A Characterization of Implementability of Allocation Rules: the Use of a Menu of Three-Part Tariffs

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Very Preliminary

Abstract

Posting a menu of three-part tariffs is a widespread phenomenon in the telecommunication industry. The paper provides a characterization of the implementability of allocation rules involving bunching in the linear framework of Mussa and Rosen [5]. I show that an allocation rule is non-decreasing if and only if it is implementable by a menu of three-part tariffs. Furthermore, I extend the notion of implementability in Rochet [7, 8] to take into account type-dependent reservation utility, introducing the notion of voluntary implementability. I show that a pair of an allocation rule and an information rent is incentive compatible and individually rational if and only if it is voluntarily implementable by a menu of three-part tariffs.

Keywords: nonlinear pricing · three-part tariffs · taxation principle · bunching

1 Introduction

I consider the canonical model of contracting by a principal and an agent under asymmetric information. I assume that private information of the agent to be one-dimensional and a single-crossing property is satisfied such as in a vertical differentiation model due to Mussa and Rosen [5]. From the principal's perspective, the agent's type is drawn according to a distribution function. A strategy of the principal is a nonlinear pricing schedules defined over a product line. Nonlinear pricing schedules are *indirect mechanisms* not indexed by agents' types. An

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allocation rule is a mapping that assigns a quality level for each type. An allocation rule is said to be *implementable* if it is given as the agent's optimization behavior induced by a nonlinear pricing schedule. The principal determines a price schedule so as to maximize his expected profit from trade, subject to the implementability constraints and the participation constraints.

The standard approach to the screening problem is reformulating the principal's strategy space. If an allocation rule is implementable by a price schedule solving the principal's optimization problem, then a transfer function is constructed as the composite of the price schedule and the allocation rule. By construction, the *direct mechanism* consisting of the allocation rule and the transfer function is *incentive compatible* in the sense that it is optimal for the agent to announce the true value of his private information. This is so-called the *revelation principle*.[1] An allocation rule is said to be *rationalizable* if there is an associated transfer function inducing truthful revelation by the agent. The problem is rewritten as an expected profit maximization problem over the set of allocations and transfers satisfying the incentive compatible constraints and the participation constraints.

Despite of prevalence in practice, the character of nonlinear pricing remains largely unexplored because of the intensive use of direct mechanisms and the application of the revelation principle. The seminar article in the literature on nonlinear pricing, Mussa and Rosen [5], predict that the principal's optimal quality allocation exhibits no-distortion for the highest type and downward distortions for all other types when the participation constraints are type-independent. When the information rent, that is, the utility the agent gets in excess of the reservation utility, is monotone, there is no pooling if the monotone hazard rate condition is satisfied. Maggi-Rodriguez-Clare [3] consider a model where the reservation utility depends on private information as well. They argue the structure of optimal contracts, in particular the occurrence of pooling, crucially depends on the shape of the reservation utility even when the monotone hazard rate condition is satisfied. The literature has been argued a pattern of distortions and a possibility of pooling or bunching. These are, however, merely properties of allocation rules.

The revelation principle states that if an allocation rule is obtained as a result of the agent's optimization problem under a nonlinear pricing, then we are always able to construct a transfer function that is a function of private information so that the menu of allocation rule and transfer function is incentive compatible. If we have a procedure to recover a nonlinear price function from such incentive compatible direct mechanism, then it means to claim that, without loss of generality, we can restrict attention to incentive compatible direct mechanisms. The reverse of the revelation principle is known as the *taxation principle*. The purpose of the paper is to establish the taxation principle for any allocation rules possibly involving bunching. It is crucial to determine a type assignment for each consumption level, instead of using the inverse function of the allocation rule. In Section 5, I extend the notion of implementability to take into account type-dependent reservation utility, introducing the notion of *voluntary implementability*. I show that a pair of an allocation rule and an information rent is incentive compatible and individually rational if and only if it is voluntarily implementable by a single price schedule (Theorem 3).

[1] For instance, see Laffont and Martimort [1, Proposition 2.2].

Sorting consumers with nonlinear pricing is a widespread phenomenon. In Section 3, I consider the case that the principal is able to post a menu of nonlinear pricing schedules. Posting a menu of *three-part tariffs* is popular in the telecommunication industry. A three-part tariff consists of an access fee (a fixed monthly fee), a usage allowance (free minutes), and a marginal price for usage beyond the allowance as shown in Figure 1.

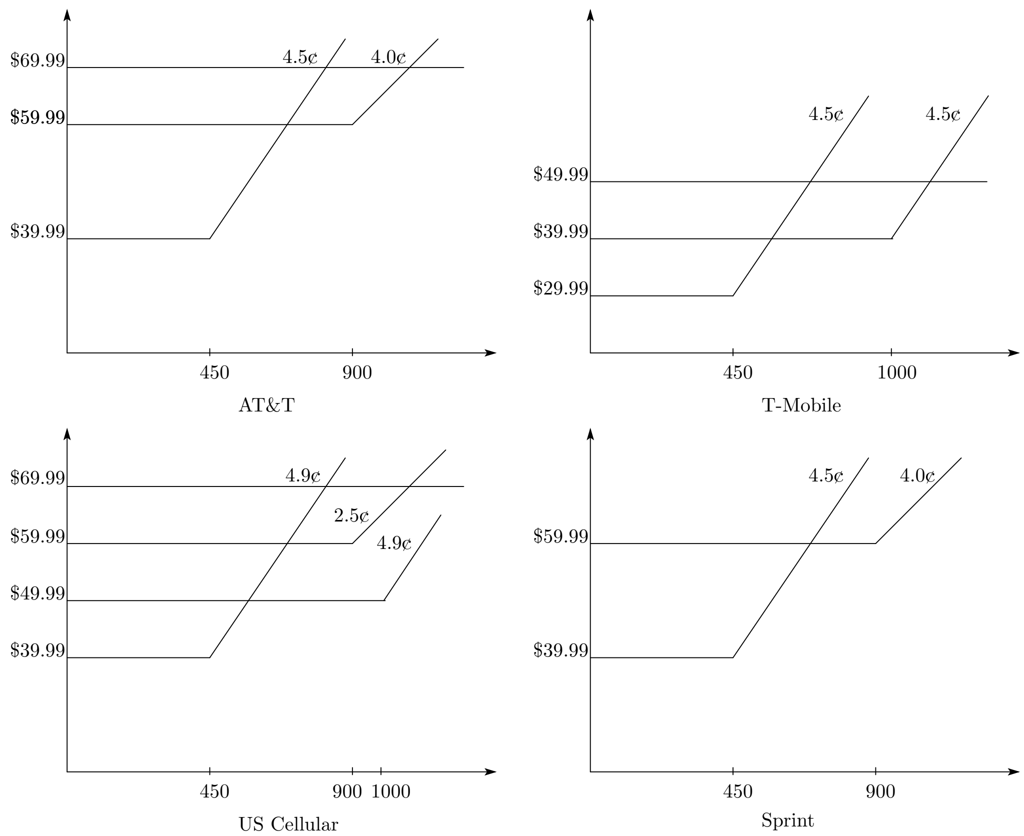


Figure 1: Calling Plans in the U.S.

Obviously, each three-part tariff is a non-decreasing, continuous and convex price schedule. My question is whether I can establish the taxation principle within a set of such tractable price schedules. The usual statement of the taxation principle does not restrict a class of price schedules: given an allocation rule, how to construct a price schedule that induces the allocation rule as a solution to utility maximization problem of the agent if the principal can use any price schedule. I show that the implementability by a menu of three-part tariffs and the monotonicity of allocation rules are equivalent (Theorem 1). Thus, any non-decreasing allocation rule is achievable by a menu of simple continuous and convex price schedules. I also argue whether I can restrict attention to convex price schedules in the standard framework of Mussa and Rosen [5]. I show that no concave price schedule satisfies the implementability requirement (Theorem 2). Furthermore, I show that a pair of an allocation rule and an information rent is incentive compatible and individually rational if and only if it is voluntarily implementable by a menu of three-part tariffs (Theorem 4).

Rochet [8] shows the equivalence between the rationalizability and a cyclical monotonicity of allocation rules in a quasi-linear context of multi-dimensional types. In particular, when private information of the agent is assumed to be one-dimensional, and the single-crossing property is satisfied, then an allocation rule is cyclically monotone if and only if it is non-decreasing. The taxation principle in the paper is valid for any non-decreasing allocation rule, so I establish the equivalence between the rationalizability and the implementability for any non-decreasing allocation rule (Corollary 1). I claim that it is reasonable to restrict attention to a class of menus of three-part tariffs for analysis because if an allocation rule is not implementable by *any* menu of three-part tariffs, then it cannot be rationalizable by *any* transfer function.

2 Model

I consider a simple economy with two commodities and consumers. A commodity in question is a one-dimensional variable x, referred to as quality, and the other commodity is a composite good y. Denote by $X \subseteq \mathbb{R}_+$ the range of qualities or the product line. Consumers' taste for quality, referred to as type, is represented by a one-dimensional parameter θ that belongs to a compact set $\Theta \subseteq \mathbb{R}_{++}$. Ignoring income effects, I consider a situation where consumers have a quasi-linear utility function $u(x, \theta) + y$, where $u(x, \theta) = x\theta$ as in Mussa and Rosen [5] and Rochet and Stole [9]. The single-crossing property is automatically satisfied. Given a non-linear price-quality function $t : X \to \mathbb{R}$, hereafter a *price schedule*, a consumer of type θ maximizes his net utility $u(x, \theta) - t(x)$ over X.

The present paper constructs a menu of price schedules under which any non-decreasing allocation rule $x : \Theta \to X$ emerges as a solution to utility maximization problem of consumers. For comparison, the literature has been focused on the following incentive compatibility through a transfer function defined over the type space.

Definition 1 (Rochet [8])**.** An allocation rule $x(\theta)$ is said to be *rationalizable* if there exists a *transfer function* $p : \Theta \to \mathbb{R}$ such that the direct mechanism $\{x(\theta), p(\theta)\}$ induces truthful revelation: $\theta \in$ argmax$[u(x(\hat{\theta}), \theta) - p(\hat{\theta}) \mid \hat{\theta} \in \Theta]$ for every $\theta \in \Theta$.

It is known that under the single-crossing property, incentive compatibility is characterized as follows.

Remark 1 (Rochet [8], Proposition 1)**.** *An allocation rule is rationalizable if and only if it is non-decreasing.*

On the other hand, my concern is the implementability in the following sense.

Definition 2 (Rochet [8])**.** An allocation rule $x(\theta)$ is said to be *implementable* if there exists a price schedule $t(x)$ such that $x(\theta) \in$ argmax$[u(x, \theta) - t(x) \mid x \in X]$.

The fact that the implementability implies the rationalizability is known as the *revelation principle*. For any invertible allocation rule, Rochet [7, Proposition 2] shows that the rationalizability implies the implementability. This result is called the *taxation principle*. The invertibility of allocation rule, however, might not be satisfied when the agent has a type-dependent reservation utility (see Maggi and Rodriquez-Clare [3] for the single-principal and single-agent contracting problem, and Watabe [10] for the case of several principals and one agent). Lewis and Sappington [2] also develop a model in which pooling is optimal.

3 Implementation by a Menu of Three-Part Tariffs

In this section, I state my version of the taxation principle by a menu of three-part tariffs. In the literature, the principal is assumed to post a *single* tariff. Posing a menu of *three-part tariffs* is popular in the telecommunication industry. A three part tariff consists of an access fee (a fixed monthly fee), a usage allowance (free minutes), and a marginal price for usage beyond the allowance. Three-part tariffs are *indirect* mechanisms not indexed by types. Denote by $Y \subseteq X$ the set of admissible levels of usage allowance.

Definition 3. A schedule $T : X \times Y \to \mathbb{R}$ is called a *three-part tariff* if there exists $(a(y), b(y)) \in \mathbb{R}^2$ such that

$$T(x, y) = \begin{cases} a(y) & \text{if } x \leqslant y \\ a(y) + b(y)(x - y) & \text{if } x > y. \end{cases}$$

When consumers are facing a menu of three-part tariffs, they have two choices: a choice *among* three-part tariffs and a choice *along* a chosen three-part tariff. I propose the following alternative implementability concept when the principal is able to post a menu of price schedules.

Definition 4. An allocation rule $x(\theta)$ is said to be *implementable by a menu of three-part tariffs* $\{T(x, y)\}_{y \in Y}$ if $x(\theta) \in \text{argmax}[u(x, \theta) - T(x, y(\theta))) \mid x \in X]$ for some $y(\theta) \in Y$.

I obtain the following characterization result on the implementability by a menu of three-part tariffs without excluding bunching. The theorem below is established by constructing a menu of three-part tariffs explicitly.

Theorem 1. *An allocation rule is implementable by a menu of three-part tariffs if and only if it is non-decreasing.*

Proof. See the Appendix. □

This theorem has an implication for the validity of the revelation principle in the context of nonlinear pricing. If an allocation rule is not implementable by *any* menu of three-part tariffs, then it cannot be rationalizable by *any* transfer function because the rationalizability and the monotonicity are equivalent by Remark 1. This is a violation of the revelation principle.

According to the context, there is a priori restriction to the class of nonlinear schedules, for instance, convex or concave. In the proof of Theorem 1, I construct a single convex price schedule (see Remark 2 in the Appendix) and a menu of three-part tariffs. It is obvious that any three-part tariff is convex as well. It is natural to seek for a solution among convex price schedules. Since the quasi-concavity of $u(x,\theta)+y$ in (x,y) is equivalent to the concavity of $u(x,\theta)$ in x, it follows that the first-order condition $u_x(x,\theta) = Dt(x)$ makes sense for *any* convex price schedule $t(x)$.

It must be emphasized that the implementability by a concave price schedule is not excluded by my argument up to here. The following theorem states that, without loss of generality, I can restrict attention to convex price schedules.

Theorem 2. *If an allocation rule $x(\theta)$ is continuous and its image of Θ is not degenerate, then it is not implementable by any concave price schedule.*[2]

Proof. Suppose, by way of contradiction, that $x(\theta)$ is implementable by a concave price schedule $t(x)$. By the revelation principle, the transfer function $p(\theta) = t(x(\theta))$ rationalizes the allocation rule $x(\theta)$. Since $x(\Theta)$ is not a singleton and $x(\theta)$ is non-decreasing by Remark 1, it follows from the fact $\underline{\theta} < \overline{\theta}$ that $x(\underline{\theta}) < x(\overline{\theta})$. Denote $\beta = \frac{t(x(\overline{\theta}))-t(x(\underline{\theta}))}{x(\overline{\theta})-x(\underline{\theta})}$.

If $\beta \leqslant 0$, then $p(\underline{\theta}) = t(x(\underline{\theta})) \geqslant t(x(\overline{\theta})) = p(\overline{\theta})$, and hence $u(x(\overline{\theta}),\underline{\theta})-p(\overline{\theta}) > u(x(\underline{\theta}),\underline{\theta})-p(\underline{\theta})$. This is a contradiction. Therefore, it must be the case $\beta > 0$. In other words, $t(x(\overline{\theta})) > t(x(\underline{\theta}))$. By the concavity of the price schedule $t(x)$, if $\underline{\theta} \geqslant \beta$ then $\{\overline{\theta}\} = \text{argmax}[u(x(\hat{\theta}),\theta) - p(\hat{\theta}) \mid \hat{\theta} \in \Theta]$ for every $\theta \neq \underline{\theta}$, as shown in Figure 2 (a). If $\underline{\theta} > \beta$, then announcing $\overline{\theta}$ is the unique best response for type $\underline{\theta}$ as well, so that $x(\Theta) = \{x(\overline{\theta})\}$, a contradiction. Suppose $\overline{\theta} = \beta$. If $t(x)$ is strictly concave, then $\{\underline{\theta},\overline{\theta}\} = \text{argmax}[u(x(\hat{\theta}),\underline{\theta}) - p(\hat{\theta}) \mid \hat{\theta} \in \Theta]$. The choice of $\underline{\theta}$ yields a discontinuity of the allocation rule $x(\theta)$ at the bottom, while the choice of $\overline{\theta}$ yields the degenerate allocation rule again. In any case, there is a contradiction. When $t(x)$ is not strictly concave, the concavity of $t(x)$, together with $\underline{\theta} = \beta$, yields that $t(x)$ is linear over $[x(\underline{\theta}), x(\overline{\theta})]$. It must be the case $[\underline{\theta},\overline{\theta}] = \text{argmax}[u(x(\hat{\theta}),\underline{\theta}) - p(\hat{\theta}) \mid \hat{\theta} \in \Theta]$. Anyway, $x(\Theta)$ cannot be the degenerate interval $[x(\underline{\theta}), x(\overline{\theta})]$, a contradiction.

Suppose, next, that $\overline{\theta} \leqslant \beta$. Then, $\{\underline{\theta}\} = \text{argmax}[u(x(\hat{\theta}),\theta) - p(\hat{\theta}) \mid \hat{\theta} \in \Theta]$ for every $\theta \neq \overline{\theta}$, as shown in Figure 2 (b). Similarly, it is shown that the allocation rule $x(\theta)$ is either discontinuous at the top or degenerate. This case is also ruled out.

It remains to consider the case $\underline{\theta} < \beta < \overline{\theta}$. In this case, type β is indifferent between $x(\underline{\theta})$ and $x(\overline{\theta})$. Similar to the above argument, $\{\underline{\theta}\} = \text{argmax}[u(x(\hat{\theta}),\theta) - p(\hat{\theta}) \mid \hat{\theta} \in \Theta]$ for every $\theta < \beta$, while $\{\overline{\theta}\} = \text{argmax}[u(x(\hat{\theta}),\theta) - p(\hat{\theta}) \mid \hat{\theta} \in \Theta]$ for every $\theta > \beta$. Hence, the allocation rule $x(\theta)$ is discontinuous at $\theta = \beta$, a contradiction. This establishes the proposition. □

[2] Mussa and Rosen [5] show the continuity of the optimal contract in their model.

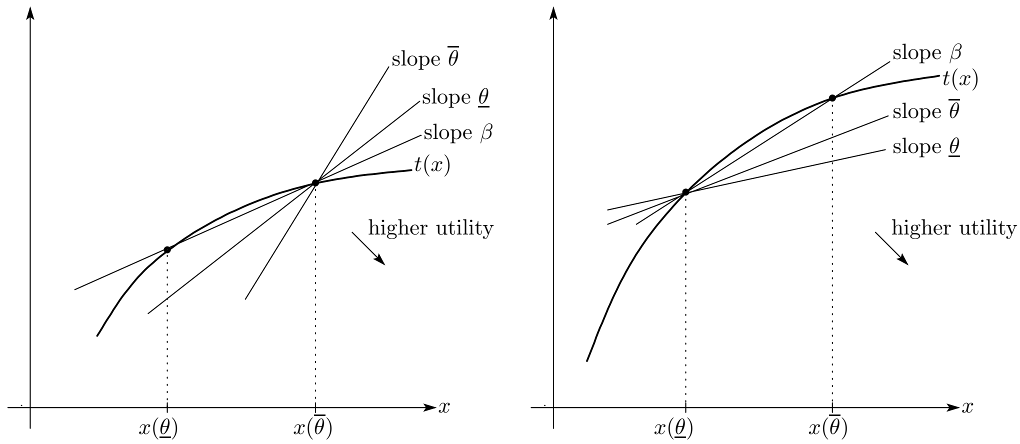


Figure 2: (a) $\underline{\theta} \geqslant \beta$ and (b) $\overline{\theta} \leqslant \beta$

The following characterization result is immediate from Theorem 1 and Remark 1.

Corollary 1. *The following statements are equivalent:*
(1) An allocation rule is non-decreasing.
(2) An allocation rule is rationalizable by a transfer function.
(3) An allocation rule is implementable by a menu of three-part tariffs.

4 An Illustrative Example

In this section, I present the menu of three-part tariffs constructed in the proof of the theorem and some remarks.

Example 1. Let $C(x)$ be a quadratic cost function of the principal with $DC(x) = a + bx$ for some $a \geqslant 0$ and $b > 0$. Taste θ is uniformly distributed over $\Theta = [\underline{\theta}, \overline{\theta}]$ such that $\underline{\theta} > a$, and $\theta^* = \frac{1}{2}(\overline{\theta} + a) > \underline{\theta}$. According to the argument by Mussa and Rosen [5], the principal's optimal allocation rule is given by

$$x(\theta) = \begin{cases} 0 & \text{for } \theta < \theta^* \\ \frac{1}{b}(2\theta - (\overline{\theta} + a)) & \text{for } \theta \geqslant \theta^* \end{cases}$$

when the reservation utility of the agent is constant over types. The interpretation is that consumers for whom $\theta < \theta^*$ are excluded from the market. Note that $x(\theta^*) = 0$ so that allocation rule is continuous.

The social surplus is defined by $u(x, \theta) - C(x)$, and the full-information allocation rule is given by $x^*(\theta) \in \text{argmax}[u(x, \theta) - C(x) \mid x \in X]$. The first-order condition yields that $x^*(\theta) = \frac{1}{b}(\theta - a)$ for all $\theta \in \Theta$. For the highest type $\overline{\theta}$, $x(\overline{\theta}) = \frac{1}{b}(\overline{\theta} - a) = x^*(\overline{\theta})$. For any $\theta < \overline{\theta}$, $x(\theta) = \frac{1}{b}(2\theta - (\overline{\theta} + a)) < \frac{1}{b}(2\theta - (\theta + a)) = \frac{1}{b}(\theta - a) = x^*(\theta)$. In other words, the allocation rule $x(\theta)$ exhibits no-distortion at the top and downward distortions for all other types. Figure 3 (a) shows the pattern of distortions and pooling at $x = 0$.

Theorem 1 is established by the following construction. For each (x, y) of element of $X \times Y$, define

$$T(x,y) = \begin{cases} t(y) & \text{if } x \leqslant y \\ t(y) + \overline{\theta}(x-y) & \text{if } x > y, \end{cases}$$

where

$$t(x) = \max \left[u(x, \hat{\theta}) - \int_{\theta^*}^{\hat{\theta}} u_\theta(x(s), s) ds \,\middle|\, \hat{\theta} \in \Theta \right].$$

Denote by $X = [x(\underline{\theta}), x(\overline{\theta})]$, where $x(\underline{\theta}) = 0$, the corresponding product line. Consider the case $Y = X$.

For each $x \in X$, the first-order condition with respect to $\hat{\theta}$ in the auxiliary price schedule $t(x)$, $0 = x - \frac{1}{b}(2\hat{\theta} - (\overline{\theta} + a))$, gives the optimal type assignment schedule $\gamma(x) = \frac{1}{2}(bx + \overline{\theta} + a)$. The resulting price schedule is

$$t(x) = u(x, \gamma(x)) - \int_{\theta^*}^{\gamma(x)} u_\theta(x(s), s) ds.$$

Here, I see that

$$\begin{aligned} &\int_{\theta^*}^{\gamma(x)} u_\theta(x(s), s) ds = \int_{\theta^*}^{\gamma(x)} x(s) ds = \tfrac{1}{b} \int_{\theta^*}^{\gamma(x)} (2s - (\overline{\theta} + a)) ds = \tfrac{1}{b}(s^2 - (\overline{\theta} + a)s) \Big|_{\theta^*}^{\gamma(x)} \\ &= \tfrac{1}{b} s(s - (\overline{\theta} + a)) \Big|_{\frac{1}{2}(\overline{\theta}+a)}^{\frac{1}{2}(bx+\overline{\theta}+a)} = \tfrac{1}{2b}(bx + \overline{\theta} + a) \cdot \tfrac{1}{2}(bx - (\overline{\theta} + a)) - \tfrac{1}{2b}(\overline{\theta} + a) \left(-\tfrac{1}{2}(\overline{\theta} + a) \right) \\ &= \tfrac{1}{4b}(bx + \overline{\theta} + a)(b - (\overline{\theta} + a)) + \tfrac{1}{4b}(\overline{\theta} + a)^2 = \tfrac{1}{4} b x^2. \end{aligned}$$

Hence, the price schedule is given by

$$t(x) = \tfrac{1}{2} x(bx + \overline{\theta} + a) - \tfrac{1}{4} bx^2 = \tfrac{1}{4} bx^2 + \tfrac{1}{2} x(\overline{\theta} + a),$$

and the marginal price schedule becomes $Dt(x) = \frac{1}{2}(bx + \overline{\theta} + a)$, which is consistent with the consequence of the envelope theorem, $Dt(x) = u_x(x, \gamma(x)) = \gamma(x)$. I conclude that the price schedule is increasing. Furthermore, the price schedule $t(x)$ is strictly convex because $D^2 t(x) = D\gamma(x) = \frac{b}{2} > 0$.

Nöldeke and Samuelson [6] reformulate the standard monopolistic screening problem as the *type-assignment* problem subject to the class of price schedules satisfying the envelope condition $Dt(x) = u_x(x, \gamma(x))$ for some non-decreasing function $\gamma : X \to \Theta$. I show that the type-assignment function $\gamma : X \to \Theta$ is non-decreasing indeed in the proof of Theorem 1.

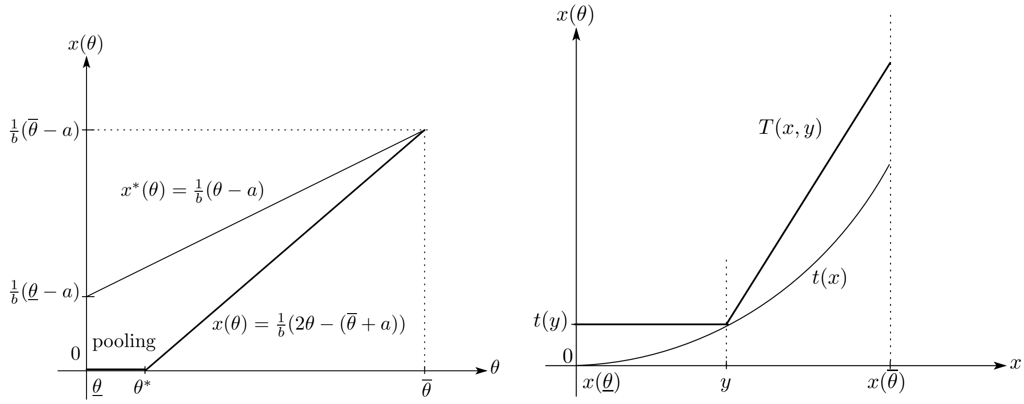
It turns out that the price schedule $t(x)$ is a lower envelope of the menu of three-part tariffs as shown in Figure 3 (b). Consider any $y \in X$. For every $x \leqslant y$, since $t(x) \leqslant t(y)$ by the monotonicity of $t(x)$, and $t(y) = T(x, y)$ by construction, it follows that $t(x) \leqslant T(x, y)$. For every $x \geqslant y$,

$$\begin{aligned} T_x(x, y) - Dt(x) &= \overline{\theta} - \tfrac{1}{2}(bx + \overline{\theta} + a) \geqslant \overline{\theta} - \tfrac{1}{2}(bx(\overline{\theta}) + \overline{\theta} + a) \\ &= \overline{\theta} - \tfrac{1}{2}(2\overline{\theta} - (\overline{\theta} + a) + (\overline{\theta} + a)) = 0. \end{aligned}$$

The above inequality yields that the price schedule does not intersect the three-part tariff $T(x, y)$ from below over the interval $[y, x(\overline{\theta})]$. Actually, for every $x \geqslant y$,

$$\begin{aligned} T(x, y) - t(x) &= \left[\tfrac{1}{4}by^2 + \tfrac{1}{2}y(\overline{\theta} + a) + \overline{\theta}(x - y)\right] - \left[\tfrac{1}{4}bx^2 + \tfrac{1}{2}x(\overline{\theta} + a)\right] \\ &= (x - y)\left(\tfrac{1}{2}(\overline{\theta} - a) - \tfrac{b}{4}(x + y)\right) = \tfrac{1}{4}(x - y)(2(\overline{\theta} - a) - b(x + y)) \\ &= \tfrac{1}{4}(x - y)(2bx(\overline{\theta}) - b(x + y)) \geqslant \tfrac{1}{4}(2bx(\overline{\theta}) - 2bx(\overline{\theta})) = 0, \end{aligned}$$

where the last inequality holds from the fact that $x, y \leqslant x(\overline{\theta})$. Therefore, $T(x, y) \geqslant t(x)$ for every $x \geqslant y$. In other words, the menu of three-part tariffs is supported by a single convex price schedule.



(a) Allocation rules (b) Price Schedules

Figure 3: (a) Allocation rules and (b) price schedules

By Theorem 1, the menu of three-part tariffs implements the allocation rule $x(\theta)$. So does the price schedule $t(x)$. Since the price schedule $t(x)$ is strictly convex, it follows that the net utility $u(x, \theta) - t(x)$ is a strictly concave function in x. Consider, first, any $\theta \geqslant \theta^*$. The first-order condition with respect to x is given by $0 = u_x(x, \theta) - Dt(x) = \theta - \left(\frac{1}{2}bx + \frac{1}{2}(\overline{\theta} + a)\right)$, and so $x = \frac{1}{b}(2\theta - (\overline{\theta} + a)) = x(\theta)$ is the solution to utility maximization problem of type $\theta \geqslant \theta^*$.

Consider, next, any $\theta \leqslant \theta^*$. Similarly, $u_x(x, \theta) - Dt(x) = \theta - \left(\frac{1}{2}bx + \frac{1}{2}(\overline{\theta} + a)\right) \leqslant \theta^* - \frac{1}{2}bx - \theta^* = -\frac{1}{2}bx \leqslant 0$ with equality at $x = 0$. Since the net utility is strictly concave in x, it follows that $x = 0$ is the only solution to utility maximization problem of type $\theta \leqslant \theta^*$.

Moreover,

$$u(x(\theta),\theta) = \frac{1}{b}\left(2\theta^2 - \theta(\overline{\theta}+a)\right)$$

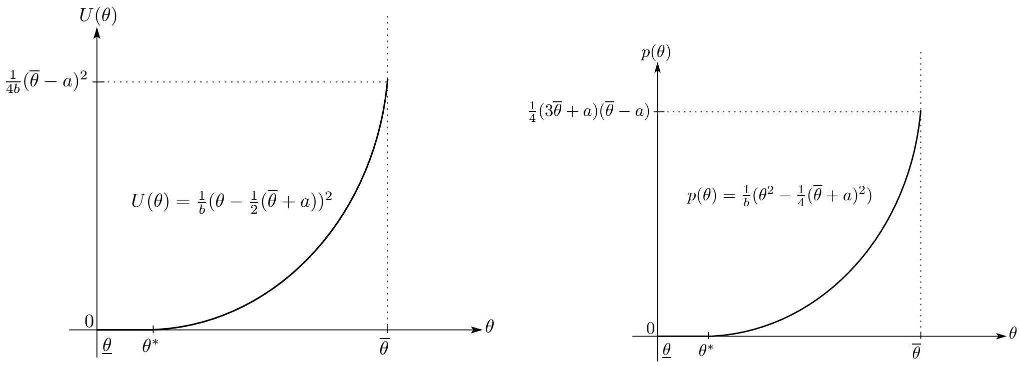
and

$$\begin{aligned} t(x(\theta)) &= \frac{1}{4b}(2\theta-(\overline{\theta}+a))^2 + \frac{1}{2b}(2\theta-(\overline{\theta}+a))(\overline{\theta}+a) = \frac{1}{b}\left(\theta^2 - \frac{1}{4}(\overline{\theta}+a)^2\right) \\ &= \frac{1}{4b}(2\theta+\overline{\theta}+a)(2\theta-\overline{\theta}-a). \end{aligned}$$

Therefore, the maximized utility is given by

$$U(\theta) = u(x(\theta),\theta) - t(x(\theta)) = \frac{1}{b}\left(\theta^2 - \theta(\overline{\theta}+a) + \frac{1}{4}(\overline{\theta}+a)^2\right) = \frac{1}{b}\left(\theta - \frac{1}{2}(\overline{\theta}+a)\right)^2.$$

When the reservation utility is zero for all of types, the information rent is non-decreasing for every θ because $\dot{U}(\theta) = \frac{2}{b}(\theta - \frac{1}{2}(\overline{\theta}+a)) \geqslant \frac{2}{b}(\theta^* - \frac{1}{2}(\overline{\theta}+a)) = 0$ for every $\theta \geqslant \theta^*$. In addition, $\dot{U}(\theta) = \frac{2}{b}(\theta - \frac{1}{2}(\overline{\theta}+a)) = x(\theta) = u_\theta(x(\theta),\theta)$, which is the usual envelope condition for incentive compatibility. I can see that the induced transfer function $p(\theta) = t(x(\theta))$ is increasing and convex because $\dot{p}(\theta) = \frac{2\theta}{b} > 0$ and $\ddot{p}(\theta) = \frac{2}{b} > 0$. Figure 4 depicts the information rent and the transfer function. Furthermore, the allocation rule $x(\theta)$ is rationalized by the transfer function $p(\theta)$. □



(c) Information rent (d) Transfer function

Figure 4: (a) information rent and (b) transfer function

5 Voluntary Implementability

The agent may have an outside opportunity, from which he can derive a utility level $\bar{\pi}(\theta)$. Let $\pi_t(\theta) = \max[u(x,\theta) - t(x) \mid x \in X]$. I replace the participation constraints with the system of inequalities, $r(\theta) = \pi_t(\theta) - \bar{\pi}(\theta) \geqslant 0$ for every $\theta \in \Theta$. Profit margin is written as $t(x(\theta)) - C(x(\theta)) = u(x(\theta),\theta) - \pi_t(\theta) - C(x(\theta)) = v(x(\theta),\theta) - r(\theta) - \bar{\pi}(\theta)$. Moreover, if $x(\theta) \in \text{argmax}[u(x,\theta) - t(x) \mid x \in X]$ under the price schedule t, then $\theta \in \text{argmax}[u(x(\hat{\theta}),\theta) - t(x(\hat{\theta})) \mid \hat{\theta} \in \Theta]$. Using the information rent $r(\theta) = \pi_t(\theta) - \bar{\pi}(\theta)$, the payment is written as $t(x(\hat{\theta})) = u(x(\hat{\theta}), \theta) - \pi_t(\hat{\theta}) = u(x(\hat{\theta}), \theta) - (r(\hat{\theta}) + \bar{\pi}(\theta))$. A profile $\{x(\theta), r(\theta)\}_{\theta\in\Theta}$ of an allocation rule and an information rent is *incentive compatible* in the sense that $\theta \in \operatorname{argmax}[u(x(\hat{\theta}), \theta) - u(x(\hat{\theta}), \hat{\theta}) + r(\hat{\theta}) + \bar{\pi}(\hat{\theta}) \mid \hat{\theta} \in \Theta]$ for every $\theta \in \Theta$.

The principal's problem can be written as

$$\max_{\{x(\cdot), r(\cdot)\}} \int_\Theta [v(x(\theta), \theta) - r(\theta) - \bar{\pi}(\theta)] f(\theta) d\theta$$

subject to the participation constraints $r(\theta) \geqslant 0$ for every $\theta \in \Theta$, and the incentive constraints $\theta \in \operatorname{argmax}[u(x(\hat{\theta}), \theta) - u(x(\hat{\theta}), \hat{\theta}) + r(\hat{\theta}) + \bar{\pi}(\hat{\theta}) \mid \hat{\theta} \in \Theta]$ for every $\theta \in \Theta$. The monotonicity of the information rent $r(\theta)$ is not guaranteed in general. Finally, a profile $\{x(\theta), r(\theta)\}_{\theta\in\Theta}$ is *individually rational* if $r(\theta) \geqslant 0$ for every $\theta \in \Theta$.

The following lemma shows the equivalence between the incentive constraints in terms of $\{x(\theta), r(\theta)\}_{\theta\in\Theta}$, and the monotonicity condition of $x(\theta)$ together with the envelope condition of $r(\theta)$.

Lemma 1. *A profile $\{x(\theta), r(\theta)\}_{\theta\in\Theta}$ of an allocation rule and an information rent is incentive compatible if and only if $x(\theta)$ is non-decreasing and $\dot{r}(\theta) = u_\theta(x(\theta), \theta) - \dot{\bar{\pi}}(\theta)$ for every $\theta \in \Theta$.*

Proof. Suppose, first, that $\{x(\theta), r(\theta)\}_{\theta\in\Theta}$ is incentive compatible. Consider any pair $(\theta, \hat{\theta})$ of elements of Θ with $\theta > \hat{\theta}$. Let $U(\hat{\theta}, \theta) = u(x(\hat{\theta}), \theta) - u(x(\hat{\theta}), \hat{\theta}) + r(\hat{\theta}) + \bar{\pi}(\hat{\theta})$ and $U(\theta) = U(\theta, \theta)$. I see that $U(\theta) = r(\theta) + \bar{\pi}(\theta) \geqslant u(x(\hat{\theta}), \theta) - u(x(\hat{\theta}), \hat{\theta}) + r(\hat{\theta}) + \bar{\pi}(\hat{\theta}) = U(\hat{\theta}) + u(x(\hat{\theta}), \theta) - u(x(\hat{\theta}), \hat{\theta})$. Similarly, $U(\hat{\theta}) = r(\hat{\theta}) + \bar{\pi}(\hat{\theta}) \geqslant u(x(\theta), \hat{\theta}) - u(x(\theta), \theta) + r(\theta) + \bar{\pi}(\theta) = U(\theta) + u(x(\theta), \hat{\theta}) - u(x(\theta), \theta)$. These imply that $u(x(\theta), \theta) - u(x(\theta), \hat{\theta}) \geqslant U(\theta) - U(\hat{\theta}) \geqslant u(x(\hat{\theta}), \theta) - u(x(\hat{\theta}), \hat{\theta})$. Thus, $(x(\theta) - x(\hat{\theta}))(\theta - \hat{\theta}) \geqslant 0$. Since $\theta > \hat{\theta}$, it follows that $x(\theta)$ is non-decreasing.

I show the envelope condition. For each $\theta \in \Theta$, define $\xi_t(x, \theta, \bar{\pi}(\theta)) = u(x, \theta) - t(x) - \bar{\pi}(\theta)$ and $\Xi_t(\theta) = \{x \in X \mid \xi_t(x, \theta, \bar{\pi}(\theta)) \geqslant \xi_t(y, \theta, \bar{\pi}(\theta)) \text{ for every } y \in X\}$. Without loss of generality, I may assume that $\Xi_t(\theta)$ is non-empty. For every $y(\theta) \in \Xi_t(\theta)$, I see that $\xi_t(y(\theta), \theta, \bar{\pi}(\theta)) = \max[u(x, \theta) - t(x) \mid x \in X] - \bar{\pi}(\theta) = \pi_t(\theta) - \bar{\pi}(\theta) = r(\theta)$. Theorem 2 in Milgrom and Segal[4] yields that $r(\theta) = r(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} D_\theta \xi_t(y(s), s, \bar{\pi}(s)) ds$, and the Leibniz rule yields the envelope condition $\dot{r}(\theta) = D_\theta \xi_t(y(\theta), \theta, \bar{\pi}(\theta)) = u_\theta(y(\theta), \theta) - \dot{\bar{\pi}}(\theta)$. Since $y(\theta)$ was arbitrary and, by the hypothesis, $x(\theta) \in \Xi_t(\theta)$, it follows that $\dot{r}(\theta) = u_\theta(x(\theta), \theta) - \dot{\bar{\pi}}(\theta)$.

Suppose, next, that the monotonicity condition and the envelope condition hold. Suppose, by way of contradiction, that there exists a pair $(\theta, \hat{\theta})$ of elements of Θ such that $U(\hat{\theta}, \theta) > U(\theta, \theta)$. Then, $u(x(\hat{\theta}), \theta) - u(x(\hat{\theta}), \hat{\theta}) + r(\hat{\theta}) + \bar{\pi}(\hat{\theta}) > r(\theta) + \bar{\pi}(\theta)$. If $\theta > \hat{\theta}$, then $u(x(\hat{\theta}), \theta) - u(x(\hat{\theta}), \hat{\theta}) > \int_{\hat{\theta}}^{\theta} [u_\theta(x(s), s) - \dot{\bar{\pi}}(s)] ds + \bar{\pi}(\theta) - \bar{\pi}(\hat{\theta}) = u(x(\theta), \theta) - u(x(\hat{\theta}), \hat{\theta}) \geqslant u(x(\hat{\theta}), \theta) - u(x(\hat{\theta}), \hat{\theta})$, a contradiction. If $\hat{\theta} > \theta$, then the same logic leads to a contradiction. This establishes the lemma. □

The participation constraints are incorporated into the implementability in the following manner.

Definition 5. A profile $\{x(\theta), r(\theta)\}_{\theta\in\Theta}$ of an allocation rule and an information rent is said to be *voluntarily implementable* if there exists a price schedule $t(x)$ such that for every $\theta \in \Theta$,
(i) $x(\theta) \in$ argmax$[u(x,\theta) - t(x) \mid x \in X]$,
(ii) $r(\theta) = u(x(\theta),\theta) - t(x(\theta)) - \bar{\pi}(\theta) \geqslant 0$.

If I obtain a profile $\{x(\theta), r(\theta)\}_{\theta\in\Theta}$ of an allocation rule and an information rent as an outcome achieved by some price schedule, then it is obviously incentive compatible and individually rational. The following theorem states that the converse implication also holds.

Theorem 3. *A profile $\{x(\theta), r(\theta)\}_{\theta\in\Theta}$ of an allocation rule and an information rent is incentive compatible and individually rational if and only if it is voluntarily implementable.*

Proof. It suffices to show the necessity. For each $x \in X$, define

$$t(x) = \max\,[-(r(\hat{\theta}) + \bar{\pi}(\hat{\theta})) + u(x,\hat{\theta}) \mid \hat{\theta} \in \Theta].$$

Step 1. *$x(\theta) \in$ argmax$[u(x,\theta) - t(x) \mid x \in X]$ for every $\theta \in \Theta$.*

Proof of Step 1. It suffices to show that $\pi_t(\theta) = u(x(\theta),\theta) - t(x(\theta))$ for every $\theta \in \Theta$. Consider any $\theta \in \Theta$.

Claim 1. *$u(x(\theta),\theta) - t(x(\theta)) = r(\theta) + \bar{\pi}(\theta)$.*

Proof of Claim 1. Let $h(x,\hat{\theta}) = u(x,\hat{\theta}) - (r(\hat{\theta}) + \bar{\pi}(\hat{\theta}))$. Substituting the envelope condition $\dot{r}(\theta) = u_\theta(x(\theta),\theta) - \dot{\bar{\pi}}(\theta)$, I obtain that $h_\theta(x,\hat{\theta}) = u_\theta(x,\hat{\theta}) - u_\theta(x(\hat{\theta}),\hat{\theta}) = x - x(\hat{\theta})$ and $h_{\theta\theta}(x,\hat{\theta}) = -\dot{x}(\hat{\theta}) \leqslant 0$ almost everywhere. The first-order condition $0 = h_\theta(x(\theta),\hat{\theta})$ yields that $x(\theta) = x(\hat{\theta})$. Therefore, $h(x(\theta),\hat{\theta})$ is maximized at $\hat{\theta} = \theta$. Thus, $t(x(\theta)) = h(x(\theta),\theta) = u(x(\theta),\theta) - (r(\theta) + \bar{\pi}(\theta))$. This establishes the claim.

Claim 2. *$\pi_t(\theta) = r(\theta) + \bar{\pi}(\theta)$.*

Proof of Claim 2. Consider any $x \in X$. By the definition of $t(x)$, I see that $t(x) \geqslant -(r(\theta) + \bar{\pi}(\theta)) + u(x,\theta)$, which implies that $u(x,\theta) - t(x) \leqslant r(\theta) + \bar{\pi}(\theta)$. Since x was arbitrary, it follows that $\pi_t(\theta) \leqslant r(\theta) + \bar{\pi}(\theta)$. It remains to show that $\pi_t(\theta) \geqslant r(\theta) + \bar{\pi}(\theta)$. I see that $\pi_t(\theta) - (r(\theta) + \bar{\pi}(\theta)) = \max[u(x,\theta) - t(x) \mid x \in X] - (r(\theta) + \bar{\pi}(\theta)) \geqslant u(x(\theta),\theta) - t(x(\theta)) - (r(\theta) + \bar{\pi}(\theta)) = 0$, where the last equality follows Claim 1. Therefore, $\pi_t(\theta) \geqslant r(\theta) + \bar{\pi}(\theta)$. This establishes the claim.

By Claims 1 and 2, $\pi_t(\theta) = r(\theta) + \bar{\pi}(\theta)$. This establishes the step.

Step 2. $r(\theta) = \pi_t(\theta) - \bar{\pi}(\theta) \geqslant 0$.

Proof of Step 2. By assumption, $r(\theta) \geqslant 0$ for every $\theta \in \Theta$. By Step 1, $\pi_t(\theta) = u(x(\theta),\theta) - t(x(\theta)) = r(\theta) + \bar{\pi}(\theta) \geqslant \bar{\pi}(\theta)$. This establishes the step.

Steps 1 and 2 establish the theorem. □

The corresponding notion of voluntary implementation by a menu of three-part tariffs is straightforward.

Definition 6. A profile $\{x(\theta), r(\theta)\}_{\theta\in\Theta}$ of an allocation rule and an information rent is *voluntarily implementable by a menu of three-part tariffs* $\{T(x,y)\}_{y\in Y}$ if for some $y(\theta) \in Y$,
(i) $x(\theta) \in \text{argmax}[u(x,\theta) - T(x, y(\theta))) \mid x \in X]$,
(ii) $r(\theta) = u(x(\theta), \theta) - T(x(\theta), y(\theta)) - \bar{\pi}(\theta) \geqslant 0$.

Just like Theorem 3, any incentive compatible and individually rational outcome is voluntarily implementable by a menu of three-part tariffs.

Theorem 4. *A profile* $\{x(\theta), r(\theta)\}_{\theta\in\Theta}$ *of an allocation rule and an information rent is incentive compatible and individually rational if and only if it is voluntarily implementable by a menu of three-part tariffs .*

Proof. See the Appendix. □

6 Appendix

Proof of Theorem 1. Suppose, first, that the allocation rule $x(\theta)$ is implementable by some menu $\{T(x,y)\}_{y\in Y}$ of three-part tariffs. Denote the maximized net utility by $\pi(\theta) = u(x(\theta), \theta) - T(x(\theta), y(\theta))$ for each $\theta \in \Theta$, where $y(\theta)$ describes the choice of type θ *among* three-part tariffs, and $x(\theta)$ describes his choice *along* the three-part tariff $T(x, y(\theta))$. Consider any pair $(\theta, \hat{\theta})$ of elements of Θ with $\theta > \hat{\theta}$. I see that

$$\begin{aligned}\pi(\theta) &= u(x(\theta), \theta) - T(x(\theta), y(\theta)) \geqslant u(x(\hat{\theta}), \theta) - T(x(\hat{\theta}), y(\hat{\theta}))\\ &= u(x(\hat{\theta}), \hat{\theta}) - T(x(\hat{\theta}), y(\hat{\theta})) + u(x(\hat{\theta}), \theta) - u(x(\hat{\theta}), \hat{\theta})\\ &= \pi(\hat{\theta}) + (\theta - \hat{\theta})x(\hat{\theta}).\end{aligned}$$

Similarly,

$$\begin{aligned}\pi(\hat{\theta}) &= u(x(\hat{\theta}), \hat{\theta}) - T(x(\hat{\theta}), y(\hat{\theta})) \geqslant u(x(\theta), \hat{\theta}) - T(x(\theta), y(\theta))\\ &= u(x(\theta), \theta) - T(x(\theta), y(\theta)) + u(x(\theta), \hat{\theta}) - u(x(\theta), \theta)\\ &= \pi(\theta) + (\hat{\theta} - \theta)x(\theta).\end{aligned}$$

These imply that

$$(\theta - \hat{\theta})x(\theta) \geqslant \pi(\theta) - \pi(\hat{\theta}) \geqslant (\theta - \hat{\theta})x(\hat{\theta}) \quad \text{or} \quad (\theta - \hat{\theta})(x(\theta) - x(\hat{\theta})) \geqslant 0.$$

Since $\theta > \hat{\theta}$, it follows that $x(\theta) \geqslant x(\hat{\theta})$.

Let $x(\Theta) = [\underline{x}, \overline{x}]$. In what follows, I denote $\underline{\theta} = \inf[\theta \in \Theta \mid x(\theta) = \underline{x}]$ and $\overline{\theta} = \sup[\theta \in \Theta \mid x(\theta) = \overline{x}]$. Without loss of generality, I may assume that $X = [\underline{x}, \overline{x}] = Y$ and $\Theta = [\underline{\theta}, \overline{\theta}]$. Finally, set $\theta^* = \sup[\theta \in \Theta \mid x(\theta) = \underline{x}]$.

Suppose, next, that $x(\theta)$ is non-decreasing. I shall construct a particular menu of three-part tariffs. For each (x, y) of element of $X \times Y$, define

$$T(x,y) = \begin{cases} t(y) & \text{if } x \leqslant y \\ t(y) + \overline{\theta}(x-y) & \text{if } x > y, \end{cases}$$

where

$$t(x) = \max \left[u(x,\hat{\theta}) - \int_{\theta^*}^{\hat{\theta}} u_\theta(x(s),s)ds \,\middle|\, \hat{\theta} \in \Theta \right].$$

Consider the menu $\{T(x, x(\theta))\}_{\theta \in \Theta}$. Let $p(x,\theta) = T(x, x(\theta))$. The proof consists of two steps. Step 1 describes an optimal choice *along* a three-part tariff. I shall show that it is weakly dominant for agent of type θ to consume the amount of $x(\hat{\theta})$ when he is facing the schedule $p(x,\hat{\theta}) = T(x, x(\hat{\theta}))$. Step 2 describes an optimal choice *among* three-part tariffs.

Step 1. $x(\hat{\theta}) \in$ *argmax*$[u(x,\theta) - p(x,\hat{\theta}) \mid x \in X]$ *for every* $\theta \in \Theta$.

Proof of Step 1. The assertion is immediate from the construction of the menu of three-part tariffs. Since $p(x(\hat{\theta}),\hat{\theta}) = p(x,\hat{\theta})$ for every $x \leqslant x(\hat{\theta})$, it follows that

$$[u(x(\hat{\theta}),\theta) - p(x(\hat{\theta}),\hat{\theta})] - [u(x,\theta) - p(x,\hat{\theta})] = \theta(x(\hat{\theta}) - x) \geqslant 0.$$

On the other hand, since $p(x(\hat{\theta}),\hat{\theta}) = t(x(\hat{\theta}))$ and $p(x,\hat{\theta}) = t(x(\hat{\theta})) + \overline{\theta}(x - x(\hat{\theta}))$ for every $x > x(\hat{\theta})$, it follows that

$$[u(x(\hat{\theta}),\theta) - p(x(\hat{\theta}),\hat{\theta})] - [u(x,\theta) - p(x,\hat{\theta})] = \theta x(\hat{\theta}) - \theta x + \overline{\theta}(x - x(\hat{\theta})) = (x - x(\hat{\theta}))(\overline{\theta} - \theta) \geqslant 0.$$

Therefore, $x(\hat{\theta})$ is a solution to utility maximization problem of type θ under $p(x,\hat{\theta})$. This establishes the step.

Step 2. $\theta \in$ *argmax*$[u(x(\hat{\theta}),\theta) - p(x(\hat{\theta}),\hat{\theta}) \mid \hat{\theta} \in \Theta]$ *for every* $\theta \in \Theta$.

Proof of Step 2. Let $q(\hat{\theta}) = t(x(\hat{\theta}))$. Then, $q(\hat{\theta}) = T(x(\hat{\theta}), x(\hat{\theta})) = p(x(\hat{\theta}),\hat{\theta})$. By Step 1, the agent of type θ faces the menu $\{(x(\hat{\theta}), q(\hat{\theta}))\}_{\hat{\theta} \in \Theta}$. It suffices to show that the transfer function $q(\theta)$ rationalizes the allocation rule $x(\theta)$.

Before proceeding the proof of the step, I shall show the properties of the type assignment function $\gamma(x)$ defined by $\gamma(x) = \inf \varGamma(x)$, where

$$\varGamma(x) = \text{argmax} \left[u(x,\hat{\theta}) - \int_{\theta^*}^{\hat{\theta}} u_\theta(x(s),s)ds \,\middle|\, \hat{\theta} \in \Theta \right].$$

Claim 1. *The composite* $x \circ \gamma$ *of* X *into* X *is the identity, that is,* $(x \circ \gamma)(y) = x(\gamma(y)) = y$ *for every* $y \in X$. *Furthermore,* $\theta \in \varGamma(x(\theta))$ *and* $\gamma(x(\theta)) \leqslant \theta$.[3]

[3] I do not claim $\gamma(x(\theta)) = \theta$.

Proof of Claim 1. The first-order condition at $\hat{\theta} = \gamma(y)$ yields that

$$0 = u_\theta(y, \gamma(y)) - u_\theta(x(\gamma(y)), \gamma(y)) = y - x(\gamma(y)),$$

whereas the second-order condition is satisfied: [4]

$$u_{\theta\theta}(y, \gamma(y)) - u_{x\theta}(x(\gamma(y)), \gamma(y))\dot{x}(\gamma(y)) - u_{\theta\theta}(x(\gamma(y)), \gamma(y)) = -\dot{x}(\gamma(y)) \leqslant 0$$

Thus, it is necessary that $y = x(\gamma(y))$. Next, I shall show that θ belongs to the set $\Gamma(x(\theta))$. For the second assertion, it suffices to show $\theta \in \Gamma(x(\theta))$ Suppose, by way of contradiction, that $\theta \notin \Gamma(x(\theta))$. By definition, $\gamma(x(\theta)) \in \Gamma(x(\theta))$. Since $\theta \in \Theta$, it must be the case that

$$u(x(\theta), \gamma(x(\theta))) - \int_{\theta^*}^{\gamma(x(\theta))} u_\theta(x(s), s)ds > u(x(\theta), \theta) - \int_{\theta^*}^{\theta} u_\theta(x(s), s)ds.$$

There are two possible cases to be considered. If $\theta > \gamma(x(\theta))$, then this inequality gives

$$\begin{aligned} 0 &> x(\theta)[\theta - \gamma(x(\theta))] - \int_{\gamma(x(\theta))}^{\theta} u_\theta(x(s), s)ds \geqslant x(\theta)[\theta - \gamma(x(\theta))] - \int_{\gamma(x(\theta))}^{\theta} u_\theta(x(\theta), s)ds \\ &= x(\theta)[\theta - \gamma(x(\theta))] - x(\theta)[\theta - \gamma(x(\theta))] = 0. \end{aligned}$$

This is a contradiction. If $\theta < \gamma(x(\theta))$, then the above inequality gives

$$\begin{aligned} 0 &> x(\theta)[\theta - \gamma(x(\theta))] + \int_{\theta}^{\gamma(x(\theta))} u_\theta(x(s), s)ds \geqslant x(\theta)[\theta - \gamma(x(\theta))] + \int_{\theta}^{\gamma(x(\theta))} u_\theta(x(\theta), s)ds \\ &= x(\theta)[\theta - \gamma(x(\theta))] + x(\theta)[\gamma(x(\theta)) - \theta] = 0. \end{aligned}$$

This is a contradiction.

The last assertion is immediate from the fact that $\theta \in \Gamma(x(\theta))$ and $\gamma(x(\theta)) = \inf \Gamma(x(\theta))$. This establishes the claim.

Claim 2. *$\gamma(x)$ is non-decreasing.*

Proof of Claim 2. Let $x > y$. Suppose, by way of contradiction, that $\gamma(x) < \gamma(y)$. The definition of $\gamma(x)$ yields that

$$\begin{aligned} 0 &\leqslant \left(u(x, \gamma(x)) - \int_{\theta^*}^{\gamma(x)} u_\theta(x(s), s)ds\right) - \left(u(x, \gamma(y)) - \int_{\theta^*}^{\gamma(y)} u_\theta(x(s), s)ds\right) \\ &= x[\gamma(x) - \gamma(y)] + \int_{\gamma(x)}^{\gamma(y)} x(s)ds \leqslant x[\gamma(x) - \gamma(y)] + x(\gamma(y))[\gamma(y) - \gamma(x)] \\ &= x[\gamma(x) - \gamma(y)] + y[\gamma(y) - \gamma(x)] = (x - y)[\gamma(x) - \gamma(y)] < 0. \end{aligned}$$

This is a contradiction. This establishes the claim.

[4] Strictly speaking, the second-order condition is satisfied almost everywhere because any non-decreasing function is differentiable almost everywhere.

Define $U(\hat{\theta},\theta) = u(x(\hat{\theta}),\theta) - q(\hat{\theta})$. I see that $U(\theta,\theta) \geqslant U(\hat{\theta},\theta)$ if and only if $\theta \in \text{argmax}[u(x(\hat{\theta}),\theta) - p(x(\hat{\theta}),\hat{\theta}) \mid \hat{\theta} \in \Theta]$. Let $\theta \geqslant \hat{\theta}$. Then,

$$\begin{aligned} U(\theta,\theta) &= u(x(\theta),\theta) - u(x(\theta),\gamma(x(\theta))) + \int_{\theta^*}^{\gamma(x(\theta))} u_\theta(x(s),s)ds \\ &= \int_{\gamma(x(\theta))}^{\theta} u_\theta(x(\theta),s)ds + \int_{\theta^*}^{\gamma(x(\theta))} u_\theta(x(s),s)ds \end{aligned}$$

and

$$\begin{aligned} U(\hat{\theta},\theta) &= u(x(\hat{\theta}),\theta) - u(x(\hat{\theta}),\gamma(x(\hat{\theta}))) + \int_{\theta^*}^{\gamma(x(\hat{\theta}))} u_\theta(x(s),s)ds \\ &= \int_{\gamma(x(\hat{\theta}))}^{\theta} u_\theta(x(\hat{\theta}),s)ds + \int_{\theta^*}^{\gamma(x(\hat{\theta}))} u_\theta(x(s),s)ds \end{aligned}$$

since $\gamma(x(\theta)) \leqslant \theta$ and $\gamma(x(\hat{\theta})) \leqslant \hat{\theta} \leqslant \theta$ by Claim 1. Moreover,

$$\begin{aligned} U(\theta,\theta) - U(\hat{\theta},\theta) &= \left(\int_{\gamma(x(\theta))}^{\theta} u_\theta(x(\theta),s)ds + \int_{\theta^*}^{\gamma(x(\theta))} u_\theta(x(s),s)ds \right) \\ &\quad - \left(\int_{\gamma(x(\hat{\theta}))}^{\theta} u_\theta(x(\hat{\theta}),s)ds \int_{\theta^*}^{\gamma(x(\hat{\theta}))} u_\theta(x(s),s)ds \right) \\ &\geqslant \int_{\gamma(x(\hat{\theta}))}^{\gamma(x(\theta))} u_\theta(x(s),s)ds + \int_{\gamma(x(\theta))}^{\theta} u_\theta(x(\hat{\theta}),s)ds - \int_{\gamma(x(\hat{\theta}))}^{\theta} u_\theta(x(\hat{\theta}),s)ds \\ &= \int_{\gamma(x(\hat{\theta}))}^{\gamma(x(\theta))} u_\theta(x(s),s)ds - \int_{\gamma(x(\hat{\theta}))}^{\gamma(x(\theta))} u_\theta(x(\hat{\theta}),s)ds, \end{aligned}$$

where the inequality follows from the fact that $\gamma(x(\theta)) \geqslant \gamma(x(\hat{\theta}))$ by Claim 2. Notice that for every $\tilde{\theta} \in [\gamma(x(\hat{\theta})), \gamma(x(\theta))]$, $x(\tilde{\theta}) \geqslant x(\gamma(x(\hat{\theta}))) = x(\hat{\theta})$ because $x(\theta)$ is non-decreasing and $x \circ \gamma$ is the identity by Claim 1. By the single-crossing property $u_{x\theta}(x,\theta) > 0$, I have $u_\theta(x(\tilde{\theta}),\tilde{\theta}) \geqslant u_\theta(x(\hat{\theta}),\tilde{\theta})$. This yields that

$$U(\theta,\theta) - U(\hat{\theta},\theta) \geqslant \int_{\gamma(x(\hat{\theta}))}^{\gamma(x(\theta))} u_\theta(x(\hat{\theta}),s)ds - \int_{\gamma(x(\hat{\theta}))}^{\gamma(x(\theta))} u_\theta(x(\hat{\theta}),s)ds = 0.$$

Hence, $U(\theta,\theta) \geqslant U(\hat{\theta},\theta)$.

Steps 1 and 2 establish the theorem. □

Remark 2. *The price schedule $t(x)$ defined in the proof of Theorem 1 is convex and increasing.*

Proof of Remark 2. The price schedule $t(x)$ is convex because it is the maximum of the affine functions. It remains to show the monotonicity of $t(x)$. Let $x > y$. Then,

$$\begin{aligned} t(x) - t(y) &\geqslant \left(u(x,\gamma(y)) - \int_{\underline{\theta}}^{\gamma(y)} u_\theta(x(s),s)ds \right) - \left(u(y,\gamma(y)) - \int_{\underline{\theta}}^{\gamma(y)} u_\theta(x(s),s)ds \right) \\ &= u(x,\gamma(y)) - u(y,\gamma(y)) = \gamma(y)(x-y) > 0 \end{aligned}$$

because $\gamma(y) \in \Theta \subseteq \mathbb{R}_{++}$. Hence, $t(x) > t(y)$. This establishes the remark. □

Proof of Theorem 3. It suffices to show the necessity. Suppose that a profile $\{x(\theta), r(\theta)\}_{\theta\in\Theta}$ is incentive compatible and individually rational. Integrating the envelope condition $u_\theta(x(\theta),\theta) = \dot{r}(\theta) + \dot{\bar{\pi}}(\theta)$ by Lemma 1,

$$\int_{\theta^*}^{\hat{\theta}} u_\theta(x(s), s)ds = \int_{\theta^*}^{\hat{\theta}} [\dot{r}(s) + \dot{\bar{\pi}}(s)]ds = r(\hat{\theta}) + \bar{\pi}(\hat{\theta}) - [r(\theta^*) + \bar{\pi}(\theta^*)].$$

For each (x, y) of element of $X \times Y$, define

$$T(x,y) = \begin{cases} t(y) & \text{if } x \leqslant y \\ t(y) + \overline{\theta}(x-y) & \text{if } x > y, \end{cases}$$

where

$$t(x) = \max\,[-(r(\hat{\theta}) + \bar{\pi}(\hat{\theta})) + u(x,\hat{\theta}) \mid \hat{\theta} \in \Theta].$$

Rewriting the price schedule,

$$t(x) = \max\left[u(x,\hat{\theta}) - \int_{\theta^*}^{\hat{\theta}} u_\theta(x(s),s)ds \,\Big|\, \hat{\theta} \in \Theta\right] - [r(\theta^*) + \bar{\pi}(\theta^*)].$$

Then, the rest is similar to the proof of Theorem 1. □

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