Word of Mouth Advertising, Credibility and Learning in Networks *

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1 Introduction

Social networks representing the pattern of social interactions - who talks to or who observes whom- play a crucial role as a medium for the spread of information, ideas, diseases, products. Someone in the population may be struck with an infection or may adopt a new technology, and it can then either die out quickly or spread throughout the population, depending possibly on the location of the initial appearance, the structure of the network - for instance, how dense it is. The dynamics of adoption -the extent to which individuals are influenced by their neighbours, the impact of "word-of- mouth" communication- also plays a role in determining the speed of diffusion. Given the large range of contexts in which social learning is important, it is not surprising that researchers from various disciplines have studied processes of diffusion from a variety of perspectives.[1]

For instance, the classic study of [3] describes the impact of the social structure on the prescription of a new drug by doctors. [8], [18] are amongst several papers that focus on social learning and the adoption of new technologies in agriculture in developing countries. Of

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[1] See [15] for an account of recent research on diffusion.

greater relevance to the present paper is the use of social networks to promote new products through “word-of-mouth" communication and viral marketing.[2] The growing popularity of this form of advertising has resulted in a corresponding increase in the attention this has received both in the academic marketing literature as well as in the more popular media.

A particularly interesting form of word-of-mouth marketing, described in a New York Times magazine article by Rob Walker on December 5, 2004, provides the main motivation for this paper. Writing about how “word of mouth” marketing was superseding more traditional modes with fracturing market segments, he wrote about the “growing number of marketers organizing veritable armies of hired “trendsetters” or “influencers” or “street teams” to execute “seeding programs,”, “viral marketing,” “guerrilla marketing” ” as “attempts to break the ...wall that used to separate the theater of commerce, persuasion and salesmanship from our actual day-to-day life.” For example, the acquaintance who recommends a book to someone might actually not have read it but have been provided talking points by a paid agent of the publisher. With the advent of web-based social networking sites, viral marketing has taken on added importance. See, for example, http://techcrunch.com/2008/04/21/facebook-publishes-insiders-guide-to-viral-marketing/, where several techniques are discussed including using the “News Feed” feature to convey information to friends of users and their friends and so on. In a somewhat different context, a government body hoping to spread a new technology might solicit important and connected individuals, in the focal community to recommend the new methods to people they know.

It might well be known to the target segment of consumers that this type of activity is going on (especially after the appearance of the New York Times article!); this might lead to such recommendations from neighbors being received with some scepticism and possibly ignored. In such cases, even a good product and a good new technology might not diffuse as fast as it should or even diffuse at all, thus leading to a socially undesirable outcome.

The problems discussed in the preceding paragraphs have two important distinguishing features from those studied in the basic models of the spread of disease in epidemiology, namely:

1. Interactions among individuals take place in an explicit network structure-individual j might not be at risk of contracting an infection from individual i, if i and j are not “neighbours” in some sense. Thus there is some network microstructure to consider, which the differential equation models of diffusion or infection ignore.

2. Whilst an individual i might have no choice as to whether he or she contracts a viral infection from the person coughing in the next seat on a trans-Atlantic flight, he or she *does* have a choice as to whether she buys a particular set of noise-cancelling headphones

[2] See [17], [14].

because she sees this person using it (or recommending it to her). There is thus a strategic informational element, (the person in the next seat might have a special interest propagating these headphones) that seems central to any analysis. [3]

Tracking how these two elements interact is the main aim of this paper. We consider a *fixed network,* given exogenously as in the motivational examples above. This contrasts with some recent work (see Goyal and Vigier ([11]), in which the question considered is the design of a robust network. In our context, such a "robust network" might be designed to limit the spread of bad information for example.

We consider a finite number of potential buyers who are connected in a fixed network. There is a seller who can choose to "seed" the network by paying an agent at any given node in the network to give a positive recommendation about the seller's product. The seller knows in advance whether his product is of good or bad quality. Buyers have ex ante beliefs about whether the product is good. Buyers are of two types. There is some probability that a buyer at a given node is an "innovator", who will try the new product immediately; with the complementary probability a buyer is normal in that she makes a rational decision on whether to buy or not. We assume (to avoid trivialities) that the ex ante belief is below the threshold required to induce the second type of buyers to purchase the product.

Buyers can also recommend buying the product to their neighbors in the network. We can also think of this as individuals using the product frequently, so they are observed to do so by their neighbours. Buyers who are not innovators and who receive recommendations from their neighbors have to form posterior beliefs about the quality of the product, and then decide whether to buy the product. Each purchase gives the seller a unit profit (prices are assumed to be fixed) and future payoffs are discounted; so if buying is optimal, buying now is better than waiting.

We model this process as a game of incomplete information and focus on the perfect Bayesian equilibria of this game. Our principal interest is in studying the conditions under which the product of good quality will diffuse with probability one throughout the network in the fastest possible time - we call this the efficient diffusion equilibrium (EDE). In particular, what are the types of network structures which are conducive to the existence of an EDE?

In order to answer this question, we have to analyze the optimal marketing strategies of the two types of firms. This is related to a popular theme in the existing marketing literature. This literature suggests that it is optimal to initially target a few "influential" members of the population since these influential members can then more easily convince others in the network to buy the product. [4] A "naive" view is that highly connected individuals are more

[3] There are some papers that consider the first aspect but not the second. See, for instance, the textbook treatment by Draief and Massoulie [4], and the survey by Jackson and Yariv [12].

[4] This has given rise to the development of algorithms to locate the most influential members in a network.

likely to be influential. However, [9] provides a more nuanced and interesting analysis by showing that "the optimality of targeting highly connected nodes depends very much on the content of social interaction."

Our analysis shows that the two types of firms will follow very different targeting strategies in equilibrium. A firm which produces a good quality product will want to buy off an agent which can reach out to all other agents in the shortest possible time. On the other hand, a firm producing a bad quality product is more myopic because it knows that its product will not sell beyond one period since no one other than the agent it has bought off will recommend the product. So, this firm will want to place its implant at nodes which have the highest number of connections. However, the bad quality firm must also ensure that its agent's recommendation is credible. For instance, if consumers know that the bad quality firm buys off a particular agent with probability one, then that agent's recommendation is less likely to be credible.

The optimal behavior of the two types of firm determine whether an EDE exists or not. Our analysis reveals some counter-intuitive results. For example, it turns out that if any individual is "too influential" in the sense of being connected to *everyone*, then an EDE cannot exist. This is because both types of firms would target such an individual, thereby destroying the credibility of her recommendation. Of course, efficient diffusion would typically be guaranteed in contexts where the *credibility of recommendations* is not at stake. This is one point in which the strategic element in the problem has bite. It also turns out for somewhat subtler reasons that *larger* (in a sense to be described later) networks are more likely to support efficient diffusion.

We provide a partial characterization of networks which support EDEs. We also have some "comparative statics" results. In particular, we focus on the role of the network structure, as well as on the probability that consumers are innovators. Of course, there can be other types of inefficient equilibria and we illustrate the nature of these equilibria for the special case where the network is a *line*. We also briefly discuss the robustness of our results if the model is changed to allow for innovators to make *negative* recommendations. Obviously, the possibility of negative recommendations will lower the expected profits from employing an "implant" for the bad quality firm, and thus make the existence of an EDE more likely. However, conditional on the bad quality firm employing an implant, there is very little qualitative change in our results.

Our model assumes that the structure of the network is common knowledge. A model where there is partial knowledge about the network (in addition to the strategic element) is left for future research.

See [16] and [13].

1.1 Related literature

There has been a voluminous literature on diffusion of innovation arising from different causes; for a game-theoretic analysis on how these different causes could lead to different observed patterns of diffusion, see Peyton Young [20]. However, Young does not explicitly consider a network structure.

The paper that considers the most closely related problem of a monopolist "seeding" a network in order to spread information about his product is [9]. This paper considers a monopolist who can choose a fraction x of the population (modelled as the unit interval [0,1]) at a cost of $c(x)$). Each individual picks a finite number of neighbors from the unit interval, so the neighbors of different individuals are in independent subsets (the probability of a common neighbour is 0). The number of neighbors varies according to a probability distribution; each individual can get information about the product either directly from the monopolist or indirectly from one of her neighbors, who is himself directly informed. The results obtained are essentially about properties of the degree distribution that facilitate spread of the monopolist's information.

Unlike our work, Galeotti and Goyal do not consider a seller with private information or Bayesian buyers who take into account how their current state of information reveals what is happening in unobserved parts of the network. Our results relate primarily to different notions of centrality in a given network structure, rather than to the average degree or the variance of the degree distribution as in Galeotti and Goyal.

Our work is also related to social learning in networks, as in [1], [2].[5] The main difference between our work and these papers is that we have a strategic seller who has private information about quality and is trying to manipulate the diffusion, whilst none of the other papers do. Also our buyers are fully rational Bayesians; even [1] do not model buyers who infer events in unobserved parts of the network, as our buyers do. It is not surprising therefore that our equilibria are quite different.

The next section describes the formal model. We then illustrate the workings of our model when the network structure is a line. This is followed by a partial characterization result on the type of networks which sustain an EDE. Our comparative statics results are contained in the subsequent section. The following sections describe the nature of inefficient equilibria on the line and extensions to the basic model.

[5] See [10] for an illuminating survey of papers on learning in networks. Other related papers are[5], [6], [7].

2 The model

In this section, we describe the basic model. The set $N = \{1, 2, \ldots, n\}$ represents the set of consumers. The structure of interactions between the set of consumers is represented by means of a graph Γ in which the nodes are elements of N and $ij \in \Gamma$ if consumers i and j can communicate with each other. There is a firm F, which is interested in selling its product. The product is either of type $G(ood)$ or $B(ad)$. Firm F knows the type of its product. We assume the graph is connected, so there is a path through which diffusion can occur between any two individuals.

All buyers have an initial probability p that the product is of the good type. There are *two* types of consumers. Buyers of the first type - we refer to them as the *innovators*- get utility g' from the G product and utility $-b'$ from the B product. These numbers are such that it is a dominant strategy for innovators to buy immediately. On the other hand, the second type of buyers (the *normal* types) get utilities g and b from the G and B type products. Thus, if

$$p < \bar{p} \equiv \frac{b}{b+g}$$

the second type of consumer will not buy the good unless she revises her probability belief about the good in subsequent periods. If $p = \bar{p}$, she will be indifferent between buying and not buying the good if she does not expect to receive any further information about the product if she does not buy. Since there are a finite number of agents, there must be a time period in which this is so (take the maximum distance from a node to any other node and suppose that information, if transmitted takes a number of periods equal to the distance to traverse the path). If future information is expected, in general a buyer i is indifferent between buying and not buying in period t for $p = \bar{p}_i^t$, where $\bar{p}_i^t$ could depend on the time period t, the position of the agent and the strategies of the good and bad types and is in general strictly greater than $\bar{p}$. When $t = 1$, we will typically write $\bar{p}_i$ instead of $\bar{p}_i^t$. We illustrate the dependence of $\bar{p}_i^t$ on these factors when the network is a line in Section 3.

Each consumer buys the product at most once. The firm gets 1 unit for each item purchased and 0 if an item is not purchased. There are no capacity constraints on the number of items sold.

Future payoffs are discounted by δ for F and for the consumers.

The time line: Nature draws the type of F and this is revealed only to F. F chooses a site i to place one "implant" at a "small" cost c or decides not to use any implants. The implant, if any, is paid to pass on a recommendation to his neighbors in Γ. If i is not an implant, she can be an "innovator" in which case she tries the new product immediately.

The probability that any site i is an innovator is $\rho < 1$ and the event that "i is an innovator" is independent of other events "$j \neq i$ is an innovator". All this takes place, in sequence, at $t = 0$. At $t = 1$, any i who is an innovator, and who has obtained a utility of g', makes a recommendation to his neighbors. An implant always makes a recommendation, but may choose the time at which she makes the recommendation for strategic reasons. The neighbors receiving the recommendation might choose to buy the product or not to buy it. At $t = \tau$, any site who is either an implant or has tried the product and found it good, after receiving a recommendation in $\tau - 1$, can make a recommendation to neighbors. A site i does not observe if neighbors have received recommendations or have chosen to buy the product- she only observes whether recommendations are made by the neighbors themselves.

There is no exogenous time limit on the game; however, since there are a finite number of neighbors and each speaks at most once, the game must end in finite time.

Strategies and equilibrium

1. The players in this game are the firm with a product, with private information about its quality G or B;an implant, whose preferences are the same as the firm's, and the consumers who are not innovators, at site i. *Innovators are not strategic players-an innovator publicly uses a good product and does not use a bad product.*

2. At time, 1, G or B chooses whether to use an implant at site i.Let the probabilities with which i is chosen be α_i for G and β_i for B, respectively, conditional on there being an implant. Let $\ominus$ represent the decision not to use an implant by either type of firm.

3. At time t, $t \geq 1$,let h_{it} denote the private history of the non-innovating agent at site i, whose direct neighbours belong to N_i. If agent i is not an implant, the history consists of a vector $x_{j\tau}, a_{i\tau}$, $\tau < t$, $j \in N_i$, where, for a particular $\tau, x_{j\tau} \in \{0, 1\}$ and a value of 1 means that j has been observed using the new product at time τ. The action $a_{i\tau} \in \{0, 1\}$, where $a_i = 1$ means that agent i has decided to buy and try the new product in time τ. If he finds it good, his neighbours observe him using it in the same period. A strategy for the non-innovating agent is then a map from h_{it} to $\Delta[\{0, 1\}]$, that is, a probability of buying the product given the history.

4. Let i now be the implant of the firm. Player i has a private history consisting of her type and $h_{it} = (x_{j\tau}, s_{i\tau})$, $\tau < t$, where $s_{i\tau}$ is 1 if implant i makes a recommendation (or "speaks") in period τ and 0 otherwise. A strategy for the implant at i is a map from (her type, h_{it}) to a probability of speaking in period $t, i.e.$ a probability that $s_{it} = 1$. Note that the implant and the firm are assumed to have identical payoffs.

Note that the firm enters (buys an implant) or not at the beginning of the game. We assume that entry is not possible at a later time, though we discuss this assumption later. However, the firm/implant can *speak* at any time. Since "speaking" is, for us, synonymous with being observed using the product, any individual can speak only once.

5. *Equilibrium* in this game is to be interpreted as Perfect Bayes Equilibrium. The requirements are: (i) Each agent, including the firm and implants but not including the innovators, updates beliefs according to Bayes' Theorem whenever possible and (ii) each agent maximises her expected payoff at each (private) history given these beliefs. Out-of-equilibrium beliefs do not play a major role here, but will be explicitly described when necessary.

Updating Beliefs

Let α and β be the mixed equilibrium strategies of types G and B respectively. That is, α_i is the probability with which type G uses consumer i as an implant. Suppose consumer i receives a recommendation from her neighbor $i-1$ in period 1. If she receives no other recommendation, what is the probability that the product is G? Let us denote this by $\eta^1_{i,i-1}$, where the superscript refers to the time the recommendation is received and the subscripts to the recipient and the sender of the recommendation.

Let $N_i(\Gamma) = \{j \in N | ij \in \Gamma\}$ denote the set of neighbors of i, and $d_i(\Gamma) = |N_i(\Gamma)|$ be the *degree* of i in Γ. (Henceforth, whenever there is no ambiguity about Γ, we will simply write d_i, d_j, etc.)

Since the derivation of the probability $\eta^1_{i,i-1}$ is somewhat tedious to check, we reproduce the actual calculation. The probability required is: Prob. [product is G| i is not an implant or innovator and none of the other neighbours other than $i-1$ has made a recommendation and $i-1$ has made a recommendation]

Let's call the conditioned event A and the conditioning event B.

Then, by Bayes' Theorem, P$(A \mid B) = P(B \mid A)P(A)/[P(B \mid A)P(A) + P(B \mid A^C)P(A^C)]$.

$=\frac{P(A \text{ and } B)}{P(A \text{ and } B)+P(A^C \text{ and } B)}$

The numerator is then $p(1-\alpha_i)(1-\rho)^{d_i-1}[\frac{\alpha_{i-1}}{1-\alpha_i} + \rho(1-\frac{\Sigma_{j\in N_i}\alpha_j}{1-\alpha_i})]$

$=p(1-\rho)^{d_i-1}[\alpha_{i-1} + \rho(1-\Sigma_{j\in N_i \cup i}\alpha_j)]$ for $\alpha_i \neq 1$. (We can take limits if $\alpha_i = 1$)

The denominator is this quantity plus another term $P(B \mid A^C)P(A^C)]$. This is the probability of all this happening if the product is a bad one. The second term in the denominator is therefore $(1-p)(1-\beta_i)\frac{\beta_{i-1}}{1-\beta_i}$ again assuming $\beta_i < 1$. Hence,

$$\eta^1_{i,i-1} = \frac{p(1-\rho)^{d_i-1}\left[\alpha_{i-1} + (1-\sum_{j\in N_i}\alpha_j - \alpha_i)\rho\right]}{p(1-\rho)^{d_i-1}\left[\alpha_{i-1} + (1-\sum_{j\in N_i}\alpha_j - \alpha_i)\rho\right] + (1-p)\beta_{i-1}} \tag{1}$$

Some "special" cases illustrate the nature of the updating process. Suppose that the type G firm uses a pure strategy so that for some site m, $\alpha_m = 1$. Suppose i is a neighbor of m, but receives only one recommendation from some $j \neq m$. Then, i must conclude that j is a bad implant - if the product had been good, then there would have been a good implant at m who would then have passed on a recommendation to her. This argument generalizes even when the type G firm uses a strategy whose support is some set M containing more than one node. Suppose now that i is a *common* neighbor of all nodes in M. Again, if i does not receive a recommendation from some member of M, she will conclude that any other recommendation comes from a bad implant. Next, suppose again that $\alpha_m = 1$ and that m receives a recommendation from some neighbor. Of course, such recommendations are not credible to m - she would have been used as an implant by the type G form if the product was good. These inferences are confirmed by equation (1) - in all cases, the numerator is 0.

Of course, if i receives a recommendation from *two* or more neighbors, then i concludes that the product is G with probability one- if there is a bad implant at j, then there cannot be a bad implant at $j' \neq j$.

Suppose next that i receives a recommendation from $i-1$ in some period $t > 1$, but no recommendation from any other neighbor. If i has not received any recommendations before period t and receives one from $i-1$ in period t, this can happen because the product is Bad, there is an implant at $i-1$ and the implant chooses to speak at period t. Alternatively, the product is Good, $i-1$ heard a recommendation from one of her neighbors in the previous period, but none of i's other neighbors received a recommendation from any of their neighbors in period $t-1$. Explicit computations of these probabilities are hard to describe since these depend on the structure of the network. But, notice that some recommendations are easy to dismiss. For instance, suppose Γ is a *line*, and let i be an extreme point of Γ, with degree one. Then, any recommendation from i coming in period $t > 1$ is not credible to i's neighbor since i could not have received a recommendation in period $t-1$.

Efficient Diffusion Equilibrium

We shall mainly, though not exclusively, limit ourselves to the consideration of "efficient diffusion equilibria" (EDE), adopting the viewpoint of, say, a development agency that wants a *good* idea or product to be spread through the entire population as quickly as possible, given the constraints of the network structure and the technology of diffusion. Note that the B type in our model can expect to sell only to the initial adopters, since the word-of-mouth will be negative (or the absence of positive). Therefore, the study of diffusion through the network is most relevant for the G type, hence the focus on EDE.

Consider an environment in which Firm G is the only type of firm, so that the issue of credibility of recommendations does not arise. For instance, consumers may be initially unaware about the existence of the product, but are willing to buy the product after receiving a recommendation. Then, Firm G will want to "seed" the network by using an implant. In the absence of any issue of credibility of recommendations, the product will diffuse throughout the network with probability one if an implant is used. Since the firm discounts the future, it will want to place its implant so as to maximize the speed of diffusion. The optimal site(s) for a Good implant is related to a measure of centrality of network structures. Let d_{ij} denote the *geodesic distance* between i and j in the graph Γ. That is, d_{ij} is the length of the shortest path between i and j.

Definition 1 *A node i maximizes decay centrality in a graph Γ if $\sum_{j \neq i} \delta^{d(i,j)} \geq \sum_{j \neq k} \delta^{d(k,j)}$ for all $k \in N$.*

Let $D(\Gamma)$ denote the set of nodes maximizing decay centrality. While it is not easy to compute this set in general graphs, the set is easily identified in special cases. For instance, if Γ is a line, then the median(s) must be maximizing decay centrality. Or if Γ is a star, then the hub (that is a node with degree $n-1$) is obviously the node maximizing decay centrality.

Notice that since consumers also discount the future, consumers' surplus is also maximized when the G-implant is placed at a node which maximizes decay centrality.

Henceforth, we are particularly interested in a PBE (the *efficient diffusion equilibrium* (EDE)) with two properties -(i) the good product will diffuse throughout the network with probability one, and (ii) the good implant is placed at some node maximizing decay centrality.

So, if (α, β) denote the probability distributions with which the G-type and B-type implants are placed at different nodes, then the support of α must be contained in the set $D(\Gamma)$. Moreover, there must be at least one sequence of recommendations originating from all nodes with $\alpha_i > 0$ which are accepted with probability one- otherwise the good product will not diffuse through the entire network with probability one. Hence, conditional on the product being Good, an EDE maximizes consumer surplus.[6]

Of course, we also need to identify when G and B will decide to use an implant. This must depend on a comparison of the increase in expected profit resulting from an implant and the cost c incurred by employing an implant. We assume that $c > 0$.[7]

[6] As mentioned earlier, if the product is Bad, the upper bound on the extent of diffusion is given by the maximal degree of any node in the network.

[7] We assume that at most one implant is chosen. However, one could interpret this as being a statement about the size of c

3 EDE on the line; an example

In this section, we provide an informal discussion of a specific network structure, - the *line* when n is odd, in order to illustrate the conditions required for an efficient diffusion equilibrium.

So, let Γ be a line, with the sites ordered so that 1 and n are the end-points of the line having degree one, while all other sites have degree 2. Since n is odd, the unique median maximizes decay centrality. Hence, in any PBE (α, β), where α and β denote the equilibrium strategies of the Good and Bad types respectively, $\alpha_m = 1$. Now, if this PBE is to be an EDE, then the recommendation coming from m has to be accepted with probability one. That is, $m-1$ and $m+1$ must accept any recommendation coming from m with probability one. Since the bad type can always mimic the Good type, this implies that wherever the bad implant is placed, her recommendation must be accepted by two neighbors.

But, of course, $\beta_m \neq 1$. For, if $\beta_m = 1$, then from equation (1),

$$\eta^1_{m-1,m} - p = \eta^1_{m+1,m} - p = p[\frac{(1-\rho)^{d_{m-1}-1} - 1)(1-p)}{p(1-\rho)^{d_{m-1}-1} + (1-p)}] < 0,$$

and neither of m's neighbors would buy the product after receiving a recommendation from m, which is a contradiction. So, while the support of β can *include* m, it cannot coincide with $\{m\}$. Over what set of nodes can the bad type "distribute" β? The answer of course is that the support of β must be contained in those nodes who can make "credible" recommendations to both their neighbours, since m's recommendations are credible. This must be the set

$$S \equiv N \setminus \{1, m-2, m-1, m+1, m+2, n\}$$

It is clear that 1 and n cannot be in S since they have only one neighbor. Although $m-2$ and $m+2$ have degree 2, notice that $m-1$ (respectively $m+1$) will not believe a single recommendation from $m-2$ (respectively from $m+2$).[8] Of course, $m-1$ and $m+1$ cannot sell to m. Also, notice that if $n < 9$, then m will be the sole member of S, and there will not be any EDE.

It is also clear when the bad type will want to use an implant when there is an EDE. Let the implant be placed at i. Since i can be an innovator with probability ρ (in which case he would buy the product anyway), the effective cost of an implant at i is

$$\rho + c$$

[8] Both $m-1$ and $m+1$ know that if the product is good, then m would have made a recommendation.

The benefit is the additional probability that $i-1$ and $i+1$ buy the product. With probability $(1-\rho)^2$, neither is an innovator. With probability $2(1-\rho)\rho$, one of the two is an innovator. Hence, the benefit is

$$2(1-\rho)^2\delta + 2(1-\rho)\rho\delta = 2\delta(1-\rho)$$

So,the net gain of an implant for B is given by

$$2\delta(1-\rho) - \rho - c$$

where c is the cost of an implant. So, B will use an implant if

$$c \leq c^B(\delta, \rho) \equiv 2\delta(1-\rho) - \rho$$

Not surprisingly, the higher the value of ρ, the lower is the expected gain from employing an implant.

Consider G. Let $\underline{\pi}^G(\delta,\rho)$ denote the profit of G in the absence of an implant, and $\bar{\pi}^G(\delta,\rho)$ denote the profit of G if an implant is used.[9]

Then, G will use an implant if

$$c \leq c^G(\delta, \rho) \equiv \bar{\pi}^G(\delta,\rho) - \underline{\pi}^G(\delta,\rho)$$

The expected profits of G with or without an implant increases in ρ. However, notice that the *difference* between the two levels of profit decreases as $\delta \to 1$ since the speed of diffusion is less important as δ increases. In the limit,

$$\lim_{\delta \to 1}(\bar{\pi}^G(\delta,\rho) - \underline{\pi}^G(\delta,\rho)) = n - 1 - n[1-(1-\rho)^n]$$

On the other hand, an increase in δ increases the value of an implant to B. So, for "high" δ, it may well be the case that only the bad type uses an implant!

Let $c_m = \min\left(c^B(\delta,\rho), c^G(\delta,\rho)\right)$.

Suppose $\beta_i > 0$ for some i. It is easy to place an upper bound on how high β_i can be in equilibrium. Since i's recommendation must be accepted with probability one by both her neighbors, their updated probability that the product is good cannot fall below $\bar{p}_{i-1}$ or $\bar{p}_{i+1}$. Note that for $m-1$ and $m+1$, $\bar{p}_{m-1} = \bar{p}_{m+1} = \bar{p}_0$. Equation 1 and the fact that $\alpha_m = 1$

[9] These expressions are difficult to compute for general n.

now readily yield these upper bounds.

$$\bar{\beta}_m = \frac{p(1-\bar{p}_0)}{\bar{p}_0(1-p)}, \text{and for } i \neq m, \bar{\beta}_i = \frac{\rho p}{1-p} \min\left[\frac{(1-\bar{p}_{i-1})}{\bar{p}_{i-1}}, \frac{(1-\bar{p}_{i+1})}{\bar{p}_{i+1}}\right]$$

It is now easy to describe what an EDE looks like when $n \geq 9$ is odd, and the cost of an implant does not exceed c_m.

(i) The type G firm puts its implant at the median of the line with probability one.

(ii) The support of β is contained in S, and for each i in the support of β, $\beta_i \leq \bar{\beta}_i$, $\beta_m \leq \bar{\beta}_m$ and $\beta_i = 0$ elsewhere.

(iii) The implant (irrespective of type) "speaks" in period 1.

(iv) Recommendations received from each site in the support of β are accepted with probability one in period 1.

(v) In subsequent periods $t > 1$, a site $i < m$ accepts a recommendation from $i+1$ with probability one if the equilibrium response of $i+1$ was to accept the recommendation from $i+2$ in period $t-1$. Similarly, a site $i > m$ accepts a recommendation from $i-1$ if the equilibrium response of $i-1$ was to accept the recommendation from $i-2$ in period $t-1$.

As we have mentioned before, Property (ii) follows from the fact that recommendations will not be accepted unless the updated belief that the product is good reaches the threshold value of $\bar{p}$. Property (iv) is consistent with this since a site that receives a recommendation (and is not an innovator or an implant) is at least indifferent between buying and not buying and strictly prefers buying if the inequality is strict.

For property (iii), note that if the product is bad, then the implant can only hope to persuade two neighbors to buy the product - her recommendation will not be passed on. If the product is good, then the product will diffuse through the entire population given (v). Since an implant's recommendation is accepted with probability one by both neighbors, the implant gets the same outcome as early as possible in either case.

Property (i) follows by noting that type G cannot be indifferent between m and any other site. At m, the implant will obtain an expected payoff of $\delta.2 + \delta^2.2 + ...$ for $m-1$ terms. At $m-k$, say, the payoff will be $\delta.2 + \delta^2.2 + ...\delta^t 2$ for $m-k-1$ terms and $\delta^{m-k}.1 + ..$ for an additional $2k$ terms, thus taking $m+k-1$ periods to diffuse completely rather than $m-1$ periods, if the good implant locates at m.[10] Thus the speed of diffusion is higher by locating at m, since there are two new buyers in each period for every period the diffusion continues, whilst at $m-k$, there is only one buyer for every period after $k-1$.

The argument for (v) includes the following: Since the B implant speaks in period 1 in

[10] This calculation does not take ρ into account, but it is obvious this wouldn't change the ranking because sites are independently innovators or non-innovators

equilibrium, any recommendation after period 1 must come from a relayed recommendation from an innovator or a good implant at some $i \neq m$ or m respectively. If $t \geq m-1$, no recommendation will occur along the equilibrium path but it is assumed that off-equilibrium beliefs also induce acceptance of the recommendation.

We note that the B implant will not deviate to speaking after $m-1$, even with this belief, because of discounting and the acceptance probabilities of 1 even if B speaks early.

In this equilibrium, a message from m in period 2 would be an event of probability 0. To check that neither type wants to deviate, we have to consider out-of-equilibrium beliefs, but all beliefs about the type sustain this equilibrium. The m implant's recommendation is accepted with probability 1 in the first period, so discounting makes it sub-optimal to wait, no matter what the belief.

Remark 1 *We note again that $\mid S \mid$ must be large enough for the B implant to put "small amounts" of probability at each site in S in his randomization. For example, if $\mid S \mid$ is very small so that there is no distribution that makes it possible for $\beta_i \leq \bar{\beta}_i$ for all sites in S and $\beta_m \leq \bar{\beta}_m$, then this equilibrium will not exist. On the other hand, notice that if a smaller "line" supports an EDE, then so must a larger line. In other words, efficient diffusion is more likely the larger the number of potential customers!*

Remark 2 *An explicit characterisation of $\bar{p}_i^t$ is tedious, but it is easy to see how it can be done. Suppose $\alpha_m = 1$ as above. Suppose site i is located at such a distance to the left of m that a message from m will arrive in period T. At period $T-1$, suppose i receives a recommendation from a neighbour on his left, which could have been relayed from an innovator $T-1$ steps to i's left. For what value of p would i accept this recommendation? If he waits for one more period, he will receive perfect information, either a message from his right neighbour, which will make $p = 1$ or no message, which will imply $p = 0$. His expected utility from waiting if his current belief that the product is good is p_{T-1}, is therefore $\delta p_{T-1} g$. If he buys now, he will get $p_{T-1}(g+b) - b$. The value of $\bar{p}_i^{T-1}$ is therefore $\frac{b}{b+(1-\delta)g}$. If $\delta = 0$, this is exactly $\bar{p}$; if δ is close to 1, this quantity is also close to 1. One can similarly calculate $\bar{p}_i^{T-2}$ at $T-2$ by comparing the utility of buying now versus waiting one period for imperfect information and two periods for perfect information, and so on.*

If i gets a recommendation before period T-1 from a neighbour on the path between him and m, perfect information is ruled out, since each player only speaks once and i's neighbour will not pass on any other message from the right. There is still some probability of imperfect information from the left (confirmatory if the product is good, none if the product is bad) coming from an innovator whose message takes the requisite number of periods to reach i. Once again, a value of $\bar{p}_i^t$, which makes i indifferent between buying and not buying, can be *calculated. On the line, both the above possibilities cannot arise together. In more general networks, one would take the maximum value of $\bar{p}_i^t$ calculated for each possible path on which additional information could travel.*

Remark 3 *As noted earlier, the game we consider has F choosing an implant at i (or choosing $\ominus$) at $t = 1$. We might ask whether there exists a profitable deviation from the strategies above, if the game is changed to allow postponing this choice past period 1. We do not have a definitive answer to this question. We can make the following points: (i) If G decides to postpone entry, he or she will postpone it to at most period 2, since the only reason for postponement is to obtain further information about where potential innovators are. Therefore, if $\rho = 0$, there is no incentive for G to deviate from the equilibrium above. (ii) Observability of innovators is an issue, since implants or other agents only observe their neighbours. To consider a deviation from the strategies mentioned above, suppose the G implant is only able to observe whether there is an innovator at m or its neighbours. If there is an innovator at m, the expected payoff for G will be the same as in equilibrium, except for saving the cost of an implant. If there is no innovator at m,there can also be innovators at $m-1$ or $m+1$. If there are innovators at both these sites, the payoff to the firm will be the same as before except for c. If there are no innovators at one or more of these sites, it will be profitable to enter at one of these sites. However, m cannot be used as an implant because of credibility issues. The payoff from waiting is therefore $(E+c)(\rho+(1-\rho)\rho^2)+\frac{1}{2}\delta E(1-\rho)^3+\delta E(1-\rho)2.\rho(1-\rho)$. Since this is continuous in ρ, there is a region for low values of ρ for which the deviation is unprofitable.*

Of course, inefficient equilibria can exist when n is not large enough. These will typically involve the bad implant delaying his recommendation for strategic reasons. In such equilibria, recommendations will not be accepted with probability one. We will discuss this issue in a later section (Section 7), where we construct an example of this result.

4 The star network

The star network, somewhat surprisingly, is one where an EDE does not exist. We will come back to this fact, more formally, in a subsequent section. Moreover, it is eaier to illustrate why an EDE might not exist, on the star rather than on the line. We label the centre of the star m and peripheral nodes by $i = 1, 2, ...n$. There are at least two peripheral nodes.

1. There is a unique node that maximizes degree, namely m. This is also the unique node that maximizes decay centrality. If a recommendation from m is accepted with

probability 1 by all the peripheral nodes, the payoff to firm F will be δn,if it locates an implant at m.This is clearly higher than locating an implant at a peripheral vertex, for either type, G or B. But if both G and B locate at m, the posterior probability of G will be less than $\overline{p}$, given it is initially this. Therefore, the recommendation is not acceptable to the peripheral nodes and hence an EDE does not exist.

2. One possibility would be for both G and B to choose m with some probability so as to make $p = \overline{p}$ and not have any implants with the remaining probability. Then a recommendation would be accepted by any peripheral site with probability γ, such that the expected payoff from an implant is exactly c.But for this to be an equilibrium (even before considering out of equilibrium moves), $c \geq 1$ otherwise there is a profitable deviation for either type to locating with probability one at a peripheral node. Typically, we think of c as being small, so this might not hold. If $\rho = 0$, a no-implant equilibrium could be sustained by beliefs that any (out-of-equilibrium) recommendations would be believed to be from B and be rejected, so the payoff from entering would be zero, whilst the cost is positive. But for $\rho > 0$,this cannot be an equilibrium.

3. This implies that the equilibrium must have implants with positive probability at some or all of the peripheral nodes.

   Suppose G locates at peripheral node i with probability 1. A recommendation from one of the other peripheral nodes j (to m) will be credible even if B puts probability β_j at the site if

$$\overline{p} \leq \frac{p[\rho(1-\rho)^{n-1}]}{p[\rho(1-\rho)^{n-1}] + (1-p)\beta_j}$$

   . conditional on m not being an innovator, of course. (If she is the probability will be either 1 or 0.) The expression holds with equality for the maximum β_j that can be placed at j. We can similarly determine the maximum β_m and β_i that are consistent with the recommendations from m and i being just acceptable. The maximal β_m is the sum of two parts-type B must randomize between speaking in period 1 (s_1) or in period 2 (s_2). B must be indifferent between speaking in period 1 or period 2. This implies that the recipients of a recommendation in period 2 must also be indifferent between accepting and rejecting the recommendation. There are two cases: (i) The sum of the maximal credible $\beta_j + \beta_m + \beta_i > 1$. Then the expected payoff to B from having an implant will be 1, with the actual β_j being less than the maximal value and β_m being set at its maximum value. A recommendation from m in the first period is then accepted by each of the peripheral nodes with probability $\frac{1}{n}$, and recommendations from the peripheral nodes are accepted with probability 1. (We shall discuss the rest of the equilibrium strategies shortly.) (ii)The sum of the maximal credible β_j $+\beta_m + \beta_i < 1$. Then the expected payoff to B must be c and the residual probability is allocatedby B to not entering at all. The acceptance probabilities are determined appropriately

4. We need to specify what happens in the second period. if there is no recommendation from m in the first period. If m is observed to speak in period 2 and not in period 1, then his recommendation can correspond to a good product, if he is passing on $i's$ recommendation or that of any innovator. It can also be that B is speaking strategically in period 2. The probability that B speaks in period 2 at m must be small enough that the Bayesian posterior that the product is G is exactly $\overline{p}$. Thus the remaining peripheral nodes must accept the second-period recommendation with enough probability to keep B indifferent.

5. Note that G is not indifferent between i and m. If she locates the implant at m, she gets exactly the same payoff as B. By locating it at i, she gets $B's$ expected payoff and the expected payoff from second period sales following $M's$ recommendations.

5 A Partial Characterization Result

In this section, we describe a sufficient condition for an EDE to exist, and then show that this condition is necessary for certain types of network structures.

Say that a link $ij \in \Gamma$ is *critical* if $\Gamma - ij$ has more components than Γ. That is, if a critical link is removed from a connected network Γ, then the network Γ no longer remains connected. Say that a node i is *critical* in Γ if *all* links of i are critical. Of course, if the network is a tree, then all links are critical, and so all nodes are critical.

Fix some set $M \subset N$. For any node $i \notin M$, let $\bar{N}_i(M, \Gamma) = N_i(\Gamma) \setminus (\cap_{j \in M} N_j(\Gamma))$, where $N_j(\Gamma)$ is the set of neighbors of j in Γ. Let $\bar{d}_i(M, \Gamma) = |\bar{N}_i(M, \Gamma)|$. We will refer to $\bar{d}_i$ as the *effective* degree of a node $i \notin M$. For any $m \in M$, let $\bar{d}_m = d_m = |N_m(\Gamma)|$. That is, if $i \in M$, then the effective degree of i coincides with $|N_i(\Gamma)|$.

Suppose α denotes the mixed strategy employed by the type G firm in any equilibrium. Let M be the support of α.

From now on, we shall restrict our attention to networks satisfying:

Assumption S: There is a unique node m maximizing decay centrality, so that M above (the support of α) consists of a single point m.[11]

[11] In an earlier version, we did not use this assumption here. This is available from the authors.

Several network structures satisfy this assumption - for example, the line when n is odd, or the star.

Then $\bar{N}_i(m, \Gamma)$ is the set of potential neighbors of i who may possibly believe that the product is good after receiving a single recommendation from i in period 1. To see this, notice that if the product is good, then a positive recommendation *must* come from m. The absence of such a recommendation signals that any other recommendation comes from a bad implant. So, any k who is a neighbor of m will find a recommendation credible only if m sends a recommendation.

In order to simplify the notation, we will simply write $\bar{N}_i, \bar{d}_i$ whenever the absence of m and Γ will not cause any confusion.

Given m, partition nodes into sets $S_1, \ldots, S_K$ such that S_1 is the set of nodes maximizing $\bar{d}_i$, S_2 is the set of nodes with the next highest value of $\bar{d}_i$, and so on.

For every node i, let h_i be the site in $\bar{N}_i$ which maximizes degree, and d_{h_i} be its degree. Notice that equation 1 implies that for any given value of β_i,

$$\eta^1_{h_i,i} \leq \eta^1_{j,i} \text{ for all } j \in \bar{N}_i$$

For each $j \in \bar{N}_i$, let β^j_i be the value of β_i which sets $\eta^1_{j,i} = \bar{p}_j$.

$$\text{For each } i, \bar{\beta}_i \equiv \min_{j \in \bar{N}_i} \beta^j_i \tag{2}$$

Remark 4 *Notice that each $\bar{\beta}_i$ depends on the vector α via equation 1. We will not explicitly indicate this dependence in order to simplify the notation.*

The next theorem identifies a sufficient condition for a network structure to support an EDE. It also shows that if all nodes in $D(\Gamma)$ are critical and (somewhat loosely speaking) have highest effective degree, then this condition is also necessary. These conditions are satisfied by the line, and so this theorem will include the line as a special case.

Theorem 1 *Suppose the cost of an implant is sufficiently low for both types of the firm to use an implant. Then, an EDE exists if there is some m contained in the set $D(\Gamma)$ such that*

$$\sum_{i \in S_1(m)} \bar{\beta}_i \geq 1 \tag{3}$$

and

$$\alpha_m = 1$$

. *Conversely, an EDE does not exist if every* $m \in D(\Gamma)$ *is critical, for* $m \subset S_1(m)$ *and equation 3 does not hold.*

Proof. Suppose there is α such that equation 3 is satisfied. Let α be the strategy employed by the good type, and choose β such that

$$\beta_i \leq \bar{\beta}_i \text{ for all } i \in S_1$$

and

$$\beta_i = 0 \text{ if } i \notin S_1$$

The response strategies are straightforward. All sites accept all recommendations in all periods.

It is easy to check that these strategies constitute a PBE. Clearly, the type G firm has no incentive to deviate since her implant is at some site maximizing decay centrality and recommendations are accepted with probability one. Similarly, the type B firm has no incentive to deviate since she obtains a payoff of $\bar{d}_1|S_1| - c$, conditional on no innovators in $\bar{N}_1$. Clearly, no other site can yield a higher payoff. The response decisions are optimal because (i) any recommendation coming from a site not in the union of the supports of α and β must be coming from an innovator, and (ii) the updated belief of any i after receiving a recommendation from a potential implant is at least as large as the threshold value $\bar{p}$.

Consider now the necessity of this condition. First, the type G firm must be choosing an α with support contained in $D(\Gamma)$ in any EDE. Fix any such α and suppose $\alpha_m > 0$. Since m is critical in Γ, all neighbors of m must accept a recommendation from m with probability one in an EDE. So, the type B firm by placing her implant at some site where $\alpha_i > 0$ can obtain $|S_1|$ "hits". Hence, in equilibrium, the support of β must be contained in S_1. Moreover, if $\beta_i > 0$, every member of $\bar{N}_i(\Gamma)$ has to accept the recommendation of i. Hence, the maximum probability weight that the type B firm can put on i cannot exceed $\bar{\beta}_i$. This is not possible if equation 3 does not hold. ■

Equation 3 is easy to interpret. If there are a sufficient number of nodes maximizing effective degree, then an EDE is easy to support since the type B firm has enough "space" to distribute his probability. Why is equation 3 not necessary without additional conditions? Suppose, for simplicity that $\alpha_m = 1$ for some node in $D(\Gamma)$, but $\alpha_m \notin S_1$, but say in S_2. Also, assume that equation 3 does not hold. It is possible then to have another equilibrium in which (i) the type B uses a mixed strategy over nodes in $S_1 \cup S_2$, (ii) the probability of acceptance of recommendations coming from nodes in S_1 is adjusted below one so as to ensure that the expected payoff from an implant located in S_1 is the same as that from an implant in S_2.[12] The freedom to distribute some probability weight over nodes in S_2 may now help in ensuring existence of equilibrium. Instead of formally deriving a sufficient condition for this type of equilibrium, we illustrate such an equilibrium in the example below.

Example 1 *Let $n = K(K+1)$ where $k > 3$. Denote $I = \{i_1, \ldots, i_K\}$, and let each $i_k \in I$ be the hubs of K stars $\Gamma_1, \ldots, \Gamma_K$, each Γ_k having $K+1$ peripheral sites. Finally, let site 1 be connected to each site in I, and to no other site. Also, no site i_k in I is connected to any site in the other stars. So,*

$$\Gamma = (\bigcup_{i \in I} \Gamma_i) \cup \{1i_1, \ldots, 1i_K\}$$

Figure 1 illustrates the graph for the case where $K = 2$.

We want to choose values of ρ and δ such that 1 is a member of $D(\Gamma)$. Then, $\bar{d}_1 = K$. But, notice that for each $i_k \in I$, $\bar{d}_{i_k} = K+1$ since none of the peripheral sites in Γ_{i_k} are connected to 1.

Suppose the type G firm places an implant at 1. Then, the expected benefit from the implant will be

$$B_1 = (1-\rho)\delta \left[\xi_K(1 + \delta\xi_{K+1})\right]$$

where for any k

$$\xi_k = \sum_{i=1}^{k} \binom{k}{i} i(1-\rho)^i \rho^{k-i}$$

On the other hand, if the type G firm places an implant at any of the sites in $\{i_1, \ldots, i_K\}$, then the expected payoff is

$$B_2 = (1-\rho)\delta \left[\xi_{K+1} + (1-\rho)\left(1 + \delta\xi_{K-1}(1 + \delta\xi_{K+1})\right)\right]$$

Evaluating the two expressions at $\rho = 0$, it is easy to check that

$$B_1 > B_2 \text{ if } \delta > \frac{2}{K^2 - 1}$$

That is, if $\delta > \frac{2}{K^2-1}$, then 1 maximizes decay centrality for $\rho = 0$ and hence for "small" values of ρ. Assume that $\rho > 0$ is such that $B_1 > B_2$. Also,

$$\delta \in (\frac{2}{K^2 - 1}, \frac{\xi_K}{\xi_{K+1}})$$

[12] However, the value of δ cannot be too high. If δ is high, then the low discounting may induce the type B implant at some site in S_1 to strategically postpone her recommendation to a later period.

Let

$$\bar{\beta}_1 = \frac{(1-\rho)^K p(1-\bar{p})}{\bar{p}(1-p)}, \bar{\beta}_i = \frac{p\rho(1-\bar{p})}{\bar{p}(1-p)} \textit{for each } i \in I,$$

Using equation 1, it is easy to verify that if $\alpha_1 = 1$*, then for each* $i \in I \cup \{1\}$*,* $\eta^1_{k,i} = \bar{p}$ *if* $\beta_i = \bar{\beta}_i$*. Suppose*

$$\sum_{i\in I} \bar{\beta}_i < 1, \text{ but } \sum_{i\in I} \bar{\beta}_i + \bar{\beta}_1 \geq 1$$

Notice that equation 3 is not satisfied. However, the pair of strategies (α, β) *along with response decisions specified below is an EDE.*

Let

(i) $\alpha_1 = 1$,

(ii) $\beta_i = \bar{\beta}_i$ *for each* $i \in I$*,* $\beta_1 = 1 - \sum_{i\in I} \bar{\beta}_i$.

(iii) Each $i \in I$ *accepts recommendation from* 1 *with probability one in period 1.*

(iv) Each $i \in \Gamma_{i_k}, i \neq i_k$ *accepts a recommendation from* i_k *with probability* $\frac{\Gamma_K}{\Gamma_{K+1}}$ *in period 1 and with probability 1 in period 2.*

(v) Both types of implants make their recommendations in period 1.

To check that these constitute an equilibrium, first notice that the type G firm has no incentive to deviate since (1) 1 *maximizes decay centrality, (2) the implant speaks immediately and the product diffuses throughout the network in 2 periods in view of (iii) and (iv) above. Consider now the type B implant. Her expected payoff from any site* $i \in I$ *is* ξ_K *given the acceptance probabilities of the peripheral sites in the star. This is also the expected payoff from the implant at* 1 *since* 1*'s recommendation is accepted with probability one. Also, note that since* $\delta \leq \frac{\xi_K}{\xi_{K+1}}$*, the implant at* $i \in I$ *has no incentive to strategically postpone her recommendation to period 2, even though her recommendation in this period would be accepted with probability one. Finally, we check that the response decisions described in (iii) and (iv) are optimal. To see this, note that the relevant agents in each case do not expect any further information flows and so* $\bar{p}$ *is the appropriate threshold for acceptance.*

If $m \in S_1$ but is not critical, then there could be an equilibrium of the following kind. The good product may diffuse throughout the network with probability one even if some neighbors of m who do not constitute critical links with m refuse m's recommendations - the fact that some link mi is not critical obviously implies that there is some path from m to i not involving the link mi. An implication of this is that the type B firm needs fewer customers in equilibrium. Now, suppose each node $i \in S_1$ has one neighbor in $\bar{N}_i(\Gamma)$ with very high degree, say h_i, while the others have relatively low degree. Then, one option for the type B firm is to put probability weights $\tilde{\beta}_i > \bar{\beta}_i$ on each i such that all nodes in $\bar{N}_i(\Gamma)$ except h_i accept i's recommendation. In other words, the freedom to dispense with h_i as a customer helps to raise the probability that type B can put on each node i in S_1 and so it may be possible to support an EDE even when equation 3 is not satisfied.

6 Comparative Statics

In this section, we discuss the role of different parameters and the network structure in sustaining an EDE.

It is easy to check that an increase in p makes it more likely that an EDE exists since it allows the type B firm to place more probability weight on nodes and still satisfy the requirement that the updated beliefs reach the threshold value of $\bar{p}$. In other words, if an EDE exists for some value of p and then p increases, there must continue to be an EDE. In what follows, we focus on the role of the network structure and of ρ.

6.1 The Role of the Network Structure

We first show that no site i can be too well-connected if the network is to sustain an EDE. In particular, no network that contains a star encompassing all nodes can support an EDE.

Theorem 2 *Suppose Γ contains a star as a subgraph. Then, Γ cannot support an EDE.*

Proof. Let $M = \{i \in N | d_i(\Gamma) = n-1\}$. If Γ contains a star, this set is non-empty. Then, all members of M maximize decay centrality. Let $m \in M$ be the site chosen by G. Take any site $i \notin m$. Then, $\bar{d}_i = 0$ since any site $j \neq i$ is connected to m. So, the support of β is also m. Then, it follows from equation 1 that

$$\eta^1_{j,i} \leq p < \bar{p}$$

So, no neighbor of the bad implant at i buys the product after receiving a recommendation from i. Since the bad type is indifferent between all sites in the support of β, no site in the support of β can get her recommendation accepted. This implies that only the good type employs an implant. However, this cannot be an equilibrium since the bad type would then deviate and place an implant at some site in M'. ■

Remark 5 *Note that Theorem 2 implies that the complete graph cannot support an EDE!*

Under Assumption S, we can also place an upper bound on the degree of m.

Theorem 3 *Let Assumption S hold, with m the unique node maximizing decay centrality in Γ. Then, if Γ is to support an EDE, $d_m \le \frac{n-1}{2}$.*

Proof. Let Γ support an EDE, and let $d_m = k$. Then, for all $i \neq m$, $\bar{d}_i \le n-k-1$. Suppose $d_m > \bar{d}_i$ for all $i \neq m$. Then the bad type would prefer to put her implant at m since recommendations from m are accepted with probability one. But, if $\beta_m = 1$, then $\eta^1_{m,j} < p < \bar{p}$ and this is not consistent with m's recommendations being accepted. Hence, $d_m = k \le \bar{d}_i \le n-k-1$ for some $i \neq m$. This implies $d_m \le \frac{n-1}{2}$. ∎

Theorems 2 and 3 describe some network structures that cannot support an EDE. In particular, nodes maximizing decay centrality cannot be too well-connected since their connections tend to reduce the effective degree of those nodes which are "close" to them. In an intuitive sense, it is also clear that "larger" networks[13] are more conducive to supporting EDE. The next example shows that an asymmetric network where the site maximizing decay centrality has low degree may actually facilitate efficient diffusion.

Example 2 *Let n be odd and $n \ge 9$. Choose any site, say 1, and divide $N \setminus \{1\}$ into two equally-sized disjoint subsets N_1, N_2. Let $2, 4$ be in N_1 and $3, 7$ be in N_2. The network is the following:*

(i) 1 is connected to just 2, 3.

(ii) No connection between nodes in N_1 and N_2.

(iii) All nodes in N_1 are connected to each other, except that 2, 4 are not connected to each other.

(iv) All nodes in N_2 are connected to each other, except that 3, 7 are not connected to each other.[14]

The effective degree of all nodes $i \neq m$ is $\frac{n-5}{2}$, while that of m is 2. So, $n \ge 9$ ensures that the effective degree of $i \neq m$ is at least as high as that of m.

Let $\alpha_1 = 1, \beta_i = 1/(n-1)$ for all $i \neq 1$. Responses are as follows:

(i) Sites 2 and 3 only accept recommendations from 1.

(ii) Site 1 does not accept any recommendations.

(iii) All other sites accept recommendations from all other sites.

[13] A larger network would be one that has a greater number of nodes of maximal effective degree, but keeping the node(s) that maximize decay centrality fixed. It is easy to define this explicitly for regular trees.

[14] Figure 2 illustrates this graph for the case $n = 9$.

This will be an equilibrium if $\beta_i = 1/(n-1)$ ensures that the posterior probability is at least as large as $\bar{p}$. Notice that $\bar{p}$ is the appropriate threshold for acceptance because the network structure ensures that no agent expects any further information flows.

Our conjecture is that given all other parameters, if this network of size n does not support an EDE, then *no network* of size n or smaller will support one. This is because this network allows Firm B to distribute its probability weight across $n-1$ sites.

6.2 The Influence of Innovators

Recall that ρ is the probability of an innovator. Here, we discuss the role of ρ on an EDE. In order to simplify the discussion, we assume throughout this section that Assumption S is satisfied. Thus, the type G firm must be using a pure strategy of placing her implant at m with probability one, where m is the unique node in $D(\Gamma)$.

Consider first the case where $\rho = 0$. This is a particularly stark case to analyze, where the equilibria of the previous sections do not exist-specifically only m can speak credibly in the first period. A bad implant at other sites will have to wait to speak. For instance, a neighbor of m can speak in period 2 by pretending to have received a recommendation from m in period 1, a site at a distance of 2 from m can speak in period 3 and so on.

The specific structure of the network will determine whether the good product will diffuse throughout the network.

Theorem 4 *Suppose $\rho = 0$, Assumption S is satisfied*[15] *and the unique node maximizing decay centrality is both critical and is in S_1. Then, Γ cannot support an EDE.*

Proof. Let m be the unique node maximizing decay centrality. If an EDE exists, $\alpha_m = 1$. Also, if m is critical, then all of m's neighbors must accept m's recommendations. If some neighbor j does not accept m's recommendation with probability one, then the criticality of m implies that the good product will not diffuse to some segment of the network. Since $m \in S_1$, the type B firm can get $|S_1|$ acceptances by putting an implant at m. Of course, $\beta_m = 1$ is not possible since all neighbors of m would then not revise their beliefs about the product. On the other hand, when $\rho = 0$, recommendations from no other site are credible initially. Hence, the expected payoff to type B from an implant at $i \neq m$ is strictly less than the payoff at m if the recommendation from m were credible. This shows that the type B firm does not have an equilibrium strategy that sustains efficient diffusion. ■

This theorem of course immediately implies that when $\rho = 0$, the line or, more generally, a tree where the node maximizing decay centrality also has maximal effective degree cannot

[15] This assumption can be relaxed but is maintained here for ease of exposition.

support an EDE. It is easy to construct examples of trees which support an EDE if m does not have maximal effective degree. Consider the following example.

Example 3 *Let $n \geq 9$ and n be odd. Let Γ be as follows. Individual 1 has just 2 links to 2 and 3. Divide the set $\{4, \ldots, n\}$ into two equal subsets, let 2 be connected to all individuals in the first subset, each of whom have no other link. Similarly, let,individual 3 be connected to all agents in the second subset, each of whom have no other link. Figure 1 illustrates the graph for the case $n = 9$.*

Assume that

$$\delta \geq \max\left(\frac{4}{n-3}, \frac{n-5}{n-3}\right)$$

First, notice that if type G places his implant at 1 and all subsequent recommendations are accepted with probability one, then his payoff is $2\delta + (n-3)\delta^2 - c$. On the other hand, if he places his implant at 2 or 3, then his payoff is $(((n-3)/2) + 1)\delta + \delta^2 + ((n-3)/2)\delta^3 - c$. The inequality $\delta \geq \frac{n-5}{n-3}$ ensures that the first sum is at least as large.

Second, assume also that p and $\bar{p}$ are such that the type B can put probability 1/2 on each of the nodes 2 and 3, and still make a credible recommendation in period 2. That is, the neighbors of 2 and 3 have to infer whether sites 2 and 3 have received a recommendation from the implant of type G placed at 1 in the previous period or whether it is the type B implant who is speaking in period 2, having strategically kept silent in period 1. A probability weight of $\beta_2 = \beta_3 = 1/2$ brings their updated belief that the recommendation is being passed on from 1 to the threshold $\bar{p}$, given the initial parameter values.

Then, the following is an EDE. The type G puts his implant at 1 with probability 1, the type B randomizes between 2 and 3 with equal probability. The type G implant speaks immediately while the type B implant speaks in period 2. All recommendations are accepted with probability one.

These constitute an equilibrium because if $\delta \geq \frac{4}{n-3}$, then the type B implant has no incentive to deviate and place his implant at 1 - he gets $\delta^2(n-3)/2 - c$ in equilibrium, whereas he would get $2\delta - c$ by placing his implant at 1. The response decisions are optimal because updated beliefs are not below the threshold.

How does a positive ρ impact on the possibility of efficient diffusion? Fix all other parameters and the network structure Γ. Now, ρ influences the nature of equilibrium in two ways. First, the higher the value of ρ, the lower is the net gain from having an implant for both firm types. So, there will be some value of $\bar{\rho}$ such that if $\rho \geq \bar{\rho}$,then one or both types of firm F will refrain from employing an implant.

Second, the value of ρ influences the updating process according to equation 1. Suppose ρ changes. How does this affect $\eta^1_{i,i-1}$ for a fixed value of β_i and distribution α? An increase in ρ makes it more likely that $i-1$ is an innovator, but also makes it less likely that none of the *other* neighbors of i are innovators. These effects move in opposite directions and an unambiguous answer is difficult to provide.

Suppose, however, that Assumption S holds, and an EDE exists where type B's equilibrium strategy places no probability weight on m, the unique node maximizing decay centrality. Then, the trade-offs are somewhat easier to discern.

In this case, the expression for $\eta^1_{i,j}$ simplifies considerably for $j \neq m$ and becomes

$$\eta^1_{i,j} = \frac{p\rho(1-\rho)^{d_i-1}}{p\rho(1-\rho)^{d_i-1} + (1-p)\beta_i}$$

Now,

$$\text{sign } \frac{\partial \eta^1_{i,j}}{\partial \rho} = \text{ sign } \frac{\partial p\rho(1-\rho)^{d_i-1}}{\partial \rho}$$

Hence,

$$\frac{\partial \eta^1_{i,j}}{\partial \rho} \geq 0 \text{ iff } (1-d_i\rho) \geq 0$$

So, if the initial value of ρ for which an equilibrium exists is "low", then an increase in ρ does not decrease $\eta^1_{i,j}$ and so the same strategies continue to be an equilibrium. On the other hand, an increase in ρ at higher values decreases $\eta^1_{i,j}$ and so the same strategy may not be an equilibrium. These suggest that there is some threshold value of ρ above which equilibrium of this type is not possible.

7 Inefficient Equilibria

From the previous sections, it is clear that some network structures may not support efficient diffusion equilibria. We examine the nature of the equilibria in the absence of an EDE. We first discuss the general issues and then illustrate them with an example, both in the case of a line with a unique median (or node maximising decay centrality). Considerations for general networks are similar but are difficult to see explicitly, whilst this is possible in the case of a line.

Recall from Section 3 that

$$S = N \setminus \{1, n, m-1, m+1, m-2, m+2\}$$

is the set of sites which can possibly have two consumers, and that an EDE may not be possible if S contains too few members for the type B firm to distribute its probability weight for acceptance probabilities of 1 to be best responses.

Possible ways out of this problem involve changing the equilibrium acceptance probabilities so that some or all nodes in $\{1, n, m-1, m+1, m-2, m+2\}$ on the line can be part of the support of β. We must consider the following, however:

1. The equilibrium acceptance probabilities have to be indexed now by time. Suppose some node i makes recommendations to nodes j in period 1 and these are accepted with probabilities less than 1. Node i, if it contains an implant from B, must not, in equilibrium, gain by deviating to being silent in period 1 but speaking in period 2. Such a deviation need not be known to be off the equilibrium path, since i might be passing on recommendations he received and accepted the preceding period. If an implant is supposed to speak in the first period with probability one, a recommendation from that site in period 2 will be accepted for sure. For players with two neighbours, the expected payoff in equilibrium, without considering the cost of an implant or a payoff from an innovator at the same site, must therefore be 2δ. Thus the incentives for implants to delay speaking must be considered seriously. (This is not an issue if all recommendations are accepted with probability 1, since in that case an implant with two neighbours would clearly do better speaking in the first period.) However, for each node i, there is a maximal time i_T at which he can speak credibly to all his neighbours. After this time the probability of acceptance goes to zero for at least one neighbour. We discuss this last point in detail later in this section. If a message is received from $m, 1$ or 9 in period 2, we assume this is from a B type and it is rejected. Messages at other sites in period 2 have positive probability. Later periods are covered similarly-though other beliefs could sustain the equilibrium.

2. The second major issue that arises with equilibrium acceptance probabilities in $(0,1)$ is that they affect the updating of probabilities. Consider $n = 9$, and an equilibrium where the G type places an implant at site 5 with probability 1 (as in the EDE). Take the case of individual 3, who is not an innovator or an implant and who does not receive any recommendation from either neighbour in period 1. Suppose this player 3 gets a recommendation in period 2. The conditional probability that the product is good given a recommendation from 2 depends on (i) the value of ρ, the probability a good recommendation can originate in 1, and (ii) the probability that 2 accepts a recommendation from 1. If the recommendation comes from 4 in period 2, the relevant probabilities include the acceptance probability with which 4 accepts 5's recommendation. If the acceptance probabilities are different, the conditional probabilities for recommendations from left or right will be different. The higher the acceptance probabilities, the more is not receiving a recommendation "bad news" for Player 2. This

means that not receiving a recommendation from the right could be worse news than not getting one from the left.

3. For small n and for given c, it might be optimal for both types to locate at m. In this case, acceptance probabilities have to make the bad type indifferent between entering and not entering and the good type strictly prefer to enter. We do not discuss this further because we consider the cost c to be relatively small so that both types will find it optimal to enter.

We now discuss i_T in more detail. Consider $i \in S$. Suppose $i = 2$, and there is an implant at 2. Suppose this implant at 2 makes a recommendation in period 2. Then, 3 can believe that site 1 is an innovator and has passed on a positive recommendation in period 1, which is then passed on by 2 in period 2. But, suppose 2 makes a recommendation in period 3. Then, 3 knows that 2 must be an implant who has not made a recommendation in an earlier period.

In general, suppose $i < m$, and is at a distance of k_l from 1 and k_r from m with $k_r < k_l$. Then, if i makes a recommendation in period $t = k_r$, then $i + 1$ will know that conditional on the product being good, a positive recommendation should have come from the right in no more $k_r - 1$ periods.

Hence, for each $i \in S$, there is a time period i_T up to which a bad implant can make recommendations and still have a positive probability of getting her recommendation accepted by both her neighbors. If the maximum time period for a credible recommendation is different for tthe left and right neighbour, we set i_T to be the minimum of the two. Note that $m_T = 1$.

For each $i \in S$, let s_i^t denote the probability with which the implant at i makes a recommendation in period t for $1 \leq t \leq i_T$. In equilibrium, the implant mixes over the timing of the recommendation only if the sum of the acceptance probabilities of his neighbors in period $t-1$ equals δ times the corresponding sum in period $(t+1)$. If i speaks in period t,then the player to $i's$ left assesses the probability that the product is good to be $\eta_{i-1,i}^t$ and the person to $i's$ right as $\eta_{i+1,i}^t$.The explicit expressions for these are complicated because they have to take into account possibly different acceptance probabilities along the way (from agents who received recommendations at prior time periods from a possible innovator or good implant to i) depending on whether the assumed path to i is from $i's$ left or right. Using the values of $\eta_{j,i}^t$, where j is $i's$ neighbour, is not an innovator and has received no recommendation from any of her other neighbours up to this point, we set each $\eta_{j,i}^t = \bar{p}_j^t$ (if j is to randomise), where $\bar{p}_j^t$ is the value that makes j indifferent between buying now or waiting.

We can similarly calculate i_T for other nodes. Note that a reccommendation is credible for *both* neighbours, if a site has two, if $t \leq i_T$, but it could remain credible for one neighbour and not for the other for $t > i_T$.

We now present an example which illustrates some of the features of an inefficient equilibrium.

Let $n = 9$ and Γ be a line. Let $\delta \geq 1/2, \rho = p = 1/2$, and $\bar{p} \equiv \frac{b}{b+g} = 0.6$.

The constructed equilibrium has the following features:

(i) $\alpha_5 = 1$.

(ii) $\beta_1 = \beta_9 = 0$, and all other $\beta_i > 0$. The values are specified below.

(iii) The implant of the G-type firm "speaks" in period 1. The B-type implant at nodes 3,4,5,6,7 also makes a recommendation immediately, while the implants at nodes 2 and 8 randomize between speaking in periods 1 and 2.

(iv) . Nodes 4 and 6 accept the recommendation from 5 with probability 1/2 in period 1. For $i = 3, 4$, the recommendation of i is accepted with probability one by $i - 1$ in period 1. Similarly, for $i = 6, 7$, the recommendation of i is accepted by $i + 1$ in period 1.

(v) Nodes 1 and 9 accept the recommendations of 2 and 8 respectively with probability one in periods 1 and 2.

(vi) Nodes 3 and 7 reject the recommendations of 2 and 8 respectively with probability one in period 1. They accept these recommendations with probability $\gamma = \frac{1-\delta}{\delta}$ in period 2.

(vii) All other "credible" recommendations are accepted with probability one. [16]

We now provide some detailed calculations to show that these constitute an equilibrium. First, note that

$$\eta^1_{45} = \eta^1_{65} = \frac{p(1-\rho)}{p(1-\rho) + (1-p)\beta_5}$$

Also, the threshold values of 4 and 6 for accepting a recommendation from 5 must be $\bar{p} = 0.6$. Setting $\eta^1_{45} = \eta^1_{65} = \bar{p}$, we get

$$\bar{\beta}_5 = 0.333$$

[16] A "credible recommendation is the following. Any recommendation in period $t \leq i_T$ is credible for both neighbours of $i \in S$. However, a recommendation from 2 in period 4 is credible for 1(since 2 may be passing on a recommendation originating from 5, but *not* credible for 3.

Note that this value of $\bar{\beta}_5$ ensures that there cannot be an EDE in this example. This is because if an EDE is to exist, all the probability mass of β has to be placed on 2,5 and 8, but since $\beta_5 \leq 1/3$ and $\beta_2 < 1/3$, $\beta_8 < 1/3$, the sum will be less than 1 and therefore an EDE is not possible.

Now,

$$\eta_{23}^1 = \frac{p(1-\rho)}{p(1-\rho)+(1-p)\beta_3}$$

Moreover, 2 cannot hope to get any further information in later periods - if 1 were an innovator and had to make a recommendation, she would have done so in period 1 itself. So, 2's threshold probability value for acceptance after receiving a recommendation from 3 is $\bar{p}$. Equating $\bar{p} = \eta_{23}^1$, we get

$$\bar{\beta}_3 = .166$$

For exactly similar reasons, we get

$$\bar{\beta}_7 = .166$$

We now calculate the threshold probability value for which 3 is indifferent between accepting and rejecting a recommendation received solely from 4 in period 1. Denote this as $\bar{p}_{34}^1$. If 3 accepts this recommendation at this value, then his expected utility is $\bar{p}_{34}^1(b+g)-b$. But, 3 could postpone his decision to purchase by one more period in the hope of receiving another recommendation from 2 next period.[17] Since 2 accepts 1 's recommendation with probability one, the probability of receiving such a recommendation equals $\bar{p}_{34}^1\rho$, and so the expected utility from waiting is $\bar{p}_{34}^1\rho\delta g$.[18] Hence,

$$\bar{p}_{34}^1 = \frac{b}{b+g-\delta\rho g} = \frac{3}{5-\delta}$$

Since

$$\eta_{34}^1 = \frac{p(1-\rho)}{p(1-\rho)+(1-p)\beta_4}$$

the value of β_4 which makes 3 indifferent between accepting and rejecting 4 's recommendation is

$$\bar{\beta}_4 = \frac{0.5-0.25\delta}{3}$$

[17] Conditional on receiving a recommendation from 4 in period 1, a second recommendation from 2 in period 2 must imply that 2 is passing on a recommendation from 1 since 2 cannot be either a bad implant or an innovator.

[18] We can check that if there is no recommendation from 2, the probability of G drops to $\frac{3}{7-\delta}$, which is less than $\bar{p}$.

Similarly,

$$\bar{\beta}_6 = \frac{0.5 - 0.25\delta}{3}$$

We now calculate $\bar{\beta}_2$ and $\bar{\beta}_8$. Let s_i^1, s_i^2 be the probabilities with which $i = 2, 8$ make recommendations over periods 1 and 2. We first calculate $\bar{p}_{32}^1$, the threshold probability which makes 3 indifferent between accepting and rejecting 2's recommendation in period 2. If 3 waits one period, then she may receive another recommendation from 4 (who would be passing on the recommendation from 5). Since 4 accepts 5's recommendation with probability 1/2, the probability that 3 receives a recommendation from 4 in period is $1/2\bar{p}_{34}^1$. Equating expected utilities, we get $\bar{p}_{32}^1 = \frac{3}{5-\delta}$.[19] Hence,

$$\bar{\beta}_2 s_2^1 = \frac{0.5 - 0.25\delta}{3}$$

In period 2, the threshold probability value of acceptance for 3 must be $\bar{p}$ since she cannot get any further recommendations. Using this, routine calculation yields

$$\bar{\beta}_2 s_2^2 = 0.021$$

Adding, and using the fact that $s_2^1 + s_2^2 = 1$, we get

$$\bar{\beta}_2 \in [0.10, 0.145] \text{ for } \delta \in [1/2, 1]$$

Similarly

$$\bar{\beta}_8 \in [0.10, 0.145] \text{ for } \delta \in [1/2, 1]$$

Hence, for all values of $\delta \geq 1/2$,

$$\sum_{i \neq 1,9} \bar{\beta}_i > 1$$

Now, for any value of $\delta \geq 1/2$, choose the probability distribution β such that

$$\beta_2 = \beta_8 = \bar{\beta}_2, \beta_i < \bar{\beta}_i \text{for } i = 3, 4, 5, 6, 7, \beta_1 = \beta_9 = 0$$

Also, choose s_i^1, s_i^2 for $i = 2, 8$ so that $\eta_{32}^1 = \bar{p}_{32}^1$, $\eta_{32}^2 = \bar{p}$, $\eta_{78}^1 = \bar{p}_{78}^1$, $\eta_{78}^2 = \bar{p}_{78}^2$. Clearly, these can be done given the choice of $\bar{\beta}_i$.

We first check that the actions specified for both the G-type and B-type implant are optimal. Note that for each $i \neq 1, 9$, the sum of the acceptance probabilities equals 1 if

[19] Here again, no recommendation from 4 is sufficiently bad news that 2 then does not buy. The conditional probability of G is again $\frac{3}{7-\delta}$.

$i \in S$, while exactly one neighbour of $i \notin S$ accepts the recommendation with probability one. Also, for $i = 2, 8$, the expected payoffs are equal in periods 1 and 2. It is easy to check that these ensure that the B-type has no profitable deviation. Similarly, the G-type is also maximizing expected payoff by placing his implant at the median with probability one.

The relevant probability weights β_i and s_i^1, s_i^2(for $i = 2, 8$) have been chosen so that except for nodes 1 and 9, all other nodes receiving credible recommendations are indifferent between accepting and rejecting recommendations. Hence, their stipulated responses are optimal. Of course, $\bar{p}_{12}^1 == \bar{p}_{98}^1 \bar{p}$ since neither 1 nor 9 can expect to get any other recommendation. This ensures that 1 and 9 have to accept the recommendations of 2 and 8 with probability one in period 1.

Suppose now that $\delta < 1/2$. Then, the foregoing is not an equilibrium because 2 and 8 have no incentive to make a recommendation in period 2. For instance, even if both 1 and 3 accept 2's recommendation in period 2 with probability one, the (gross) expected payoff is still less thn 1 - the expected payoff from recommending in period 1. However, it is easy to check that the following modification to the preceding specification is an equilibrium

(i) The B-type implant at nodes 2 and 8 set $s_i^1 = 1$. That is, they recommend with probability 1 in period 1.

(ii) Set $\beta_2 = \beta_8 = \frac{0.5-0.25\delta}{3}$. This ensures that 3 and 7 are indifferent between accepting and rejecting first-period proposals from 2 and 8 respectively. They both reject with probability one, while 1 and 9 accept with probability one.

(iii) Set $\beta_m = \bar{\beta}_m = 1/3$.

(iv) Set $\beta_i < \bar{\beta}_i$ for $i = 3, 4, 5, 6$.

(v) Other response decisions are the same as before.

Notice that this specification would not be an equilibrium when $\delta > 1/2$ because 2 (or 8) would then have an incentive to deviate by postponing a recommendaton to period 2. This would be accepted with probability 1 by both neighbours, hence giving an expected utility of $2\delta > 1$.

8 Extensions

We consider some possible extensions of the basic model.

8.1 Negative recommendations

We have assumed that "recommendations" can only be positive. The intuition motivating this is that a neighbour can observe someone using the product over some period of time;

if the product has been tried and found unacceptable, it will be discarded. To consider the chicken sausage example, someone who dislikes the particular brand will not use it at pot luck parties with her neighbours, unless she is acting as an implant for the firm selling the product. Also, individuals may not want to admit that they have been "duped" into buying a bad product, and so may not pass on negative information.

However, one needs to consider negative recommendations as well if only to consider the robustness of the model. It is easy to check that theorems 2 and 3 continue to remain valid. Theorem 4 will remain valid for regular networks such as the *circle*.

It also turns out that the possibility of negative recommendations will actually simplify calculations in one respect in that the probability calculations would not now depend on the potential recipient's degree. So, the analogue of equation 1 will now be

$$\eta^1_{i,i-1} = \frac{p\left[\alpha_{i-1} + (1 - \sum_{j\in N_i} \alpha_j - \alpha_i)\rho\right]}{p\left[\alpha_{i-1} + (1 - \sum_{j\in N_i} \alpha_j - \alpha_i)\rho\right] + (1-p)\beta_{i-1}} \tag{4}$$

However, it would complicate expected payoff calculations for B, where a high degree recipient of an implant's (positive) recommendation would be more likely to have countervailing negative information than one of low degree. Such a problem would not arise for regular graphs, but the general issue is illustrated below.

So, suppose B places implants at i and j with some positive probability. Then, his expected payoff from i is

$$E_i = \sum_{k\in\bar{N}_i} (1-\rho)^{d_k-1} P_k$$

where P_k is the probability with which the offer is accepted by k. The updated belief for all $k \in \bar{N}_i(\Gamma)$ will now be the same - it will just depend on β_i and not on the degrees of k.

So, since $E_i = E_j$, we need

$$\sum_{k\in\bar{N}_j} (1-\rho)^{d_k-1} P_k = \sum_{k\in\bar{N}_i} (1-\rho)^{d_k-1} P_k$$

The specification of a general sufficient condition is now more difficult because the derivation of the support of β is now more complicated. However, the qualitative result that a larger network is more conducive for efficient diffusion remains unchanged for the line and regular networks.

The possibility of negative recommendations may also help in sustaining efficient diffusion, particularly in dense networks where nodes have high degree. This is because the expected payoff of the type B firm will now be lower- and it will be lower the larger are the degrees of different sites that can be recipients of implant recommendations. So, the type B firm may simply not employ implants.

8.2 Multiple implants

If the firm can choose multiple implants, the qualitative features of the analysis will be similar. Clearly it does not make sense for the multiple implants to have overlapping supports (for the firm's randomised strategies). This suggests that for large networks, the firm will partition the networks in such a way as to have one implant randomly located (for B) in each element of the partition. If δ is close to 1 and an EDE exists as above, there is very little incentive for G to incur the cost of an additional implant, since this can only speed up the diffusion and the benefit from this might be low compared to the cost. Therefore, for low discounting, we would expect to have several B implants but only one G implant. This suggests that the B implants would either have to rely on a relatively high ρ for credibility or speak only at sufficiently late time periods for the message from a supposed good implant to have reached the particular site concerned.

8.3 The bad type's probability of producing a good product.

We have assumed so far that Firm F knows its type, where *type* is identified with the quality of the product that is produced. Let us redefine type as follows. The type G firm produces a good product with probability one, while type B produces a good product with small positive probability ϵ and the bad product with probability $1 - \epsilon$. Suppose as before that firm F knows its type in the modified sense.[20]

In this case, the following cases could arise (this is not an exhaustive description):

(i) There is a unique node m maximizing decay centrality, which also maximizes degree centrality. In this case, an EDE will not exist. The reason is that both G and B will care about speed of diffusion, though B will care less, and therefore both will prefer to locate at m rather than at any other node. As pointed out earlier, both types locating with probability 1 at m cannot be an equilibrium.

(ii) There is a unique node m maximizing decay centrality but it does not maximize effective degree centrality. Now B will be better off not locating at m for ε small enough. If he locates at a site that has effective degree at least 1 more than m, he can get some additional payoff. If he locates at m, he loses at least 1 for sure and obtains some additional

[20] Alternatively, suppose type is identified with quality of the product as before, but consumers who buy the bad quality product make a "mistake" with small probability - they make a positive recommendation with probability ϵ.

payoff depending on ε and δ. For ϵ small enough, this is not a best response for B. In this case, the analysis from the firm's point of view will not change from that discussed earlier in this paper. Hence, an EDE will exist under the same conditions as before.

9 Conclusions

Our motivation was to explore some of the implications of viral marketing within a social network. Consumers are aware that firms may "seed" the network, and also know that both "good" and "bad" quality firms may take recourse to this form of advertising. In the presence of imperfect information regarding the quality of a specific product, consumers cannot take recommendations from their social neighbours at face value - the credibility of recommendations is at stake. A crucial ingredient of our analysis is that customers are rational, and update beliefs using Bayes Rule. Within this framework, we show that a priori notions about what network structure is conducive to efficient diffusion may be misleading. In particular, "small" networks and highly-connected agents may actually deter the diffusion of the good product.

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Figure 1:

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Figure 2: