean–variance portfolio weights are extremely sensitive to changes in asset means when no constraints are imposed. Broadie (1993) points out errors in computing efficient frontiers and stresses that estimates of mean returns using historical data should be used with caution. Chopra and Ziemba (1993) go one step further to analyze the relative impact of estimation errors in means, variances, and covariances.

These studies naturally led to the advancement of robust portfolio optimization. Costa and Paiva (2002) optimize robust tracking-error optimization when the expected returns and covariance matrix are not exactly known. Similarly under uncertain situations, El Ghaoui et al. (2003) solve the worst-case value-at-risk problem when only bounds on the inputs are known. In addition, Goldfarb and Iyengar (2003) introduce a way to formulate robust optimization problems as second-order cone programs using ellipsoidal uncertainty sets, and similar approaches are investigated by Tüttüncü and Koenig (2004). As summarized, much effort has been put into defining uncertainty in input parameters and formulating the problems using worst-case optimization methods.\footnote{For further contributions on robust portfolio optimization, refer to Fabozzi et al. (2007a,b), Fabozzi et al. (2010), and Kim et al. (accepted for publication-b).}

On the same topic but on a different track, there have been recent developments looking into the behavior of robust portfolios formed from worst-case optimization (Kim et al., accepted for publication-a; Kim et al., 2012; Kim et al., 2013). They look for unexpected properties of robust formulations in order to not only expand our understanding on robust portfolios but also to increase its practical use. Focusing on the worst-case formulation with ellipsoidal uncertainty sets, Kim et al. (2012) mathematically show how robustness leads to higher correlation with fundamental factors. Kim et al. (accepted for publication-a) further investigate this behavior using worst-case formulations with ellipsoidal and box uncertainty sets under various settings to confirm the relationship between robustness and increased correlation with factors. Even though some worst-case formulations may result in portfolios with unintended factor loadings, robust portfolios are still required to protect portfolio returns from unexpected market movements. Kim et al. (2013) find that under the existence of market regimes, a portfolio that focuses on asset returns during the worst regime of the stock market results in a portfolio that is not only robust but has an improved performance based on measures such as Sharpe ratio, drawdown, and value-at-risk.

Based on these research findings, we know that investors can achieve robustness through robust optimization techniques but at the same time lose control of the factor dependency of the optimal portfolio. Factors explain the underlying movement of the
market and widely studied factors include the market index, size (market capitalization), and book-to-market ratio factors (Fama and French, 1993). In addition to these three fundamental factors, portfolio managers look for macroeconomic factors and also identify statistical factors through principal component analysis. Investors and portfolio managers need to control the factor exposure for managing the portion of total risk generated from each factor because it allows them to better understand the overall risk of their portfolios. Therefore, in this paper, we introduce formulations that form robust portfolios which are not tilted towards factors. Investors may consider robust counterparts of mean–variance portfolios for achieving robust performance. However, if the factor exposure of the robust counterparts is affected by the robust formulations, investors need to compare not only the performance between the classical mean–variance and robust portfolios but also their factor exposures. By forming portfolios from our proposed optimization problems, investors will be able to control the risk associated with each factor of robust portfolios. In other words, investors can hold portfolios that are robust and at the same time have a better understanding of how the portfolio will be affected by movements in market factors.

Before we introduce and analyze the revised robust formulations, we need a measure to compare the robustness of portfolios. This robustness measure allows us to compare the optimal portfolios from the new formulations with the existing mean–variance and robust portfolios. Therefore, we begin our analysis by defining a robustness measure and exploring its properties. Next, we derive new robust formulations to form optimal portfolios that match a target factor exposure by adding constraints to the original robust formulation. Since the additional constraints are developed based on factor exposure measures of portfolios, the new formulations will not only add robustness but also keep the factor dependency to a specified level. Furthermore, we perform empirical tests to confirm the factor exposures of the new robust portfolios and find that the new approach forms portfolios that are more robust than Markowitz portfolios without affecting the factor exposure.

The organization of the paper is as follows. Section 2 defines the robustness measure and its properties, and the new robust formulations that do not affect factor exposure are introduced in Section 3. Section 4 includes empirical tests using industry-level returns to confirm our development, and Section 5 concludes.

2. Robustness measure

In this section, we define a robustness measure prior to developing new robust formulations. By defining a measure for portfolio robustness, investors can compare the robustness among portfolios and also use this measure to set the desired robustness level when choosing an optimal portfolio. Once we define a robustness measure, we analyze its properties for the case when the geometry of the uncertainty set for the expected asset returns is an ellipsoid.

2.1. Defining a robustness measure

We begin defining a robustness measure by identifying the meaning of robustness. In general, \( X(e) \), which is a function of an uncertain parameter \( e \), being robust means that the value of \( X(e) \) is relatively stable even when the uncertain parameter \( e \) fluctuates. Therefore, we can measure the robustness of \( X(e) \) by the movement of \( X(e) \) with respect to \( e \).

The robustness in portfolio management can be defined in a similar fashion. We first determine a suitable uncertain function \( X(e) \) for measuring portfolio robustness. Since the optimal portfolio weights are a function of uncertain parameters \( \mu \) and \( \Sigma \), expected return and covariance of returns, respectively, the weights of the optimal portfolio are one possible choice for the function \( X(e) \). However, this measure does not reflect the portfolio robustness desired by investors because investors want robust portfolio performance but not necessarily robust portfolio weights. In other words, the stability of the portfolio performance is more important to investors than the stability of the optimal weights. Thus, it is more reasonable to use the portfolio performance for \( X(e) \) when we define portfolio robustness. Consequently, we can say that the optimal portfolio \( \omega^* \) is robust if the performance of the portfolio \( \omega^* \) is stable with respect to \( \mu \) and \( \Sigma \).

Based on the above definition of robustness in portfolio optimization, we derive the robustness measure for portfolios. The stability level of portfolio performance determines the degree of robustness. Therefore, the robustness measure can be defined by the performance fluctuation from the change in uncertain parameters. Portfolios that are robust will have small performance fluctuations and therefore low levels for the robustness measure. For simplicity, we assume that the only uncertain parameter is the mean of asset returns \( \mu \) and the uncertainty is revealed at time \( T \). Since the portfolio performance depends on the mean of asset returns \( \mu \) and the portfolio weight \( \omega^* \), it can be represented as \( f(\omega^*, \mu) \). The expected value of \( \mu \) at time 0 is \( \mu_0 \) and the realized value at time \( T \) is \( \mu_T \). Hence, investors make investment decisions based on \( \mu_0 \) at time 0 but they evaluate the portfolio performance using \( \mu_T \) at time \( T \). Thus, performance fluctuation is the difference between the expectation and the realization of portfolio performance,

\[
D(\omega^*) = \max_{\mu_T \in \Omega} d(\omega^*, \mu_T) - d(\omega^*, \mu_0).
\]

However, we cannot evaluate \( D(\omega^*) \) at time 0 because \( \mu_T \) is unknown until time \( T \). So, instead of \( D(\omega^*) \), the maximum of \( d(\omega^*) \) on the uncertainty set \( U \) of \( \mu_T \) should be the robustness measure,

\[
D(\omega^*) = \max_{\mu_T \in U} d(\omega^*, \mu_T) - d(\omega^*, \mu_0).
\]

When the realization \( \mu_T \) of the uncertain parameter \( \mu \) varies in a given set \( U \), the fluctuation of the portfolio performance cannot exceed the robustness measure \( D(\omega^*) \). It follows that portfolios with smaller values of \( D(\omega^*) \) will have less fluctuation in portfolio performance and can be considered to be more robust.

The generalized form of the robustness measure is

Robustness measure = \[ \max_{\theta \in \Theta} f(\omega^*, \theta) - f(\omega^*, \theta) \]

- \( f(\omega^*, \theta) \): portfolio performance,
- \( \omega^* \): an optimal portfolio,
- \( \theta \): uncertain parameters,
- \( \Theta \): the expected value of uncertain parameters,
- \( \Theta \): an uncertainty set for \( \theta \).

2.2. Properties of the robustness measure

For the robustness measure of portfolio performance, we assume the uncertain parameter \( \mu \) to be in an uncertainty set \( U \). Since ellipsoidal uncertainty sets are often used in robust optimization, we also assume that the uncertainty set of \( \mu \) to be in an ellipsoid. The robust portfolio formulation with an ellipsoidal uncertainty set is,

\[
\min_{\mu, \omega} \quad \frac{1}{2} \omega' \Sigma \omega - \lambda \mu' \omega = \min_{\mu, \omega} \quad \frac{1}{2} \omega' \Sigma \omega - \lambda \mu' \omega + \lambda d^2 \sqrt{\mu' \Sigma \mu},
\]

where \( \Omega = \{ (\omega, \mu) \} \), \( U(\tilde{\mu}) = \left\{ \mu | (\mu - \tilde{\mu})' \Sigma (\mu - \tilde{\mu}) < d^2 \right\} \), where \( \Sigma \) is the covariance of expected returns, \( i \) is a vector of ones.

\( \lambda \) is the risk coefficient, and \( \Sigma_p \) is the covariance matrix of estimation errors for the expected returns. Among many possibilities for representing portfolio performance, we focus on the objective function of the robust formulation. Even though the objective function of the robust formulation is not always a conventional measure of performance, measuring the stability of the objective function is applicable when investors use portfolio optimization models for constructing portfolios because the objective function reflects what investors seek to optimize. In other words, investors will consider the portfolio robust if the objective value is stable with respect to changes in input values. Accordingly, from the objective function of the above formulation, we define \( f(\omega, \mu) = \frac{1}{2} \omega' \Sigma \omega - \lambda \mu' \omega \) as the measure for portfolio performance. In addition, since \( \delta \) determines the confidence level of the uncertainty set, higher values of \( \delta \) form portfolios that are more robust. Therefore, an important property which the robustness measure should satisfy is the decreasing pattern with respect to \( \delta \). We show this property of the robustness measure in the following two claims.

2.2.1. Claim 1

**Claim 1.** Let \( \omega_0 \) be the optimal solution to the optimization problem (R.). If \( \omega_0 \) converges to \( \omega^* \) as \( \delta \) goes to infinity, then \( \omega^* \) is the portfolio that minimizes the robust measure. That is,

\[
\arg \min_{\omega \in \Omega} \max_{\mu \in U_0} |f(\omega, \mu) - f(\omega_0, \mu)| = \lim_{\delta \to \infty} \left( \arg \min_{\omega \in \Omega} \max_{\mu \in U_0} \frac{1}{2} \omega' \Sigma \omega - \lambda \mu' \omega \right),
\]

where

\[
f(\omega, \mu) = \frac{1}{2} \omega' \Sigma \omega - \lambda \mu' \omega, \quad \Omega = \{ \omega | \omega' i = 1 \},
\]

\[
U_0(\mu) = \{ \mu | (\mu - \bar{\mu})' \Sigma_1^{-1} (\mu - \bar{\mu}) \leq \delta_0^2 \},
\]

\[
U_\delta(\mu) = \{ \mu | (\mu - \bar{\mu})' \Sigma_1^{-1} (\mu - \bar{\mu}) \leq \delta^2 \}.
\]

**Proof.** From the definition of the robustness measure,\(^3\)

\[
\max_{\mu \in U_0(\mu)} |f(\omega, \mu) - f(\omega_0, \mu)| = \max_{\mu \in U_0(\mu)} \left| \frac{1}{2} \omega' \Sigma \omega - \lambda \mu' \omega - \frac{1}{2} \omega_0' \Sigma \omega + \lambda \mu_0' \omega \right|
\]

\[
= \max_{\mu \in U_0(\mu)} \lambda |(\mu - \mu_0)' \omega| = \delta_0 \delta \sqrt{\omega' \Sigma \omega}.
\]

Therefore,

\[
\arg \min_{\omega \in \Omega} \max_{\mu \in U_0} \left| f(\omega, \mu) - f(\omega_0, \mu) \right| = \arg \min_{\omega \in \Omega} \delta \delta \sqrt{\omega' \Sigma \omega}.
\]

On the other hand, the robust formulation can be derived as,

\[
\arg \min_{\omega \in \Omega} \max_{\mu \in U_0(\mu)} \left| \frac{1}{2} \omega' \Sigma \omega - \lambda \mu' \omega \right| = \min_{\omega \in \Omega} \left| \frac{1}{2} \omega' \Sigma \omega - \lambda \mu' \omega \right|
\]

\[
+ \lambda \sqrt{\omega' \Sigma \omega} = \min_{\omega \in \Omega} \left( \arg \min_{\omega \in \Omega} \frac{1}{2} \omega' \Sigma \omega - \lambda \mu' \omega \right)
\]

\[
+ \lambda \sqrt{\omega' \Sigma \omega}.
\]

So proving the following equation will suffice our proof for Claim 1.

\[
\arg \min_{\omega \in \Omega} \delta \delta \sqrt{\omega' \Sigma \omega} = \lim_{\delta \to 1} \left( \arg \min_{\omega \in \Omega} \frac{1}{2} \omega' \Sigma \omega - \lambda \mu' \omega \right)
\]

\[
+ \lambda \sqrt{\omega' \Sigma \omega}.
\]

In order to show the above, we prove (i) and (ii).

\( \delta \geq n \Rightarrow |f(\omega) - f(\omega_0)| \leq e, \quad \forall \omega \in D. \)

By the uniform convergence of \( f_\delta(\omega) \), the following inequalities are satisfied for all \( \delta \geq n \).

\[
|f_\delta(\omega_0^*) - f_\delta(\omega_0^*)| \leq e, \quad |f_\delta(\omega_0^*) - f_\delta(\omega'_0)| \leq e,
\]

\[
|f_\delta(\omega_0^*) - f_\delta(\omega'_0)| \leq e.
\]

Therefore, it follows that (i) holds

\( \delta \geq n \Rightarrow |f_\delta(\omega_0^*) - f_\delta(\omega^*)| \leq \frac{1}{3} e. \) (i)

Because limit \( f_\delta(\omega_0^*) \) is \( f_\delta(\omega_0^*) \) by (i), for all \( e > 0 \) there exists \( n_2 \in N \) such that

\[
\delta \geq n_2 \Rightarrow |f_\delta(\omega_0^*) - f_\delta(\omega'_0)| \leq \frac{1}{3} e. \) (ii)

In addition, \( f_\delta(\omega) \) uniformly converges to \( f_\infty(\omega) \) on the set \( D \) as \( \delta \to \infty \), and therefore there exists \( n_3 \in N \) for all \( e > 0 \) such that

\[
\delta \geq n_3 \Rightarrow |f_\delta(\omega) - f_\infty(\omega)| \leq \frac{1}{3} e, \quad \forall \omega \in D. \) (iii)

By defining \( n_0 = \max(n_1, n_2, n_3) \), the inequalities (i), (ii), and (iii) can be expressed by \( \omega_{n_0} \) as,

\[
|f_{n_0}(\omega_{n_0}) - f_{n_0}(\omega)| \leq \frac{1}{3} e, \) (iv)

\[
|f_{n_0}(\omega_{n_0}) - f_{n_0}(\omega_{n_0})| \leq \frac{1}{3} e, \) (v)

\[
|f_{n_0}(\omega_{n_0}) - f_{n_0}(\omega_{n_0})| \leq \frac{1}{3} e. \) (vi)

Now, adding the three inequalities (iv), (v), and (vi) gives us,

\[
|f_{n_0}(\omega_{n_0}) - f_{n_0}(\omega_{n_0})| \leq \frac{1}{3} e.
\]
Thus, \( f_\omega(\omega) = f_\omega(\omega^*) \) and because \( f_\omega(\omega) \) is a convex function, \( \omega^* \) is equal to \( \omega_{\min} \). In conclusion, if \( \omega_{\min}^j \) converges to \( \omega^* \), \( \omega^* \) is the optimal solution of \( \min_{\omega \in \Omega} f_\omega(\omega) \). By verifying \( \Box \), we show that

\[
\arg\min_{\omega \in \Omega} \max_{\mu \in U(\omega)} f(\omega, \mu) - f(\omega, \theta) = \lim_{\delta \to -\infty} \arg\min_{\omega \in \Omega} \max_{\mu \in U(\omega)} \frac{1}{2} \omega^T \Sigma \omega - \lambda \mu^T \omega.
\]

Hence, we prove Claim 1. \( \Box \)

### 2.2.2. Claim 2

**Claim 2.** The robustness measure is a decreasing function of \( \delta \). That is, the optimal portfolio becomes more robust as \( \delta \) increases and the robustness measure \( \max_{\omega \in \Omega} f(\omega, \theta) - f(\omega^j, \theta) \) decreases.

**Proof.** We provide a proof under the stylized assumption used in Kim et al. (2012) due to analytical limitations for analyzing the general case. They show that \( \langle R \rangle \) and the below optimization problem \( (QP_a) \) have the same optimal portfolio under the proper choice of \( a \).

\[
(QP_a) \quad \min_{\omega} \quad \frac{1}{2} \omega^T (\Sigma + a \Sigma \mu) \omega - \lambda \mu^T \omega
\]

\[
\text{s.t.} \quad \omega^T 1 = 1.
\]

The analytic solution for \( (QP_a) \) can be found under the below stylized assumptions.

- Let \( \Sigma \) have the same value for all diagonal terms (\( \sigma^2 \)) and also the same value for all off-diagonal terms (\( \rho \sigma^2 \))

\[
\Sigma = \begin{bmatrix}
\sigma^2 & \rho \sigma^2 \\
\rho \sigma^2 & \sigma^2 \\
\end{bmatrix}
\]

Then, since the inverse of \( \Sigma \) will have identical diagonal terms and identical off-diagonal terms, it can be expressed as

\[
\Sigma^{-1} = \begin{bmatrix}
\alpha & \beta \\
\beta & \alpha \\
\end{bmatrix}
\]

- Let \( \Sigma = \Sigma + a \Sigma \mu \), therefore, \( \Sigma \) for \( a > 0 \) represents a covariance matrix with extra penalization on the diagonal terms imposed on the Markowitz problem.

Under the stylized assumptions, Kim et al. (2012) find the optimal portfolio \( \omega_q^a \) for \( (QP_a) \),

\[
\omega_q^a = (x_a - \beta_a) u_0 + v_0, \quad \text{where} \quad u_0 = \lambda (\mu - \mu_t) \quad \text{and} \quad v_0 = \frac{1}{n} t.
\]

Now, we use the analytic optimal portfolio \( \omega_q^a \) for the calculation of the robustness measure, and show that the robustness measure decreases as \( a \) increases. The robustness measure becomes
be a diagonal matrix whose diagonal is the same as and calculate the robustness measure for the optimization steps for the simulation.

Table 1
Summary of the new robust formulations.

<table>
<thead>
<tr>
<th>Classical robust formulation</th>
<th>( \min_{\omega} \alpha \omega' \Sigma_0 \alpha - \lambda \mu' \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>s.t. ( \omega' \Sigma_0 \omega = 1 )</td>
<td>[ (R) \min_{\omega, \max_{\omega}} \alpha \omega' \Sigma_0 \alpha - \lambda \mu' \alpha ]</td>
</tr>
</tbody>
</table>

Additional constraints

Based on mean

\[ (R1) \quad \omega_0 B' - \omega B' \leq \epsilon B' \leq \omega_0 B' + \epsilon \]

Based on variance

\[ (R2) \quad \frac{\epsilon_0^2 \Sigma_0 \epsilon_0}{\epsilon} \leq \frac{\epsilon_{\text{max}} \Sigma_0 \epsilon_{\text{min}}}{\epsilon_{\text{min}}} \]

\[ (R3) \quad (\omega_{\text{max}} - \omega_{\text{min}})' (\omega - \omega_{\text{min}}) = 0 \]

\[ (R4) \quad (\omega_{\text{max}} - \omega_{\text{min}})' (\omega - \omega_{\text{min}}) \leq 0 \]

where \( \omega_{\text{min}} = \omega_{\text{max}} + (1 - \alpha) \omega_0 \)

We repeat the above three steps for various values of \( \delta \) to observe any pattern. Fig. 1 presents results when \( n = 10, \alpha = 0.5 \), and \( \delta \) from 0 to 3 with 0.1 increments. Results for 10 simulations are shown and all cases clearly show that the robustness measure decreases as \( \delta \) increases from 0 to 3. Therefore, we can conclude that Claim 2 is also satisfied even without the stylized assumptions.

3. New robust portfolio formulation

In the previous section, we introduced the robustness measure and its properties. The main purpose for presenting the robustness measure is to compare the robustness of portfolios from the original robust formulation with the robust portfolios from the new formulations. Therefore, we return to our focus of this paper and derive robust formulations that do not influence the factor exposure of portfolios.

Kim et al. (accepted for publication-a) and Kim et al. (2012) observe that robust formulations may indirectly affect the factor loading of portfolios. Therefore, although mean–variance investors can protect the uncertainty risk by using the robust formulations based on the mean–variance model, they will not be able to sustain the expected factor exposure of the mean–variance portfolios. Consequently, we introduce new robust formulations that control the dependency on fundamental factors by having the same factor exposure relative to a target portfolio. We again base our model on the robust formulation with an ellipsoidal uncertainty set since it is one of the first and most researched formulations.

3.1. Parameter assumptions

For simplicity, we assume the following case,

4 We use this simplified assumption because the scaled covariance matrix of returns provides the estimation error covariance in a stationary return process and the diagonals of the covariance matrix of estimation errors are known to contain enough information for forming robust portfolios. For detailed approaches employed in practice, see Stubbs and Vance (2005).

Fig. 4. Relax (R4) by using \( \omega_0 \) to expand the feasible region.
3.2. Measuring the proportion invested in factors

The portfolio return of \( \omega \) can be expressed as \( r = Bf + \varepsilon \) by the factor model, where \( B \) is the matrix of stock betas and \( f \) is the factor returns. Therefore, the expected portfolio return and the variance of returns become,

\[
E[r] = \mu_f \\
\text{Var}(r) = \Sigma
\]

where \( \mu_f = E[f] \) and \( \Sigma_f = \text{Var}(f) \).

The exposure to factors can be measured by how much the mean and variance of portfolio returns are affected by factors, respectively. In other words, we can use the amount of return invested on factors, \( \omega'B \), or the proportion of variance depended on factors, \( \frac{\text{Var}(\omega'r)}{\text{Var}(r)} = \frac{\omega'B\Sigma_fB'\omega}{\text{Var}(r)} \). Therefore, by controlling the factor exposure with constraints that incorporate these values, we can construct portfolios that possess the same level of factor dependency as the target portfolio \( \omega_0 \).

3.3. New robust formulations

Consider the worst-case robust optimization of the form,

\[
(R) \quad \min_{\omega} \max_{\mu \in \Omega(\mu)} \quad \frac{1}{2} \omega'\Sigma\omega - \lambda \mu'\omega \\
\text{s.t.} \quad \omega'1 = 1,
\]

where \( \Omega(\mu) = \left\{ \mu | (\mu - \mu')'\Sigma^{-1}(\mu - \mu') \leq \delta^2 \right\} \). Based on (R), we introduce new robust portfolio formulations that include additional constraints on factor exposure.

3.3.1. Formulation based on the expected return of a portfolio

We consider a constraint that matches the factor loading with the target portfolio \( \omega_0 \). The expected return of the target portfolio \( \omega_0 \) is \( \omega_0'B\mu_f \), where \( \omega_0'B \) is the part of the return that depends on the factors. Thus, an arbitrary portfolio \( \omega \) will have the same factor exposure as portfolio \( \omega_0 \) if \( \omega \) satisfies the following constraint,

\[
\omega' B = \omega_0' B.
\]

Notice that the above constraint has a convex feasible set because of linearity. By including this constraint into the original robust optimization problem (R), the new robust formulation (R1) can be derived.
3.3.2. Formulation based on the variance of portfolio returns

Next, we look at the variance of portfolio returns, which can be represented using the factor model as

$\omega^\prime \Sigma \omega - \lambda \mu^\prime \omega$

where $\omega$ is a vector that sets the bound, which may be represented as a desired percentage of $e_0 B$. The new robust formulation maintains not only the factor exposure but also the robustness of the original formulation. Furthermore, even though (R1) and (R1') have additional constraints, the optimal solution can be found efficiently since the extra constraints are linear in both cases.

**Fig. 6.** Yearly value of proportion of variance from factors (top: $\lambda = 0.01$, bottom: $\lambda = 0.05$).

Although the above robust formulation fulfills our goal, the non-convexity from the new constraint makes it difficult to find the optimal solution even when using optimization solvers. Therefore,
we look for a relaxed modification that can still achieve similar outcomes.

We first define $g(x)$ to be the proportion of the portfolio variance that depends on factor movement, 

$$g(x) = \frac{x' \Sigma x}{x' \Sigma x}$$

Then, the portfolio whose variance depends the most and the least on factors can be expressed respectively as,

$$x_{\text{max}} = \arg \max g(x) \quad \text{such that} \quad x_0 = 1$$

$$x_{\text{min}} = \arg \min g(x) \quad \text{such that} \quad x_0 = 1$$

Using the Rayleigh’s principle (Strang, 1988), the two values, $x_{\text{max}}$ and $x_{\text{min}}$, can be easily found by calculating the maximum and minimum eigenvalues. Then the following inequality is satisfied for the given portfolio $x_0$,

$$g(x_{\text{min}}) \leq g(x_0) \leq g(x_{\text{max}})$$

and this can be used to find a constraint that relaxes constraint ($\ast$). Even if the problem (R2) cannot be solved directly, the alternative approach should at least find an optimal portfolio that is not too close to either $x_{\text{max}}$ or $x_{\text{min}}$. In other words, the factor exposure of portfolio $x$ should not be maximized or minimized unless that is the case for portfolio $x_0$. As shown in Fig. 2, portfolio $x$ will not be close to either $x_{\text{max}}$ or $x_{\text{min}}$ if $(x_{\text{max}} - x_{\text{min}})$ is orthogonal. This approach gives the third robust formulation that does not have extreme factor exposures,

$$\text{(R3)} \quad \min_{\omega \in \mu(x)} \frac{1}{2} \omega' \Sigma \omega - \lambda \mu' \omega$$

s.t. \quad $\omega' x = 1$

$$\omega_{\text{max}} - \omega_{\text{min}} (\omega - x_0) = 0.$$

Since (R3) has only linear constraints, it can be solved efficiently. Although it will not control factor exposure as well as (R2), it will be more effective than the original robust formulation.

We can modify (R3) based on the findings by Kim et al. (accepted for publication-a) and Kim et al. (2012) that robust formulations tend to increase the dependency on factors. Hence, it is more important for the optimal portfolio to not move towards $x_{\text{max}}$ than $x_{\text{min}}$. In this context, the orthogonality of $(\omega_{\text{max}} - \omega_{\text{min}})$ restrains the convergence of the optimal portfolio to $x_{\text{max}}$ as shown in Fig. 3.

Therefore, the final robust formulation that only focuses on $x_{\text{max}}$ is written as,

$$\text{(R4)} \quad \min_{\omega \in \mu(x)} \frac{1}{2} \omega' \Sigma \omega - \lambda \mu' \omega$$

s.t. \quad $\omega' x = 1$

$$\omega_{\text{max}} - \omega_{\text{min}} (\omega - x_0) = 0.$$

Furthermore, as we have shown in (R1), the additional constraint of (R4) to control factor exposure may be overly restrictive for finding optimal portfolios. Therefore, the feasible region can be expanded using the modification of the constraint as shown in Fig. 4 and the resulting robust formulation becomes,
4. Empirical analysis with stock market returns

In the previous section, we propose new robust formulations having the same factor exposure relative to a given target portfolio. We now empirically confirm the effect on factor exposure of our proposed formulations using historical market returns. We analyze robust portfolios formed from (R1) and (R4') using several test measures to compare the performance and factor exposure with the portfolios constructed from the original robust model (R) as well as the classical mean–variance portfolios from (MV).

4.1. Test measures

We define the following measures to evaluate the new robust optimal portfolios.

1. Robustness measure

To compare portfolio robustness, we use the robustness measure defined in Section 2. Portfolios that are more robust have smaller values of this measure, $\text{RM} = \lambda \sqrt{\omega^* \Sigma_{\omega^*} \omega^*}$.

2. Factor exposure based on portfolio mean

The factor loading of the optimal portfolio $\omega^*$ is $\omega^* B$, so the 2-norm of $\omega^* B$ can be used to measure the magnitude of the factor exposure, $F_1 = ||\omega^* B||_2$.

3. Factor exposure based on portfolio variance

The proportion of portfolio movement that depends on factor movement can be represented as $\frac{\omega^* B \Sigma_{\omega^*} B^T \omega^*}{\omega^* \Sigma_{\omega^*} \omega^*}$ as shown in the previous section. Therefore, we use this ratio as the second measure of the factor exposure of portfolios, $F_2 = \frac{\omega^* B \Sigma_{\omega^*} B^T \omega^*}{\omega^* \Sigma_{\omega^*} \omega^*}$.

3.3.3. Summary

Table 1 summarizes the results on the new robust formulations presented in this section.

Fig. 8. Average standard deviation of returns of 10 portfolios using past 10 years data (top: $\lambda = 0.01$, bottom: $\lambda = 0.05$).

\[ (R4') \quad \min_{\omega \in \Omega} \max_{x \in \mathbb{R}} \frac{1}{2} \omega^T \Sigma (\omega - 2\mu x) \]

s.t. $\omega' = 1$,

$\omega_{\max} - \omega_a \leq 0$,

where $\omega_a = \omega_{\max} \ast (1 - x) \omega_{\theta}$ for $x \in [0, 1]$ and larger values of $x$ result in optimal portfolios closer to $\omega_{\max}$.
4.2. Data description

In addition to using simulated returns, we use historical returns in the US stock market to check how the new robust portfolios perform against the classical mean–variance portfolio and the original robust portfolio with an ellipsoidal uncertainty set. The three-factor model proposed by Fama and French (1993, 1995) is used for the factor returns and the 49 industry portfolios proposed by Fama and French (1997) to validate their factor model are considered as the asset universe. We collect daily returns from January 1970 to December 2011 for both factor and industry-level returns.6

4.3. Empirical results

Using actual market data, we compare the portfolios from the four formulations: \((MV)\), \((R)\), \((R1)\), and \((R4^*)\) with \(\alpha = 0.5\). We run the empirical tests using a 1-year and 5-year rebalancing periods, values of \(\lambda\) between 0.01 and 0.1, and confidence levels between 10% and 95%. We use the test measures defined in Section 4.1 for comparing the portfolios and we present the results for a few cases but the overall observation is the same in all settings. All figures included use a 1-year rebalancing period since it gives most data points, one portfolio every year.

The values of \(F_1\) representing the magnitude of factor loading are plotted for the four formulations for each of the 42 years. It is clearly observed, as shown in Fig. 5, that the solid line, which represents the portfolios from \((R)\), has values different from the other three portfolios. Even though we did not directly apply a constraint for this measure, we see that the portfolios from \((R4^*)\) also have values that match the portfolios from \((MV)\) although not as exact as the ones from \((R1)\). The pattern is shown more visibly when looking at the bottom figure, which is when the value of \(\lambda\) is set to 0.05. The first set of results demonstrate that the two proposed robust formulations in this paper can indeed control the factor exposure of portfolios.

We next check the second measure of factor exposure \(F_2\), which is the proportion of portfolio variance that is explained by the variance of factors. Similar to the results for \(F_1\), levels of \(F_2\) that match the classical mean–variance portfolios from \((MV)\) are desired. As shown in Fig. 6, since the additional constraint for \((R4^*)\) is based on this ratio of variance, its portfolios have values that are closer to the portfolios from \((MV)\), but not exact since a relaxed version of the constraint is used in \((R4^*)\). In addition, although \((R1)\) does not restrict its weights based on variance, its portfolios are shown to have values much closer to the portfolios from \((MV)\) than \((R)\); the characteristic of \((R1)\) to control its factor loading is shown even when analyzed using another measure of factor exposure. This set

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6 These are the same data used in the empirical analysis by Kim et al. (accepted for publication-a). Both sets of data are available online from French’s data library (http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html).

Fig. 9. Average Sharpe ratio of 10 portfolios using past 10 years data (top: \(\lambda = 0.01\), bottom: \(\lambda = 0.05\)).

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of results also confirms that the two formulations (R1) and (R4') match the factor exposure with a target portfolio.

To measure robustness of the four formulations as defined by the deviation in the objective function of the portfolio optimization model as introduced in Section 2, we calculated the robustness measure $RM$ defined in Section 4.1. As shown in Fig. 7, the portfolios from (R) are the most robust and the ones from (MV) are the least robust. Although the portfolios from (R1) and (R4') have lower robustness than the portfolios from (R), they protect the uncertainty risk better than (MV). Therefore, we can conclude that (R1) and (R4') have the target factor exposure as well as improved robustness.

We also use standard performance measures to test the robustness of portfolio returns and the risk-return trade-off of portfolio performance. We use out-of-sample data to see whether the proposed robust models satisfy the basic objectives of robust portfolios. More specifically, for year $t$, 10 portfolios are created from year $t-10$ to $t-1$ using each year's returns and the returns of these 10 portfolios at year $t$ are observed. This directly shows how a portfolio optimization technique performs on average at year $t$ when different sets of inputs are used.

As a measure of robustness of portfolio returns, we investigate the annualized standard deviation of portfolio returns. In our setting, for year $t$, the volatility of the 10 portfolios created from returns in year $t-10$ to $t-1$ is measured to see how much portfolio returns change with respect to the change in input. As shown in Fig. 8, the portfolios from (R) are the most robust. Portfolios from (R1) and (R4') are less robust than the ones from (R) because they each have one additional constraint, which restricts the choice of feasible solutions, and the extra constraint is used to limit factor exposure but not included for further strengthening the robustness.

Similar to robustness, the average Sharpe ratio of the 10 portfolios for each year is calculated to compare the performance of the four formulations. Note that this is not a pure measure of performance but rather incorporates robustness since it is looking at the average performance of 10 portfolios, each formed using separate inputs. Again, as shown in Fig. 9, the portfolios from (R) dominate other portfolios and they are able to capture the rally of 1985 and 1995, which can be explained by its robust nature and its tendency to move towards market factors. More importantly, even though the factor exposures of (R1) and (R4') are controlled, the two robust portfolios do not perform any worse than the portfolios from (MV), mainly because they are less sensitive to input changes.

In summary, the portfolios from (R1) and (R4') more closely match the target factor exposure when compared against (MV). Even though the additional constraints of (R1) and (R4') result in portfolios that are not as robust as the portfolios from (R), they still form portfolios that are relatively more robust than (MV).

5. Conclusion

In this paper, we derive new robust formulations which maintain a desired level of factor exposure while conserving the robust effect of robust optimization. It is previously shown by Kim et al. (accepted for publication-a) and Kim et al. (2012) that robust portfolios from worst-case optimization have higher factor dependency than the classical mean–variance model. For solving this problem, we introduce several approaches by adding constraints into the well-known robust formulation with an elliptoidal uncertainty set. We define two measures for calculating the level of factor exposure of portfolios that are based on the mean and variance of portfolio returns, and derive formulations that include restrictions on these measures. Our empirical results clearly show that the improved robust portfolios resolve the tilting effect of current robust formulations while still making the portfolios more robust than mean–variance portfolios.

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