Minimal Converse Consistent Extension of the Men-Optimal Solution*

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January 8, 2011

Abstract

In this paper we study two-sided, one-to-one matching problems and consider the most wellknown solution concept: the men-optimal solution. The men-optimal solution fails to satisfy consistency as well as converse consistency. Furthermore, the minimal consistent extension of the men-optimal solution equals the core. In this paper, we compute the minimal converse consistent extension of the men-optimal solution as a correspondence which associates with each problem the set consisting of the men-optimal matching, and all stable and men-barterproof matchings for this problem.

Key Words: Matching, Men-Optimal Solution, Consistency, Converse Consistency,

JEL Classification Number: C78

*I thank Çağatay Kayı for his helpful comments. Of course, all possible errors are mine. A preliminary version of this paper has been presented at the departmental seminar of Caen University on May 2010. I thank all the audiences for their comments. This work is part of the project $\neq 107K560$ supported by the Scientific and Technological Research Council of Turkey (TUBITAK).

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1 Introduction

Consistency and converse consistency are two fundamental properties of solutions for allocation problems, in variable population models. In this study, we mainly focus on converse consistency, which is some kind of decentralization axiom. Thomson (2004) illustrates the notion of converse consistency by a jigsaw puzzle as "correct positioning of pieces two-by-two guarentees correct positioning altogether." This axiom allows us to deduce from the simpler calculations on all of the meaningfully smallest subproblems whether an alternative should be chosen for the original big problem. So converse consistency axiom is also useful from the computational viewpoint. Some versions of converse consistency are studied in many different fields. We know that several solutions, including the men-optimal solution in matching problems, fail to satisfy converse consistency. In that case, we wonder how serious the violations of converse consistency are. Thomson (2006) introduces the concept of minimal converse consistent extension of a solution in the general framework. By computing it, one actually evaluates the minimal extent to which the solution would have to be expanded in order to satisfy converse consistency.[1]

In this paper we study two-sided, one-to-one matching problems and consider the most well-known solution concept: the men-optimal solution. Thanks to Sasaki and Toda (1992)[2] and Toda (2006), we know that the minimal consistent extension of the men-optimal solution equals the core; as well as its minimal Maskin monotonic extension (Kara and Sönmez (1996)). Collaterally, the main aim of our paper is to compute its minimal converse consistent extension. Specifically, in the setting of matching, converse consistency is defined as follows: Consider some problem p and some solution φ. Take any matching μ. The requirement is the following: If the reduced matching of μ with respect to each subgroup of two matched pairs is among the recommendations made by the solution φ for the reduced problem of p with respect to the subgroup of two matched pairs, then μ must be a matching recommended by φ at the original problem p. That is the original definition of Sasaki and Toda (1992) where they show that the core is the unique correspondence which satisfies Pareto optimality, anonymity, consis-

[1] This extension is in the same spirit as the minimal consistent extension introduced by Thomson (1994*a*) and the minimal Maskin monotonic extension introduced by Sen (1995).

[2] Here the model is restricted so that we have equal number of men and women, and no agent stays selfmatched.

tency and converse consistency.[3] As far as we know, our study is one of the first attempts to compute the minimal converse consistent extension of some solution. However, there are a few papers in the literature computing the minimal consistent extension of solutions[4], as well as the minimal Maskin monotonic extension of solutions.[5]

To compute the minimal converse consistent extension of the men-optimal solution, we introduce the concept of men-barterproofness. We will consider two versions of men-barterpoofness. A matching is strongly men-barterproof whenever there is no pair of men who benefit from switching their mates among themselves. Here, we allow that any pair of men can bilaterally change their mates without the need for permission from their mates. All matchings resulting from serial men-dictatorship rules are strongly men-barterproof. If we impose that a pair of men can bilaterally change their mates only if they are acceptable to their later mates, we end up with men-barterproofness. Throughout the paper, we discuss the relationship between the men-barterproof solution and the core. It turns out that the minimal converse consistent extension of the men-optimal solution associates with each problem the set consisting of the men-optimal matching, and all stable and men-barterproof matchings for this problem.

The paper proceeds as follows: Section 2 presents the basic notions and axioms. Section 3 introduces the men-barterproof solution and gives our results. Finally, section 4 makes some closing remarks.

2 Models and Basic Axioms

Let M and W be two disjoint universal sets. Let M be a nonempty and finite subset of M. Similarly, let W be a nonempty and finite subset of W. Let $A = M \cup W$ be the set of agents.

For each agent $i \in A$, the set of potential mates of i, denoted by $A(i)$, is defined as

$$A(i) = i \cup \begin{cases} W \text{ if } i \in M \\ M \text{ if } i \in W. \end{cases}$$

Each agent $i \in A$ has a strict preference relation over $A(i)$, denoted by P_i.

[3] Recall that the model is restricted so that we have equal number of men and women and no agent stays selfmatched.

[4] See Thomson (1994*a*), Bevia (1996) and Korthues (2000).

[5] One can refer, for instance, to Thomson (1999), and Erdem and Sanver (2005).

Let $\mathcal{P}$ denote the set of all possible preference profiles $P = (P_i)_{i \in A}$. For each agent $i \in A$, the set of acceptable mates of i, denoted by $C(P,i) \subseteq A(i)$, is defined as $C(P,i) = \{j \in A(i) : j \ P_i \ i\}$.

A **matching** is a function $\mu : A \longrightarrow A$ such that for each $i \in A$, we have $\mu(i) \in A(i)$ and for each pair $\{j,k\} \subset A$, $\mu(j) = k$ implies $\mu(k) = j$. Agent $\mu(i)$ is the mate of agent i under matching μ. Let $\mathcal{M}(A)$ denote the set of all matchings for A. A **(matching) problem** is a pair $p = (A,P)$, where A is the set of agents and P is the preference profile. Let $\mathbf{P}$ denote the set of all problems.

A matching $\mu \in \mathcal{M}(A)$ is **individually rational** for p if for all $i \in A$, $\mu(i) \ P_i \ i$ or $\mu(i) = i$. A pair of agents $\{i,j\}$ **blocks** a matching $\mu \in \mathcal{M}(A)$ if $j \ P_i \ \mu(i)$ and $i \ P_j \ \mu(j)$. A matching $\mu \in \mathcal{M}(A)$ is **stable** for p if it is individually rational for p and there is no pair $\{i,j\} \subset A$ blocking μ. Let $S(p)$ denote the set of all stable matchings. Given a problem $p = (A,P) \in \mathbf{P}$ and two matchings $\mu, \mu' \in \mathcal{M}(A)$, μ **dominates** μ' if there exists a coalition $K \subseteq A$ such that for all $i \in K$, $\mu(i) \in K$ and $\mu(i) \ P_i \ \mu'(i)$. A matching μ is **undominated** if there exists no matching $\mu' \in \mathcal{M}(A)$ which dominates μ. The **core** of p is the set of undominated matchings. Recall that the set of stable matchings equals the core (Roth and Sotomayor,(1990)). Since the core has a lattice structure, for each problem $p = (A,P) \in \mathbf{P}$, there exists a matching $\mu \in S(p)$ such that for all $\mu' \in S(p)$ and all $i \in M$, we have $\mu(i) \ P_i \ \mu'(i)$ whenever $\mu'(i) \neq \mu(i)$ (Gale and Shapley, (1962)). Furthermore, this matching is unique. We call it the **men-optimal matching** for p and denote by $MO(p)$. We define $WO(p)$, the women-optimal matching for p, similarly.

A **solution** is a correspondence φ that associates with each problem $p = (A,P)$ a non-empty subset of $\mathcal{M}(A)$. Let Φ be the set of solutions. The **core solution** is the correspondence S that associates with each problem p its set of stable matchings $S(p)$. The **men-optimal solution** is the correspondence MO that associates with each problem p the men-optimal matching $MO(p)$. We similarly define the women-optimal solution WO.

Given a problem $p = (A,P)$ and a subset of set of agents A', a **reduced problem** of p with respect to A' is a problem where the preference profile P is restricted to agents in A'. Formally; for all $p = (A,P) \in \mathbf{P}$ and all $A' \subseteq A$, $p' = (A', P\mid_{A'}) \in \mathbf{P}$ is the reduced problem of p with respect to A'. Write $\mu(N) = \{\mu(i) : i \in N\}$. Given a matching $\mu \in \mathcal{M}(A)$ and a subset of agents $N \subseteq M \cup W$, a **reduced matching** of μ with respect to N is a matching $\mu\mid_N : N \cup \mu(N) \longrightarrow N \cup \mu(N)$ such that for all $i \in N \cup \mu(N)$,

$\mu|_N(i) = \mu(i)$.

To prove our main result, we need to define the consistency axiom. Consider some problem p and some solution φ. Take any matching μ recommended by φ at p. If the reduced matching of μ with respect to each subgroup of matched pairs is among the recommendations made by the solution φ for the reduced problem of p with respect to this subgroup of matched pairs, then we say that the solution φ is consistent.

Consistency: For each $p = (A, P) \in \mathbf{P}$, each $\mu \in \varphi(p)$, and each $N \subseteq A$, we have $\mu|_N \in \varphi(N \cup \mu(N), P|_{N\cup\mu(N)})$.

The next property is central to our analysis. Consider some problem p and some solution φ. Take any matching μ. The requirement is that if the reduced matching of μ with respect to each subgroup of two matched pairs is among the recommendations made by the solution φ for the reduced problem of p with respect to the subgroup of these two matched pairs, then μ must be a matching recommended by φ at the original problem p.

Converse Consistency: For each $p = (A, P) \in \mathbf{P}$ and each $\mu \in \mathcal{M}(A)$, if for each proper subset $N \subsetneq A$ with $|N| = 2$, $\mu|_N \in \varphi(N\cup\mu(N), P|_{N\cup\mu(N)})$, then $\mu \in \varphi(p)$.

It is a well-known result that the men-optimal solution fails to satisfy converse consistency.[6]

Example 2.1 *The men-optimal solution does not satisfy converse consistency.*

Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Let P be defined as follows:

P_{m_1}	P_{m_2}	P_{m_3}	P_{w_1}	P_{w_2}	P_{w_3}
w_1	w_3	w_2	m_3	m_2	m_1
w_2	w_1	w_3	m_2	m_1	m_3
w_3	w_2	w_1	m_1	m_3	m_2

Let $p = (M \cup W, P)$. The $MO(p)$ is the matching μ where $\mu(m_1) = w_1, \mu(m_2) = w_3, \mu(m_3) = w_2$. Consider the matching where $\mu^*(m_1) =$

[6] The men-optimal solution also fails to satisfy consistency.

$w_2, \mu^*(m_2) = w_1, \mu^*(m_3) = w_3$. There are three different subproblems with $|N| = 2$. Let $N_1 = \{m_1, m_2\}$, $N_2 = \{m_2, m_3\}$, $N_3 = \{m_1, m_3\}$. One can easily check that $MO(N_1 \cup \mu^*(N_1), P|_{N_1 \cup \mu^*(N_1)}) = \mu^*|_{N_1}$, $MO(N_2 \cup \mu^*(N_2), P|_{N_2 \cup \mu^*(N_2)}) = \mu^*|_{N_2}$, $MO(N_3 \cup \mu^*(N_3), P|_{N_3 \cup \mu^*(N_3)}) = \mu^*|_{N_3}$.

Given a solution φ, the minimal converse consistent extension of φ, denoted by $MCCE_\varphi$, is defined by Thomson (2006) as follows:

$$MCCE_\varphi = \bigcap_{\psi \in \Psi} \psi \text{ where } \Psi = \{\psi \in \Phi : \psi \supseteq \varphi,\ \psi \text{ is converse consistent}\}$$

By the definition of converse consistency, we can write this as, for each $p = (A, P) \in \mathbf{P}$, we have[7]

$$MCCE_\varphi(p) = \varphi(p) \cup \left\{ \begin{array}{l} \mu \in \mathcal{M}(A) : \mu|_N \in \varphi(N \cup \mu(N), P|_{N \cup \mu(N)}) \\ \textit{for all subsets } N \subseteq M \cup W \text{ with } |N| = 2. \end{array} \right.$$

3 Men-Barterproofness and The Minimal Converse Consistent Extension of the Men-Optimal Solution

A matching is strongly men-barterproof whenever there is no pair of men who benefit from switching their mates among themselves. Here, we allow that any pair of men can bilaterally change their mates without the need for permission from their mates. More formally, a pair of men $\{m, m'\} \subset M$ **barters** at a matching $\mu \in \mathcal{M}(A)$ if $\{\mu(m), \mu(m')\} \subset W$, $\mu(m')\ P_m\ \mu(m)$ and $\mu(m')\ P_{m'}\ \mu(m)$. A matching $\mu \in \mathcal{M}(A)$ is **strongly men-barterproof** for $p = (A, P) \in \mathbf{P}$, if there exists no pair of men $\{m, m'\} \subset M$ bartering at μ. Let $SMB(p)$ denote the set of the strongly men-barterproof matchings

[7] Letting $K(p) = \{\mu \in \mathcal{M}(A) : \mu|_N \in \varphi(N \cup \mu(N), P|_{N \cup \mu(N)})$ for all subsets $N \subseteq A$ with $|N| = 2\}$, for each $p = (A, P) \in \mathbf{P}$, we have $MCCE_\varphi(p) = \varphi(p) \cup K(p)$. Clearly, $\varphi \cup K$ is a converse consistent extension of φ. To see that $\varphi \cup K$ is the minimal one, take any other converse consistent extension of φ, and call it ψ. Suppose that there exists some $p = (A, P) \in \mathbf{P}$ such that $\psi(p) \subsetneq \varphi(p) \cup K(p)$. That means there exists a matching $\widetilde{\mu}$ such that $\widetilde{\mu} \in K(p)$ and $\widetilde{\mu} \notin \psi(p)$. Since for each $p = (A, P) \in \mathbf{P}$, $\varphi(p) \subseteq \psi(p)$, we have $\widetilde{\mu}|_N \in \psi(N \cup \widetilde{\mu}(N), P|_{N \cup \widetilde{\mu}(N)})$ for all subsets $N \subseteq A$ with $|N| = 2$, contradicting that ψ is converse consistent.

for p. It is easy to see that the set of strongly men-barterproof matchings include all the matchings resulting from serial men-dictatorship rules, hence this set is always nonempty.[8] The **strongly men-barterproof solution** is the correspondence SMB that associates with each problem p the set of men-barterproof matchings $SMB(p)$. A strongly women-barterproof matching for p and the strongly women-barterproof solution SWB are defined similarly.

To compute the minimal converse consistent extension of the men-optimal solution, we need to utilize from a weaker version of this axiom. Now, we allow that a pair of men can change their mates only if they are acceptable to their later mates. A matching is men-barterproof whenever there is no such a pair of men who benefit from switching their mates among themselves. More formally, a matching $\mu \in \mathcal{M}(A)$ is **men-barterproof** for $p = (A, P) \in \mathbf{P}$, if there exists no pair of men $\{m, m'\} \subset M$ with $m' \in C(P, \mu(m))$ and $m \in C(P, \mu(m')$ bartering at μ. Let $MB(p)$ denote the set of the men-barterproof matchings for p. The **men-barterproof solution** is the correspondence MB that associates with each problem p the set of men-barterproof matchings $MB(p)$. A women-barterproof matching for p and the women-barterproof solution WB are defined similarly.

First note that, if we restrict our model so that nobody is allowed to be single, i.e. for any $p = (A, P) \in \mathbf{P}$ with $|M| = |W|$ and for any $i \in A$, $C(P, i) = A(i)$, we have $MB(p) = SMB(p)$.

One can easily note that the intersection of the core and the set of men-barterproof matchings may be empty.

Example 3.1 *There exists a problem $p = (A, P) \in \mathbf{P}$ where $S(p) \cap MB(p) = \emptyset$.*

Let $M = \{m_1, m_2, m_3, m_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Let P be defined as follows:

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[8] The **serial men-dictatorship rule $\mathbf{D}_{m_1,m_2,...,m_n}$** can be found by using the following algorithm: Let the men be placed in the order, $m_1, m_2, ..., m_n$. Let m_1 be matched with his top mate, next let m_2 be matched with his top mate among the remaining potential mates. Then, let m_3 be the next dictator, and so forth. This process stops since the number of men is finite assigning each man either a woman or himself. Hence, it is well defined.

P_{m_1}	P_{m_2}	P_{m_3}	P_{m_4}	P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}
w_1	w_2	w_2	w_4	m_2	m_3	m_4	m_1
w_4	w_4	...	w_1	m_4	m_2	...	m_2
...	w_1	...	w_3	m_1	...	...	m_4

Let $p = (M \cup W, P)$. One can easily check that $S(p) = \mu$ where $\mu(m_1) = w_4$, $\mu(m_2) = w_1$, $\mu(m_3) = w_2$, $\mu(m_4) = w_3$, and $\mu \notin MB(p)$ since $\mu(m_2)$ P_{m_1} $\mu(m_1)$ and $\mu(m_1)$ P_{m_2} $\mu(m_2)$.

The next example illustrates the fact that the men-optimal matching may fail to be men-barterproof; even when the core and the set of men-barterproof matchings are not disjoint. Furthermore, the women-optimal matching may be strongly men-barterproof.

Example 3.2 *There exists a problem* $p = (A, P) \in \mathbf{P}$ *where* $MO(p) \nsubseteq S(p) \cap MB(p)$ *and* $WO(p) \subseteq S(p) \cap SMB(p)$*.*

Let $M = \{m_1, m_2, m_3, m_4, m_5, m_6\}$ and $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$. Let P be defined as follows:

P_{m_1}	P_{m_2}	P_{m_3}	P_{m_4}	P_{m_5}	P_{m_6}	P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}	P_{w_5}	P_{w_6}
w_2	w_1	w_2	w_4	w_5	w_6	m_6	m_5	m_1	m_2	m_4	m_3
w_1	w_2	w_3	w_5	w_2	w_1	m_1	m_2	...	...	...	...
w_3	w_4	w_6	...	...	...	m_4	m_3				
...	...	...				m_2	m_1				

Let $p = (M \cup W, P)$. Consider the matching $\mu \in MO(p)$ where $\mu(m_i) = w_i$ for all $i \in \{1, 2, ..., 6\}$. Since $\mu(m_2)$ P_{m_1} $\mu(m_1)$ and $\mu(m_1)$ P_{m_2} $\mu(m_2)$, we have $MO(p) \nsubseteq MB(p)$. Next consider the matching $\mu' = WO(p)$ where $\mu'(m_1) = w_3$, $\mu'(m_2) = w_4$, $\mu'(m_3) = w_6$, $\mu'(m_4) = w_5$, $\mu'(m_5) = w_2$, $\mu'(m_6) = w_1$. One can easily check that $WO(p) \subseteq SMB(p)$. Since $WO(p) \subseteq S(p)$, then $WO(p) \subseteq S(p) \cap SMB(p)$.[9]

However, if we have at most two men and at most two women in the society, then the men-optimal solution equals the set of all stable and men-barterproof matchings.

[9] Note that there is no order of men σ in A such that $\mathbf{D}_\sigma(p) = WO(p)$. Let Π be the set of possible orders of men in A. Example 3.3 also shows that $\cup_{\sigma \in \Pi} \mathbf{D}_{\prec}(p)$ may be a strict subset of $SMB(p)$.

Proposition 3.1 *For any* $p = (A, P) \in \mathbf{P}$ *with* $\mid M \mid \leq 2$ *and* $\mid W \mid \leq 2$, *we have* $MO(p) = S(p) \cap MB(p)$.

Proof. First consider any $p = (A, P) \in \mathbf{P}$ with $\mid M \mid= 1$ and $\mid W \mid \leq 2$. We trivially have $MB(p) = \mathcal{M}(A)$. Note also that $\mid S(p) \mid= 1$, hence for any $p = (A, P) \in \mathbf{P}$ with $\mid M \mid= 1$ and $\mid W \mid \leq 2$, we have $MO(p) = S(p) \cap MB(p)$.

Next consider any $p = (A, P) \in \mathbf{P}$ with $\mid M \mid= 2$ and $\mid W \mid= 1$. We trivially have $MB(p) = \mathcal{M}(A)$. Note also that $\mid S(p) \mid= 1$, hence for any $p = (A, P) \in \mathbf{P}$ with $\mid M \mid= 2$ and$\mid W \mid= 1$, we have $MO(p) = S(p) \cap MB(p)$.

Finally, consider any $p = (A, P) \in \mathbf{P}$ with $\mid M \mid= 2$ and $\mid W \mid= 2$. Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. First we will show that $MO(p) \subset MB(p)$. Let $MO(p) = \{\mu\}$. Without loss of generality, let $\mu(m_1) = w_1$ and $\mu(m_2) = w_2$. So $w_1 \in C(P, m_1)$ and $w_2 \in C(P, m_2)$. Suppose that $\mu \notin MB(p)$. Then $w_2\ P_{m_1}\ w_1\ P_{m_1}\ m_1$, $w_1\ P_{m_2}\ w_2\ P_{m_2}\ m_2$, $m_1 \in C(P, w_2)$ and $m_2 \in C(P, w_1)$. Consider $\widetilde{\mu}$ with $\widetilde{\mu}(m_1) = w_2$ and $\widetilde{\mu}(m_2) = w_1$. Since $\widetilde{\mu} \in S(p)$ and $\widetilde{\mu}(m_i)\ P_{m_i}\ \mu(m_i)$ for all $i \in \{1, 2\}$, we have a contradiction that $\{\mu\} = MO(p)$. Hence for any $p = (A, P) \in \mathbf{P}$ with $\mid M \mid= 2$ and $\mid W \mid= 2$, we have $MO(p) \subset MB(p)$. If $\mid S(p) \mid= 1$, we have $MO(p) = S(p) \cap MB(p)$. Next suppose that $\mid S(p) \mid= 2$. Note that $\mid S(p) \mid= 2$ if and only if we have $w_k\ P_{m_1}\ w_l\ P_{m_1}\ m_1$, $w_l\ P_{m_2}\ w_k\ P_{m_2}\ m_2$, $m_2\ P_{w_k}\ m_1\ P_{w_k}\ w_k$ and $m_1\ P_{w_l}\ m_2\ P_{w_l}\ w_l$ where $(k, l) \in \{(1, 2), (2, 1)\}$. Without loss of generality, let $k = 1$ and $l = 2$. One can easily check that $S(p) = \{\mu, \widetilde{\mu}\}$ and $MO(p) = \{\mu\}$. One can also easily check that $\widetilde{\mu} \notin MB(p)$. So, $MO(p) = S(p) \cap MB(p)$. Hence, for any $p = (A, P) \in \mathbf{P}$ with $\mid M \mid= 2$ and $\mid W \mid= 2$, we have $MO(p) = S(p) \cap MB(p)$. ■

Note that if agents are allowed to stay selfmatched, the core solution S still satisfies consistency (Toda (2006)) and converse consistency (Nizamogullari and Özkal-Sanver (2011)). We will also utilize from the fact that the men-barterproof solution satisfies consistency, as well as converse consistency.

Lemma 3.1 *The men-barterproof solution satisfies consistency.*

Proof. Take any $p = (A, P) \in \mathbf{P}$ and any $\mu \in MB(p)$. Suppose that there exists some $N' \subseteq A$ such that $\mu\mid_{N'} \notin MB(N' \cup \mu(N'), P\mid_{N' \cup \mu(N')})$. Then there exists a pair $\{m_1, m_2\} \subset M \cap (N \cup \mu(N))$ with $\{\mu(m_1), \mu(m_2)\} \subset W \cap (N \cup \mu(N))$ such that $\mu(m_2)\ P\mid_{N \cup \mu(N)\ m_1}\ \mu(m_1)$, $\mu(m_1)\ P\mid_{N \cup \mu(N)\ m_2}$

$\mu(m_2)$, $m_1 \in C(P\mid_{N\cup\mu(N)}, \mu(m_2))$, and $m_2 \in C(P\mid_{N\cup\mu(N)}, \mu(m_1))$. Straightforwardly, we have $\mu(m_2)\ P_{m_1}\ \mu(m_1)$, $\mu(m_1)\ P_{m_2}\ \mu(m_2)$, $m_1 \in C(P, \mu(m_2))$, and $m_2 \in C(P, \mu(m_1))$. So, it contradicts that $\mu \in MB(p)$. ■

Lemma 3.2 *The men-barterproof solution satisfies converse consistency.*

Proof. Take any $p = (A, P) \in \mathbf{P}$ and any $\mu \in \mathcal{M}(A)$ such that for all proper subsets $N \subsetneq A$ with $\mid N \mid = 2$, $\mu\mid_N \in MB(N \cup \mu(N), P\mid_{N\cup\mu(N)})$. Suppose that $\mu \notin MB(p)$, then there exists a pair $\{m_1, m_2\} \subset M$ and a pair $\{\mu(m_1), \mu(m_2)\} \subset W$ with $\mu(m_2)\ P_{m_1}\ \mu(m_1)$, $\mu(m_1)\ P_{m_2}\ \mu(m_2)$, $m_1\ P_{\mu(m_2)}\ \mu(m_2)$ and $m_2\ P_{\mu(m_1)}\ \mu(m_1)$. Taking $N' = \{m_1, m_2\}$, it contradicts that $\mu\mid_{N'} \in MB(N' \cup \mu(N'), P\mid_{N'\cup\mu(N')})$. ■

Now, we are ready to show that the minimal converse consistent extension of the men-optimal solution associates with each problem p the set consisting of the men-optimal matching $MO(p)$ and all stable and men-barterproof matchings $S(p) \cap MB(p)$.

Theorem 3.1 *The minimal converse consistent extension of the men-optimal solution is*

$$MCCE_{MO}(p) = MO(p) \cup [S(p) \cap MB(p)] \text{ for all } p \in \mathbf{P}.$$

Proof. By Proposition 3.1, for any $p = (A, P) \in \mathbf{P}$ with $\mid M \mid \leq 2, \mid W \mid \leq 2$, $[S(p) \cap MB(p)] = MO(p)$, and there is nothing to prove.

Let $p = (M \cup W, P)$ with $\max\{\mid M \mid, \mid W \mid\} > 2$. First note that, replacing φ with MO, we have

$$MCCE_{MO}(p) = MO(p) \cup \left\{ \begin{array}{l} \mu \in \mathcal{M}(A) : \mu\mid_N = MO(N \cup \mu(N), P\mid_{N\cup\mu(N)}) \\ \textit{for all subsets } N \subseteq M \cup W \text{ with } |N| = 2. \end{array} \right.$$

Letting $K(p) = \{\mu \in \mathcal{M}(A) : \mu\mid_N = MO(N \cup \mu(N), P\mid_{N\cup\mu(N)})$ for all subsets $N \subseteq M \cup W$ with $|N| = 2\}$, we need to show that $K(p) = S(p) \cap MB(p)$. Take any $\mu \in K(p)$. By its definition, for all subsets$N \subseteq M \cup W$ with $|N| = 2$, we have $\mu\mid_N = MO(N \cup \mu(N), P\mid_{N\cup\mu(N)}) \in \mathcal{M}(A)$. Again, by Proposition 3.1, we have $\mu\mid_N \in S(N \cup \mu(N), P\mid_{N\cup\mu(N)})$ and $\mu\mid_N \in MB(N \cup \mu(N), P\mid_{N\cup\mu(N)})$ for all subsets $N \subseteq M \cup W$ with $|N| = 2$. By converse consistency of S, $\mu \in S(p)$; and by converse consistency of MB, $\mu \in MB(p)$. So, we have $K(p) \subseteq [S(p) \cap MB(p)]$.

Next we want to show that $K(p) \supseteq S(p) \cap MB(p)$. Take any $\mu \in \mathcal{M}(A)$ with $\mu \mid_{N'} \neq MO\left(N' \cup \mu(N'), P \mid_{N' \cup \mu(N')}\right)$ for some subset $N' \subseteq M \cup W$ with $|N'| = 2$. Then either $\mu \mid_{N'} = WO(N' \cup \mu(N'), P \mid_{N' \cup \mu(N')})$ or $\mu \mid_{N'} \notin S(N' \cup \mu(N'), P \mid_{N' \cup \mu(N')})$. Suppose $\mu \mid_{N'} \notin S(N' \cup \mu(N'), P \mid_{N' \cup \mu(N')})$. Then, by consistency of S, $\mu \notin S(p)$. Next suppose that $\mu \mid_{N'} = WO(N' \cup \mu(N'), P \mid_{N' \cup \mu(N')}) \neq MO(N' \cup \mu(N'), P \mid_{N' \cup \mu(N')})$. By Proposition 3.1., $\mu \mid_{N'} \notin MB(N' \cup \mu(N'), P \mid_{N' \cup \mu(N')})$. Then, by consistency of MB, $\mu \notin MB(p)$. So we have $K(p) \supseteq [S(p) \cap MB(p)]$, which completes the proof. ■

Analogously, the minimal converse consistent extension of the women-optimal solution is a correspondence which associates with each problem p the set consisting of the women optimal matching $WO(p)$ and all stable and women-barterproof matchings $S(p) \cap WB(p)$.

Recall that for any $p = (A, P) \in \mathbf{P}$ with $|M| = |W|$ and for any $i \in A$, $C(P, i) = A(i)$, we have $MB(p) = SMB(p)$.

Corollary 3.1 *For any $p = (A, P) \in \mathbf{P}$ with $|M| = |W|$ and for any $i \in A$, $C(P, i) = A(i)$, the minimal converse consistent extension of the men-optimal solution is*

$$MCCE_{MO}(p) = MO(p) \cup [S(p) \cap SMB(p)].$$

Remark 3.1 *Through a slight modification of the problem in the example 3.2., one can easily check that $MO \cup [S \cap SMB]$ is not converse consistent if we allow that agents stay selfmatched. To see that, let $M = \{m_1, m_2, m_3, m_4, m_5, m_6\}$ and $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$. Let P be defined as follows:*

P_{m_1}	P_{m_2}	P_{m_3}	P_{m_4}	P_{m_5}	P_{m_6}	P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}	P_{w_5}	P_{w_6}
w_2	w_1	w_2	w_4	w_5	w_6	m_6	m_5	m_1	m_2	m_4	m_3
w_1	w_2	w_3	w_5	w_2	w_1	m_1	m_2	...	...	...	...
w_4	w_3	w_6	...	...	...	m_4	m_3	w_3	w_4		
w_3	w_4	...				m_2	m_1	m_2	m_1		

Let $p = (M \cup W, P)$. Consider the matching $WO(p) = \mu \neq MO(p)$ where $\mu(m_1) = w_3$, $\mu(m_2) = w_4$, $\mu(m_3) = w_6$, $\mu(m_4) = w_5$, $\mu(m_5) = w_2$, $\mu(m_6) = w_1$. Note that for each proper subset $N \subseteq M \cup W$ with $|N| = 2$, $\mu \mid_N \in MO(N \cup \mu(N), P \mid_{N \cup \mu(N)})$. However, $\mu \subsetneq SMB(p)$ since $w_4 \; P_{m_1} \; \mu(m_1)$ and $w_3 \; P_{m_2} \mu(m_2)$.

As the minimal converse consistent extension of the men-optimal solution always picks a subset of the core, one natural question to ask is whether $MCCE_{MO}$ prevail the lattice structure of the core. The answer turns out to be negative:

Let $p = (M \cup W, P)$. For any two matchings μ and μ', let $\lambda = \mu \vee_M \mu'$ be defined by

$$\lambda(m) = \begin{cases} \mu(m) & \text{for all } m \in M \text{ with } \mu(m)\ P_m\ \mu'(m) \text{ or } \mu(m) = \mu'(m) \\ \mu'(m) & \textit{for all } m \in M \text{ with } \mu'(m)\ P_m\ \mu(m) \end{cases}$$

Example 3.3 *There exists a problem* $p = (A, P) \in \mathbf{P}$ *where* $\{\mu, \mu'\} \subset MCCE_{MO}(p)$*, but* $\lambda = (\mu \vee_M \mu') \notin MCCE_{MO}(p)$*.*

Let $M = \{m_1, m_2, m_3, m_4, m_5, m_6\}$ and $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$. Let P be defined as follows:

P_{m_1}	P_{m_2}	P_{m_3}	P_{m_4}	P_{m_5}	P_{m_6}	P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}	P_{w_5}	P_{w_6}
w_5	w_6	w_4	w_3	w_1	w_2	m_3	m_1	m_2	m_6	m_4	m_5
w_1	w_2	w_3	w_4	w_5	w_6	m_1	m_2	m_3	m_4	m_5	m_6
w_2	w_3	w_1	w_5	w_6	w_4	...	...	...	...	...	...

Let $p = (M \cup W, P)$. Consider the matching $\mu \in S(p) \cap MB(p)$ where $\mu(m_1) = w_1$, $\mu(m_2) = w_2$, $\mu(m_3) = w_3$, $\mu(m_4) = w_5$, $\mu(m_5) = w_6$, $\mu(m_6) = w_4$. Next consider the matching $\mu' \in S(p) \cap MB(p)$ where $\mu'(m_1) = w_2$, $\mu'(m_2) = w_3$, $\mu'(m_3) = w_1$, $\mu'(m_4) = w_4$, $\mu'(m_5) = w_5$, $\mu'(m_6) = w_6$. Letting $\lambda = \mu \vee_M \mu'$, we have $\lambda(m_i) = w_i$ for all $i \in \{1, 2, ..., 6\}$. First note that $\lambda \neq MO(p)$. Since $\lambda(m_4)\ P_{m_3}\ \lambda(m_3)$ and $\lambda(m_3)\ P_{m_4} \lambda(m_4)$, we have $\lambda \notin MB(p)$. Hence $\lambda \notin MCCE_{MO}(p)$.[10]

4 Concluding Remarks

In this paper, we compute the minimal converse consistent extension of the men-optimal solution $MCCE_{MO}$, as a correspondence which associates with each problem p the set consisting of the men optimal matching $MO(p)$ and all stable and men-barterproof matchings $S(p) \cap MB(p)$.

[10] This result prevails if we restrict our model so that we have equal number of men and women and no agent stays selfmatched.

By definition, $MCCE_{MO}$ satisfies converse consistency. Since $MCCE_{MO}$ always picks a subset of the core, it satisfies Pareto optimality, as well. Considering the problem in Example 2.1, $MCCE_{MO}$ can pick a strict subset of the core, hence it fails to satisfy consistency (Toda 2006).[11] One can easily show that it is an anonymous correspondence. However, it fails to satisfy men-barterproofness, since for some problems men-optimal matching may not be men-barterproof. (See Examples 3.2 and 3.3).

The axiomatic characterization of the minimal converse consistent extension of the men-optimal solution may be interesting since men-optimal and men-barterproof matchings have very different characteristics.

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[11] Through the example 2.1., one can easily check that the set of men-barterproof matchings may be a strict subset of the core. Here, we have $S(p) = \{MO(p), \mu^*, WO(p)\}$ and $MB(p) = \{MO(p), \mu^*\}$.

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