Incentive Compatible Market Design with an Application to Matching with Wages[*]

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Abstract: This paper studies markets for heterogeneous goods using mechanism-design theory. For each combination of desirable properties, I derive an assignment process with these properties in the form of a corresponding direct-revelation game, or I show that it does not exist. Each participant's preferences are privately known, quasi-linear in money, and depend upon the allocation that he gets — thus, a participant's 'type' is multidimensional. The key properties are individual rationality, incentive compatibility, budget balance, efficiency, and stability against coalitional deviations. The main results characterize mechanisms that are ex post incentive compatible in combination with other properties.

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1 Introduction

In this paper, we study a general heterogenous goods market with transfers. This environment encompasses several classic models considered in market design literature: one-to-one matching markets, housing markets, roommate problems, seller-buyer markets with discrete heterogeneous goods, and partnership dissolution problems. Each agent's preferences are privately known, and quasi-linear in money. In most of the paper, we focus on direct revelation mechanisms, and we are interested in finding mechanisms that satisfy the following properties at the ex post stage.

- Agents find it optimal to report their types truthfully (incentive compatibility).

- The mechanism does not run a deficit (budget balance).

- The mechanism does not run a deficit or create a surplus (exact budget balance).

- The sum of allocation utilities is maximized (efficiency).

- Agents benefit from participation (individual rationality).

Our main results characterize necessary and sufficient conditions for the existence of ex post incentive compatible mechanisms in combination with the properties listed above. Ex post incentive compatibility states that each agent prefers truth-telling even when the agent knows the reports of other agents assuming these agents report truthfully. This is equivalent to an ex post no regret property, as no agent would like to change her report even if she were to know the reports of other agents. This property is desirable for at least three reasons. First, it allows the mechanism to perform well even when agents do not have common knowledge of the distribution of types (which is crucial for mechanisms that are only interim incentive compatible, and in many applications, it is an unrealistic assumption). Second, it allows designing the mechanism with minimal assumptions about the distribution of types. Third, it removes any incentives to "game" the mechanism by either spying on other agents (to learn what they plan to report) or trying to delay reporting.

In the case of private values, ex post incentive compatibility is equivalent to dominant strategy incentive compatibility for direct mechanisms. Dominant strategy incentive compatibility states that each agent prefers truth-telling regardless of what other agents report. This is an additional

desirable property of mechanisms since it allows agents to optimize without forming any beliefs about the behavior of others.

Our first result (Theorem 1) provides a necessary and sufficient condition for the existence of an ex post incentive compatible, individually rational, and budget balanced mechanism.[1] Theorem 1 characterizes when it is possible to implement any given allocation rule, not necessarily efficient, with a mechanism that satisfies these properties.

Previous results for particular models have shown that in markets with asymmetric information, efficient mechanisms may require either running a deficit (i.e., having the social planner subsidize the mechanism) or creating a surplus (to be collected by the social planner or a third party). For example, Myerson and Satterthwaite (1983) show that in a seller-buyer market with an overlap in valuations, it is necessary to subsidize the mechanism to achieve efficiency. On the other hand, it is well known that in a public goods model, an efficient mechanism may run a surplus (Green and Laffont (1979)). Theorem 1 may be viewed as a general characterization of environments in which a mechanism can achieve the desired properties without requiring a subsidy, and covers all allocation rules (not only the efficient ones).

Our second result (Theorem 2) provides a necessary and sufficient condition for the existence of a mechanism that is ex post incentive compatible and exact budget balanced without the individual rationality constraint. To explain further, Theorem 2 characterizes when it is possible to implement any given allocation rule with a mechanism that satisfies these properties.

These two general results apply to both models with private and interdependent values, and they accommodate arbitrary correlation of types. However, the conditions we provide may be difficult to verify in a general interdependent values environment. It turns out that if agents have private values, then we can use existing results about Vickrey-Clarke-Groves mechanisms to rewrite the conditions for the efficient allocation rule. These conditions are easier to check for some applications.

In the second part of the paper, we apply our general results to a labor market in which a set of firms with one job opening each is matched with a set of workers. Firms care who they hire, and likewise, workers care what positions they get. We assume that in this market, each match may be

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[1]When we write individual rationality, budget balance, and exact budget balance without any qualifier, we mean "ex post."

beneficial to any firm-worker pair, that is, for all types, there exists a wage level such that these agents prefer to be matched at that wage rather than not participating, the so called "gap case."

As an application of Theorem 1, we show that there exists a dominant strategy incentive compatible, individually rational, efficient, and budget balanced mechanism for the labor market (Theorem 3). Moreover, we construct such a mechanism in Proposition 1 to serve as a benchmark. This result seems contrary to the result of Myerson and Satterthwaite (1983) who state that in a two-agent market it is necessary to subsidize the mechanism to obtain efficiency. The difference stems from our "gap case" assumption.

Our next result (Theorem 4) shows that if the type space is rich, then it is impossible to strengthen the previous result by requiring exact budget balance, even without individual rationality. This result may seem surprising because a second-price auction is a mechanism that has all the desired properties in a labor market with one firm (which is a special case of our model), if the firm cares only about the wage and considers all the workers to be equally desirable. However, this is possible because the seller is indifferent between buyers, which violates the rich type space assumption. More generally, we show that if the set of agents can be divided into two subsets, which satisfy the condition in Theorem 1, such that types of the agents in one subset does not change the efficient allocation of the agents on the other subset, then there exists a mechanism that is dominant strategy incentive compatible, individually rational, efficient, and exact budget balanced. In other words, if the type space is sufficiently restricted, it may be possible to obtain exact budget balance together with the other properties.

For the labor market, we consider an additional property called no blocking that rules out the existence of a firm-worker pair who can do better as a coalition of two rather than participating in the mechanism. Stability requires both individual rationality and no blocking, so that neither an agent nor a pair of agents blocks the mechanism.[2] Stability is ex post in nature. Agents can form blocking pairs at the ex post stage after they learn their types and their assigned matches. It turns out that ex post stability is too strong a condition to be satisfied. We prove that if there are more than two agents, and if the type space is rich, then there exists no mechanism which is dominant strategy incentive compatible, ex post stable, and budget balanced.

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[2]Stability has been studied extensively by the matching literature, see Roth and Sotomayor (1990).

	No Blocking	IR	IC	BB	exact BB	EFF
Yes		✓	✓	✓		✓
Yes	(*Ex Ante*) ✓	✓	✓		(*Ex Ante*)✓	✓
No			✓		✓	✓
No	✓	✓	✓	✓		✓

Table 1: Summary of Results for the Matching Problem with Wages. Each column refers to a property (IR for individual rationality, IC for incentive compatibility, BB for Budget Balance, and EFF for efficiency), whereas each row represents a result: the first entry states whether the result is positive or negative, other entries show the set of properties required for the result.

Ex post stability, albeit desirable, does not need to be necessary for a good performance of a mechanism in practice, if agents are not able to block the mechanism after they learn the outcome. For example, in some countries, it is customary for universities to hire only their own former graduate students as assistant professors. Thus, agents can only form blocking pairs at the ex ante stage when the qualities of students are not yet known.

We define ex ante stability for such environments: pairs of agents can block the mechanism at the ex ante stage before types are realized. Moreover, any agent can still block the mechanism ex post by quitting. If workers and firms are each symmetric ex ante, we construct a dominant strategy incentive compatible and ex ante stable mechanism, which also balances the budget on average (Theorem 5). In Table 1, we present a summary of the results for the labor market.[3]

In the last part of the paper, we consider other applications of the general theory. The first application is a seller-buyer market for multiple heterogeneous goods. Buyers have general preferences over bundles of goods allowing substitutes or complements. In this market, there exists a dominant strategy incentive compatible, individually rational, efficient, and exact budget balanced mechanism. The second application is a housing market, where each agent has a unique good to trade with unit demand. For the housing market, we show that there does not exist a dominant strategy incentive compatible, efficient, and budget balanced mechanism. The last example is a roommates problem with transfers, where trading is restricted to pairs of agents. We show that if agents prefer to trade with anyone rather than keeping their endowments, then there exists a dominant strategy incentive compatible, individually rational, efficient, and budget balanced mechanism.

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[3]We also define interim stability for environments where a blocking pair may form after types are realized, but before the outcome is known. We show that if there is only one agent on the short side of the market, then a dominant strategy incentive compatible, interim stable, efficient, and budget balanced mechanism exists.

Most of the matching literature studies markets with complete information. For example, Gale and Shapley (1962) study stability in the marriage model, a one-to-one matching market without transfers under complete information. Similarly, Shapley and Shubik (1971) add the possibility of making transfers in their assignment game, a one-to-one matching model. In their setup, the magnitude of benefits from trade is commonly known. Roth (1982), Roth (1989), Majumdar (2003), Ehlers and Massó (2007), Lee and Schwarz (2008), Chakraborty et al. (2009), and Niederle and Yariv (2009) study ex post stability and various notions of incentive compatibility in different matching markets without transfers. They either show that ex post stable and incentive compatible mechanisms do not exist, or make harsh assumptions to get existence.

Our paper is also related to the ex post implementation literature which tries to reduce the common knowledge assumptions in the spirit of 'Wilson Doctrine' (Wilson (1987)). In particular, Bergemann and Morris (2005) study when ex post implementation is equivalent to interim implementation for all type spaces. Bergemann and Morris (2008) study the full ex post implementation problem.

In Appendix 1, we consider the special case that there is one agent on the short side of the market. In auctions, there is usually one seller, so auctions are specific examples of such markets. For this special case, we characterize the mechanism that maximizes the expected revenue of the solo agent. We also introduce new ascending auctions to implement this optimal mechanism and the benchmark efficient mechanism.

2 General Model

In a **heterogenous goods market with transfers**, there is a finite set of agents N and k types of goods. Each agent $i \in N$ has an **endowment** $e_i \in \mathbb{Z}_+^k$ which specifies non-negative integer quantities for all types of goods. Let e be the **profile of endowments** consisting of e_i for all $i \in N$. Agent i's **type** is $\theta_i \in \Theta_i$ where Θ_i is a compact and connected subset of some Euclidean space. Suppose that $\theta \equiv \underset{i \in N}{\times} \theta_i$ is distributed according to distribution D. This formulation allows types to be correlated. Each agent's utility function is quasi-linear over allocation and monetary transfer. Hence, if agent i receives $x_i \in \mathbb{Z}_+^k$ and makes a payment of t_i, her utility is $u_i(x_i, \theta) + t_i$, where $u_i(x_i, \theta)$ is continuous in θ_i. Therefore, we allow values to be interdependent, i.e., they

may depend on signals of other agents. Let $\mathcal{A}^f$ denote a finite **set of feasible allocations** given exogenously. For example, $\mathcal{A}^f$ can be the set of allocations where each agent gets one good at most.[4]

A **direct revelation mechanism** for our formulation is a pair (μ, t) where $\mu : \Theta \to \mathcal{A}^f$ is an **allocation function** and $t : \Theta \to \mathbb{R}^{|N|}$ is a **transfer function**. For a given type profile θ, $\mu_i(\theta) \in \mathbb{Z}_+^k$ is the allocation of agent i and $t_i(\theta)$ is the transfer to agent i. The **net utility** of agent i from participation is,

$$v_i(\theta) \equiv [u_i(\mu_i(\theta), \theta) - u_i(e_i, \theta)] + t_i(\theta).$$

By the revelation principle, it is sufficient to consider direct revelation mechanisms to check the existence of mechanisms with the desired properties listed below.

Definition 1. *A direct revelation mechanism (μ, t) satisfies **ex post incentive compatibility** if, for all $i \in N$ and $\theta \in \Theta$,*

$$v_i(\theta) \geq [u_i(\mu_i(\theta_i', \theta_{-i}), \theta) - u_i(e_i, \theta)] + t_i(\theta_i', \theta_{-i})$$

for all $\theta_i' \in \Theta_i$.

Ex post incentive compatibility states that each agent prefers truth-telling even when the agent knows the reports of other agents assuming these agents report truthfully. This is equivalent to an ex post no regret property, as no agent would like to change her report even if she were to know the reports of other agents.[5] If (μ, t) is ex post incentive compatible, then we say that this mechanism implements μ (with ex post incentives).

Definition 2. *A direct revelation mechanism (μ, t) satisfies (ex post) **individual rationality** if, for all $i \in N$,*

$$v_i(\theta) \geq 0$$

for all $\theta \in \Theta$.

That is, a mechanism is individually rational if agents have non-negative net utilities for all realized types. If this is violated for agent i, we say that i blocks (μ, t).

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[4]The choice of modeling is inspired by Sonmez (1999). He studies a general indivisible goods exchange problem without transfers.

[5]Ex post incentive compatibility was first introduced as uniform equilibrium in d'Aspremont and Gérard-Varet (1979) and as uniform incentive compatibility in Holmstrom and Myerson (1983).

Definition 3. *A direct revelation mechanism* (μ, t) *satisfies (ex post)* **efficiency** *if*

$$\mu(\theta) \in \arg\max_{x \in \mathcal{A}^f} \sum_i u_i(x_i, \theta),$$

for all $\theta \in \Theta$.

In words, a mechanism is efficient if the allocation function maximizes the sum of allocation utilities over the set of feasible allocations.

Definition 4. *A direct revelation mechanism* (μ, t) *satisfies (ex post)* **budget balance** *if*

$$\sum_i t_i(\theta) \leq 0$$

for all $\theta \in \Theta$.

Budget balance means that the mechanism does not run a deficit.

We only use the above definitions of individual rationality, efficiency, and budget balance. Therefore, when we use them, we do not write "ex post" in front of the notion.

Definition 5. *A direct revelation mechanism* (μ, t) *satisfies (ex post)* **exact budget balance** *if*

$$\sum_i t_i(\theta) = 0$$

for all $\theta \in \Theta$.

Definition 6. *A direct revelation mechanism* (μ, t) *satisfies* **ex ante exact budget balance** *if*

$$\sum_i E_\theta[t_i(\theta)] = 0.$$

Two separate exact budget balance notions are considered. Ex post exact budget balance means that the sum of transfers is zero for all realized types, whereas ex ante exact budget balance means the sum of transfers is zero on average. When we use the former notion, we do not write "ex post" explicitly, whereas if we use the latter, we write "ex ante".

3   General Results

Let (μ, t) be an ex post incentive compatible direct revelation mechanism. In this section, we provide necessary and sufficient conditions for the existence of a transfer function t' such that

(μ, t') is 1) ex post incentive compatible, individually rational, budget balanced and 2) ex post incentive compatible, and exact budget balanced.

We use the following lemma to prove these results.

Lemma 1. *Suppose (μ, t) is an ex post incentive compatible direct revelation mechanism. Then (μ, t') is an ex post incentive compatible direct revelation mechanism if and only if $t_i(\theta) = t'_i(\theta) + g_i(\theta_{-i})$ for some function g_i for all i.*

All omitted proofs are in Appendix 2.

The lemma tells us that if two mechanisms are ex post incentive compatible with the same allocation function, then the difference in transfers for any agent in two mechanisms cannot depend on the type of this agent. Moreover, it also states that changing the transfer function of an agent by a function which depends on the types of other agents preserves ex post incentive compatibility.

Definition 7. *Let (μ, t) be an ex post incentive compatible direct revelation mechanism. Then, $\theta_i^*(\theta_{-i})$ is the type of agent i which minimizes $v_i(\theta_i, \theta_{-i})$, called the **worst-off type** for agent i given θ_{-i}.*

We need to prove that the worst-off type exists in general. The next lemma establishes the existence.

Lemma 2. *Suppose that (μ, t) is ex post incentive compatible. Then there exists a worst-off type, for all $i \in N$ and $\theta_{-i} \in \Theta_{-i}$.*

By Lemma 1, if (μ, t) and (μ, t') are ex post incentive compatible, then the worst-off types of agent i given θ_{-i} for both mechanisms are the same. Therefore, the worst-off type does not depend on the particular transfer function as long as it implements the allocation function with ex post incentives. The existence of a worst-off type is not straightforward from definition. However, ex post incentive compatibility guarantees its existence.

If there exists an ex post incentive compatible mechanism (μ, t) which is not necessarily individually rational or budget balanced, the next theorem gives a necessary and sufficient condition using the utilities derived from (μ, t) for the existence of a transfer function t' such that (μ, t') is ex post incentive compatible, individually rational, and budget balanced.

Theorem 1. *Suppose that (μ, t) is an ex post incentive compatible direct revelation mechanism. Let $v_i(\theta)$ be the net utility of agent i and $\theta_i^*(\theta_{-i})$ be the worst-off type in this mechanism. Then there exists a transfer function t' such that (μ, t') is ex post incentive compatible, individually rational, and budget balanced if and only if the following inequality holds:*

$$\sum_i v_i(\theta_i^*(\theta_{-i}), \theta_{-i}) \geq \sum_i t_i(\theta), \ \forall \theta \in \Theta. \tag{1}$$

Proof. "If" part: Let $t_i'(\theta) = t_i(\theta) - v_i(\theta_i^*(\theta_{-i}), \theta_{-i})$. By Lemma 1, (μ, t') is ex post incentive compatible. Add $u_i(\mu_i(\theta), \theta) - u_i(e_i, \theta)$ to both sides of this equation to get $v_i'(\theta) \equiv [u_i(\mu_i(\theta), \theta) - u_i(e_i, \theta)] + t_i'(\theta) = v_i(\theta) - v_i(\theta_i^*(\theta_{-i}), \theta_{-i})$ which must be non-negative by definition of the worst-off type, so individual rationality is also satisfied. Finally, $\sum_i t_i'(\theta) = \sum_i [t_i(\theta) - v_i(\theta_i^*(\theta_{-i}), \theta_{-i})]$ which is less than or equal to zero by assumption. Therefore, (μ, t') is budget balanced as well as incentive compatible and individually rational.

"Only if" part: Suppose that (μ, t') is ex post incentive compatible, individually rational, and budget balanced. By Lemma 1, $t_i'(\theta) = t_i(\theta) + g_i(\theta_{-i})$. Add $u_i(\mu_i(\theta), \theta) - u_i(e_i, \theta)$ to both sides to get $v_i'(\theta) = v_i(\theta) + g_i(\theta_{-i})$. Since (μ, t') is ex post individually rational, $g_i(\theta_{-i}) \geq -v_i(\theta_i^*(\theta_{-i}), \theta_{-i})$. Finally, ex post budget balance implies $0 \geq \sum_i t_i'(\theta) = \sum_i [t_i(\theta) + g_i(\theta_{-i})] \geq \sum_i [t_i(\theta) - v_i(\theta_i^*(\theta_{-i}), \theta_{-i})]$. Therefore, we get $\sum_i v_i(\theta_i^*(\theta_{-i}), \theta_{-i}) \geq \sum_i t_i(\theta)$. ∎

The intuition for this result is as follows. The highest amount that we can charge an agent cannot depend on her type because of ex post incentive compatibility. Therefore, the highest amount we can charge preserving ex post individual rationality is the utility that she gets if she were to be the worst-off type, which depends on types of other agents. If the sum of these charges can cover the sum of transfers that we have to make to agents, then there exists such a mechanism. Otherwise, there exists none. That is exactly what (1) checks.

Theorem 2. *Let (μ, t) be an ex post incentive compatible direct revelation mechanism and $\tau(\theta)$ be the sum of transfers, i.e., $\tau(\theta) = \sum_i t_i(\theta)$. For each i fix a type $\underline{\theta}_i \in \Theta_i$. Then there exists a transfer function t' such that (μ, t') is ex post incentive compatible and exact budget balanced if and only if*

the following holds:

$$\sum_{j=0}^{|N|}(-1)^j \sum_{\{i_1,\ldots,i_j\}\subseteq N} \tau(\underline{\theta}_{i_1},\ldots,\underline{\theta}_{i_j},\theta_{-i_1,\ldots,i_j}) = 0, \ \forall \theta \in \Theta.^6 \tag{2}$$

The above statement is valid for any particular choice of $\underline{\theta}_i$.

Proof. "If" part: Suppose that (2) is satisfied. We construct t' such that (μ, t') is ex post incentive compatible and exact budget balanced.

$$t'_1(\theta) = t_1(\theta) + \sum_{j=1}^{|N|}(-1)^j \sum_{1\in\{i_1,\ldots,i_j\}} \tau(\underline{\theta}_{i_1},\ldots,\underline{\theta}_{i_j},\theta_{-i_1,\ldots,i_j})$$

$$t'_2(\theta) = t_2(\theta) + \sum_{j=1}^{|N|}(-1)^j \sum_{1\notin\{i_1,\ldots,i_j\},2\in\{i_1,\ldots,i_j\}} \tau(\underline{\theta}_{i_1},\ldots,\underline{\theta}_{i_j},\theta_{-i_1,\ldots,i_j})$$

$$\vdots$$

$$t'_n(\theta) = t_n(\theta) + \sum_{j=1}^{|N|}(-1)^j \sum_{1,\ldots,n-1\notin\{i_1,\ldots,i_j\},n\in\{i_1,\ldots,i_j\}} \tau(\underline{\theta}_{i_1},\ldots,\underline{\theta}_{i_j},\theta_{-i_1,\ldots,i_j})$$

By Lemma 1 (μ, t') is ex post incentive compatible. To show that it is also exact budget balanced, sum the above equalities to get:

$$\sum_{i\in N} t'_i(\theta) = \sum_{i\in N} t_i(\theta) + \sum_{j=1}^{|N|}(-1)^j \sum_{\{i_1,\ldots,i_j\}\subseteq N} \tau(\underline{\theta}_{i_1},\ldots,\underline{\theta}_{i_j},\theta_{-i_1,\ldots,i_j}).$$

By (2), $\sum_{j=1}^{|N|}(-1)^j \sum_{\{i_1,\ldots,i_j\}\subseteq N} \tau(\underline{\theta}_{i_1},\ldots,\underline{\theta}_{i_j},\theta_{-i_1,\ldots,i_j}) = -\tau(\theta)$. Moreover, $\sum_{i\in N} t_i(\theta) = \tau(\theta)$ by definition. Therefore, we get that $\sum_{i\in N} t'_i(\theta) = \tau(\theta) - \tau(\theta) = 0$.

"Only if" part: Suppose that (μ, t') is ex post incentive compatible and exact budget balanced. Then by Lemma 1, $t_i(\theta) = t'_i(\theta) + g_i(\theta_{-i})$. If we sum this over all agents we get $\sum_i[t_i(\theta)] = \sum_i[t'_i(\theta) + g_i(\theta_{-i})]$ which is equivalent to $\tau(\theta) = \sum_i g_i(\theta_{-i})$.

Define $g_i^{(1)}(\theta_{-i})$ as follows: $g_i^{(1)}(\theta_{-i}) = g_i(\theta_{-i}) - g_i(\underline{\theta}_1,\theta_{-1,i})$ for $i \neq 1$ and $g_1^{(1)}(\theta_{-1}) = g_1(\theta_{-1}) +$

$\sum_{i \neq 1} g_i(\underline{\theta}_1, \theta_{-1,i})$. Therefore, $\sum_i g_i^{(1)}(\theta_{-i}) = \sum_i g_i(\theta_{-i})$. Hence,

$$\tau(\theta) = \sum_i g_i^{(1)}(\theta_{-i}). \tag{3}$$

Plug in $\theta_1 = \underline{\theta}_1$ to get $\tau(\underline{\theta}_1, \theta_{-1}) = g_1^{(1)}(\theta_{-1}) + \sum_{i \neq 1} g_i^{(1)}(\underline{\theta}_1, \theta_{-1,i})$. By the above transformation, $g_i^{(1)}(\underline{\theta}_1, \theta_{-1,i}) = 0$, $i \neq 1$. Therefore, we get that $\tau(\underline{\theta}_1, \theta_{-1}) = g_1^{(1)}(\theta_{-1})$. Using this equality, (3) can be rewritten as

$$\tau(\theta) - \tau(\underline{\theta}_1, \theta_{-1}) = \sum_{i \neq 1} g_i^{(1)}(\theta_{-i}).$$

Define $g_i^{(j)}(\theta_{-i})$ inductively on j by $g_i^{(j)}(\theta_{-i}) = g_i^{(j-1)}(\theta_{-i}) - g_i^{(j-1)}(\underline{\theta}_j, \theta_{-i,j})$ for $i \neq j$ and $g_j^{(j)}(\theta_{-j}) = g_j^{(j-1)}(\theta_{-j}) + \sum_{i \neq j} g_i^{(j-1)}(\underline{\theta}_j, \theta_{-i,j})$. By induction on m, it is easy to see that $\tau(\theta) - \sum_{\{i_1\} \subseteq N'} \tau(\underline{\theta}_{i_1}, \theta_{-i_1}) + \ldots + (-1)^m \sum_{\{i_1, \ldots, i_m\} \subseteq N'} \tau(\underline{\theta}_{i_1}, \ldots, \underline{\theta}_{i_m}, \theta_{-i_1, \ldots, i_m}) = \sum_{i \in N \backslash N'} g_i^{(m)}(\theta_{-i})$ where $N' = \{1, \ldots, m\}$. Take $N' = N$ to get:

$$\sum_{j=0}^{|N|} (-1)^j \sum_{\{i_1, \ldots, i_j\} \subseteq N} \tau(\underline{\theta}_{i_1}, \ldots, \underline{\theta}_{i_j}, \theta_{-i_1, \ldots, i_j}) = 0.$$

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To achieve exact budget balance, we need to charge each agent an extra amount such that the sum of the charges is equal to the sum of transfers, $\tau(\theta)$. However, for agent i, this extra charge can only be a function of θ_{-i} to preserve ex post incentive compatibility. Therefore, we can achieve exact budget balance if and only if we can decompose $\tau(\theta)$ additively into $|N|$ functions where function i depends only on θ_{-i}. If such a decomposition is possible, then we can find a decomposition using only τ by algebraic manipulations as in the proof. Condition 2 gives the exact decomposition.

For a general environment, it may be hard to check (2). However, if one of the agents' type has a degenerate distribution, i.e., when the support of the type has only one value, then the condition is satisfied. Thus, this result can be applied to the important special case of auctions, where it is usually assumed that the auctioneer has a constant type.

Corollary 1. *Let (μ, t) be an ex post incentive compatible direct revelation mechanism. Suppose*

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that one of the agents' type has a degenerate distribution. Then there exists a transfer function t' such that (μ, t') is ex post incentive compatible and exact budget balanced.

If agent i's type has a degenerate distribution, then this agent does not face any incentive issues. Therefore, agent i can be charged the residual balance. This preserves ex post incentive compatibility and achieves exact budget balance.

3.1 Efficient Allocation with Private Values

In this subsection, we consider implementing the efficient allocation function. Both Theorems 1 and 2 can be applied to environments with interdependent values and correlated types. However, to apply these results, one needs to find a transfer function to implement the efficient allocation function with ex post incentives. This may be hard for environments with interdependent values. However, if values are private, then we can use a Vickrey-Clarke-Groves (henceforth VCG) payment function.[7] This enables us to rewrite the conditions given in Theorems 1 and 2 for the efficient allocation rule.

Definition 8. (Private Values) *Agent $i \in N$ has private values if, for all $\theta \in \Theta$ and $x_i \in \mathbb{Z}_+^k$, $u_i(x_i, \theta) = u_i(x_i, (\theta_i, \theta'_{-i}))$, for all $\theta'_{-i} \in \Theta_{-i}$. When this holds the utility of agent i is denoted by $u_i(x_i, \theta_i)$.*

When values are private, ex post incentive compatibility is equivalent to the stronger notion of dominant strategy incentive compatibility that we define below.

Definition 9. *A direct revelation mechanism (μ, t) satisfies **dominant strategy incentive compatibility** if, for all $i \in N$, $\theta'_{-i} \in \Theta_{-i}$ and $\theta \in \Theta$,*

$$[u_i(\mu_i(\theta_i, \theta'_{-i}), \theta) - u_i(e_i, \theta)] + t_i(\theta_i, \theta'_{-i}) \geq [u_i(\mu_i(\theta'), \theta) - u_i(e_i, \theta)] + t_i(\theta')$$

for all $\theta'_i \in \Theta_i$.

Dominant strategy incentive compatibility requires each agent to play truthfully regardless of what other players are doing.

Suppose μ^e is an efficient allocation function and $SS(\theta)$ be the maximum value of sum of allocation utilities when the type profile is θ, i.e., $SS(\theta) = \sum_i [u_i(\mu_i^e(\theta), \theta_i) - u_i(e_i, \theta_i)]$. Let $t_i(\theta) =$

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[7]Thus, it follows the idea of efficient mechanism design by Vickrey (1961), Clarke (1971) and Groves (1973).

$\sum_{j \neq i} [u_j(\mu_j^e(\theta), \theta_j) - u_j(e_j, \theta_j)]$. Therefore, $v_i(\theta) = [u_i(\mu_i^e(\theta), \theta_i) - u_i(e_i, \theta_i)] + t_i(\theta) = \sum_{j \in N} [u_j(\mu_j^e(\theta), \theta_j) - u_j(e_j, \theta_j)] = SS(\theta)$. Moreover, $\sum_i t_i(\theta) = (|N| - 1)SS(\theta)$. Thus, Theorem 1 reduces to the following.

Corollary 2. *Let $SS(\theta)$ be the sum of allocation utilities for the efficient allocation function. If agents have private values, then there exists a dominant strategy incentive compatible, individually rational, efficient, and budget balanced mechanism if and only if the following holds:*

$$\sum_i SS(\theta_i^*(\theta_{-i}), \theta_{-i}) \geq (|N| - 1)SS(\theta), \forall \theta \in \Theta. \tag{4}$$

A similar result is shown for interim implementation with exact budget balance in Krishna and Perry (1998) and Williams (1999) where they have expectations of terms in (4).

For the efficient allocation function with private values, Theorem 2 reduces to the following.

Corollary 3. *Suppose that agents have private values.*

1. *For each i fix a type $\underline{\theta}_i \in \Theta_i$. Then there exists a dominant strategy incentive compatible, efficient, and exact budget balanced mechanism if and only if the following holds:*

$$\sum_{j=0}^{|N|} (-1)^j \sum_{\{i_1,\ldots,i_j\} \subseteq N} SS(\underline{\theta}_{i_1}, \ldots, \underline{\theta}_{i_j}, \theta_{-i_1,\ldots,i_j}) = 0, \forall \theta \in \Theta. \tag{5}$$

2. *Let μ^e be an efficient allocation function. Suppose that for each agent i there exists an agent j such that for all $\theta \in \Theta$ and $\theta_j' \in \Theta_j$, $\mu_i^e(\theta) = \mu_i^e(\theta_j', \theta_{-j})$. Then there exists a dominant strategy incentive compatible, efficient, and exact budget balanced mechanism.*

Part 1 characterizes environments with private values for which there exists a dominant strategy incentive compatible, efficient, and exact budget balanced mechanism. Part 2 gives a sufficient condition for this to hold. This condition is weaker than the condition given in Corollary 1.

We ignore the individual rationality constraint in the corollary stated above. However, we can provide a sufficient condition for the existence of a mechanism with the properties listed above and individual rationality.

Corollary 4. *Suppose the set of agents can be divided into two subsets N_1 and N_2 such that in each subset the efficient allocation is a reallocation of their endowments, and that this reallocation*

does not change with the type profile of agents in the other subset. Suppose also that (4) holds for N_1 and N_2, separately. If agents have private values, then there exists a mechanism which is dominant strategy incentive compatible, individually rational, efficient, and exact budget balanced.

By assumption, there are two separate submarkets, and for each submarket (4) holds. By Corollary 2, there exists a dominant strategy incentive compatible, individually rational, and efficient mechanism for each submarket that creates a surplus. We can distribute the surplus from one submarket to the agents on the other submarket in any way. This does not thwart incentives, and individual rationality is preserved. Moreover, the final allocation is efficient for the whole economy by assumption, and exact budget balance is satisfied.

Now, let us consider some examples to see when (5) holds.

Example 1. There are two agents, seller S and buyer B. The seller owns a good that she wants to sell. The seller's value for the good, $\theta_S \in [0, 1]$, is her private value and the buyer's value, $\theta_B \in [2, 3]$, is his private value. Let $\underline{\theta}_S = 0$ and $\underline{\theta}_B = 2$. Hence, the left hand side of (5) is equivalent to:

$$SS(\theta_S, \theta_B) - SS(\underline{\theta}_S, \theta_B) - SS(\theta_S, \underline{\theta}_B) + SS(\underline{\theta}_S, \underline{\theta}_B) = (\theta_B - \theta_S) - (\theta_B - 0) - (2 - \theta_S) + (2 - 0)$$

$$= 0.$$

Therefore, there exists a dominant strategy incentive compatible, efficient, and exact budget balanced mechanism. In this example, we can construct such a mechanism easily. Each agent reports a value. Regardless of the reports, we transfer the good at $1\frac{1}{2}$. This mechanism satisfies all the properties.

In the first example, both agents' types do not change the efficient allocation and (5) holds.

Example 2. There are three agents, seller S and buyers B_1 and B_2. The seller owns a good. The seller's value for the good, $\theta_S \in [0, 1]$, is her private information. Similarly, buyer B_i's value for the good, $\theta_{B_i} \in [2, 3]$, is his private information, $i = 1, 2$. Let $\underline{\theta}_S = 0$, $\underline{\theta}_{B_1} = 2$ and $\underline{\theta}_{B_2} = 3$. The left

hand side of (5) reduces to:

$$
\begin{aligned}
SS(\theta_S, \theta_{B_1}, \theta_{B_2}) \quad - \quad & SS(\underline{\theta}_S, \theta_{B_1}, \theta_{B_2}) - SS(\theta_S, \underline{\theta}_{B_1}, \theta_{B_2}) - SS(\theta_S, \theta_{B_1}, \underline{\theta}_{B_2}) + SS(\underline{\theta}_S, \underline{\theta}_{B_1}, \theta_{B_2}) \\
+ \quad & SS(\underline{\theta}_S, \theta_{B_1}, \underline{\theta}_{B_2}) + SS(\theta_S, \underline{\theta}_{B_1}, \underline{\theta}_{B_2}) - SS(\underline{\theta}_S, \underline{\theta}_{B_1}, \underline{\theta}_{B_2}) \\
= \quad & \max\{\theta_{B_1} - \theta_S, \theta_{B_2} - \theta_S\} - \max\{\theta_{B_1}, \theta_{B_2}\} - \max\{2 - \theta_S, \theta_{B_2} - \theta_S\} \\
- \quad & \max\{\theta_{B_1} - \theta_S, 3 - \theta_S\} + \max\{2, \theta_{B_2}\} + \max\{\theta_{B_1}, 3\} + \max\{2 - \theta_S, 3 - \theta_S\} - \max\{2, 3\} \\
= \quad & \max\{\theta_{B_1}, \theta_{B_2}\} - \theta_S - \max\{\theta_{B_1}, \theta_{B_2}\} - (\theta_{B_2} - \theta_S) - (3 - \theta_S) + \theta_{B_2} + 3 + (3 - \theta_S) - 3 \\
= \quad & 0.
\end{aligned}
$$

Therefore, there exists a dominant strategy incentive compatible, efficient, and exact budget balanced mechanism. We can use a second-price auction to achieve these properties.

In the second example, the seller's type does not change the efficient allocation and (5) holds.

Example 3. There are three agents, seller S, buyer B_1, and buyer B_2. The seller owns a good that she does not value (or values at 0). Buyer B_i's value for the good, $\theta_{B_i} \in [2, 3]$, is his private information, $i = 1, 2$. The seller cares about who buys the good. Therefore, if B_i buys the good, then the social surplus is $\theta_{S_i} + \theta_{B_i}$, where θ_{S_i} denotes how much the seller wants B_i to buy the good. Thus, seller's private information is $\theta_{S_1} = (\theta_{11}, \theta_{12}) \in [0, 1]^2$. Let $(\underline{\theta}_{11}, \underline{\theta}_{12}) = (0, 0)$, $\underline{\theta}_{B_1} = 2$ and $\underline{\theta}_{B_2} = 2$. The left hand side of (5) is:

$$
\begin{aligned}
\max\{\theta_{11} + \theta_{B_1}, \theta_{12} + \theta_{B_2}\} \quad - \quad & \max\{\theta_{B_1}, \theta_{B_2}\} - \max\{\theta_{11}, \theta_{12} + \theta_{B_2}\} - \max\{\theta_{11} + \theta_{B_1}, \theta_{B_2}\} \\
+ \quad & \theta_{B_2} + \theta_{B_1} + \max\{\theta_{11}, \theta_{12}\}.
\end{aligned}
$$

For $(\theta_{11}, \theta_{12}) = (0, 1/2)$, $\theta_{B_1} = 2$, and $\theta_{B_2} = 2\frac{3}{4}$, the above expression is equal to $-\frac{1}{4}$. Hence, (5) does not hold. Therefore, there does not exists a dominant strategy incentive compatible, efficient, and exact budget balanced for this example.

In the third example, each agent's type changes the allocation, and (5) fails.

Example 4. There are six agents, two sellers S_1 and S_2, and four buyers, B_1, B_2, B_3, and B_4. S_1 is selling a Macintosh computer whereas S_2 is selling a PC. B_1 and B_2 are only interested in getting a Mac and value the PC at zero. On the other hand, B_3 and B_4 are only interested in getting

a PC and value the Mac at zero. S_1 wants B_1 or B_2 to buy the Mac and S_2 wants B_3 or B_4 to buy the PC. Therefore, $\theta_{S_1} = (\theta_{11}, \theta_{12}, \theta_{13}, \theta_{14})$ where $\theta_{11}, \theta_{12} \in [0, 1]$ and $\theta_{13} = \theta_{14} = 0$. Similarly, $\theta_{S_2} = (\theta_{21}, \theta_{22}, \theta_{23}, \theta_{24})$ where $\theta_{21} = \theta_{22} = 0$ and $\theta_{23}, \theta_{24} \in [0, 1]$. θ_{B_i} is the non-zero value that B_i has for the computer of preference stated above. For this problem, the left hand side of (5) has 64 terms, so we do not write them at all. However, all the terms cancel out because for each agent i, there exists another agent j such that the type of agent j does not change the allocation of agent i (Part 2 of Corollary 3). Therefore, there exists a dominant strategy incentive compatible, efficient, and exact budget balanced mechanism.

In the last example, although each agent's type changes the efficient allocation, (5) still holds. The reason is that the market is separable in two submarkets. In general, it is hard to see when (5) holds.

4   Matching with Wages

We now consider the following special case of our general model.

In a **matching with wages problem**, the set of agents N can be divided into two disjoint sets, the set of firms, $F = \{f_1, \ldots, f_m\}$, and the set of workers, $W = \{w_1, \ldots, w_n\}$, $(m, n \geq 1)$ with generic elements f and w.

Each agent is endowed with a unique good. The set of feasible allocations $\mathcal{A}^f$ is such that each agent keeps her endowment or gets a good from an agent on the other side of the market. Moreover, if firm f gets the good owned by worker w, then worker w gets the good of firm f and vice versa. Formally, $\mathcal{A}^f = \{x = (x_{f_1}, \ldots, x_{f_m}, x_{w_1}, \ldots, x_{w_n}) | \forall f \in F \; x_f \in \{e_f, e_{w_1}, \ldots, e_{w_n}\}, \forall w \in W \; x_w \in \{e_w, e_{f_1}, \ldots, e_{f_m}\}$ and $\forall i, j \in N \; x_i = e_j \Leftrightarrow x_j = e_i\}$. We interpret this as a one-to-one matching of firms and workers where agents may remain single, that is, unmatched.

Each agent's type is the vector of private values for potential partners. Therefore, utility functions return the value corresponding to the match. To ease notation, we assume that worker w's type is a function $u_w : F \to \mathbb{R}^+$ and firm f's type is a function $u_f : W \to \mathbb{R}^+$. Hence, the type of agent i is the utility function u_i which takes values on a connected domain Θ_i. If i is a firm, then $\Theta_i \subseteq \mathbb{R}^n_+$. Otherwise, if i is a worker, then $\Theta_i \subseteq \mathbb{R}^m_+$. Assume that distribution D is such that u_i has positive support on Θ_i regardless of θ_{-i}. Finally, suppose that for agent $i \in N$ the utility of

being unmatched, which is denoted by $u_i(i)$, is zero. Therefore, we implicitly assume that for all $(f, w) \in F \times W$, $u_f(w) \geq 0 = u_f(f)$ and $u_w(f) \geq 0 = u_w(w)$.[8]

For any firm-worker pair (f, w), $\phi(f, w) = u_f(w) + u_w(f)$ is the **joint production** of this pair, interpreted as the gain from trade of a contract between them.

For this problem, we show that (4) is satisfied.

Theorem 3. *There exists a dominant strategy incentive compatible, individually rational, efficient, and budget balanced mechanism for the matching with wages problem.*

The deriving force behind this theorem is the assumption that utility values cannot be negative. If this assumption is violated, then by Myerson and Satterthwaite (1983) we know that such a mechanism does not exist.

Theorem 3 states the existence of a mechanism. We can actually construct such a mechanism using the "if" part of the proof of Theorem 1.

Proposition 1. *Let $\mu^e(u)$ be a matching function which maximizes the sum of match utilities for all agents and $SS(u)$ be the corresponding maximum value function. Then (μ^e, t) is a dominant strategy incentive compatible, individually rational, efficient, and budget balanced direct revelation mechanism where t is given by*

$$t_i(u) = [SS(u) - u_i(\mu_i^e(u))] - SS(0, u_{-i}).$$

Moreover, for all $i \in N$, $t_i(u) \leq 0$, for all $u \in \Theta$.

Let us see how this mechanism works in a simple example.

Example 5. There are three agents in the economy, one firm, Capital, and two workers, Ann and Bob. Suppose it is common knowledge that Capital prefers to hire any worker for 50K rather than not hiring anyone, and each worker prefers to work at Capital for 50K rather than not working. Each worker reports the wage which makes them indifferent between not working and working at that wage. Similarly, Capital reports a wage for each worker such that it is indifferent between not hiring and hiring that worker at the corresponding wage. Suppose that Capital reports 70K and 80K for Ann and Bob respectively. Ann reports 30K, and Bob reports 20K. To determine

---

[8]This assumption holds, for example, if it is common knowledge that there exists a wage p_{ij} for which firm f_i is willing to hire worker w_j at p_{ij} and the worker is willing to work for the firm at this wage.

who is hired, we look at the total surplus created by hiring. If Ann is hired then the total surplus is $70K - 30K = 40K$, whereas if Bob is hired it is $80K - 20K = 60K$. Therefore, Capital should hire Bob. Ann stays single and does not make any transfers. The benchmark mechanism sets the wage paid by Capital so that its net utility is equal to the difference of the social surplus under the original report and the minimum report that it could have made, which is 50K for both workers. The social surplus under the original report is 60K. If Capital had reported 50K for the workers, then the social surplus would be 30K. Therefore, Capital's payment can be calculated by $60K - 30K = 80K - x$, so $x = 50K$. However, Bob's wage is lower than this amount which can be calculated similarly. If Bob had reported 50K, then the social surplus would be 40K. Hence, Bob's wage can be found by $60K - 40K = x - 20K$. Thus, Bob's wage is 40K. In this instance, the mechanism creates a surplus of 10K.

Theorem 3 establishes a benchmark result. It shows that there exists a mechanism which is dominant strategy incentive compatible, individually rational, efficient, and budget balanced. Budget balance means that the sum of transfers should not be positive. Therefore, the mechanism does not need any outside subsidy to run but may create some surplus. Then an important question is what happens with the surplus.

One possible answer is that the market manager receives the residual money. For example, in a seller-buyer market the payment that the seller receives can be smaller than the price that the buyer pays. The difference goes to the market maker. Similarly, a contractor working in a company receives less than the company pays. The difference goes to the managed services provider. In the absence of such a person, one may try to redistribute the surplus between agents provided that this does not thwart incentive compatibility.

For the rest of this section, we show that the surplus cannot be redistributed among the agents without destroying incentive compatibility or efficiency even if we ignore individual rationality. Therefore, if we want to have a mechanism with exact budget balance, then we need a claimant for the residual money.

Theorem 5.3 in Green and Laffont (1979) is a similar result for a public goods model. However, their result cannot be applied here since they require valuation functions to be unrestricted over the whole allocation. This assumption is not satisfied here because our economy is a private goods

economy.

Theorem 4. *Suppose that there are at least three agents in the matching with wages problem and that there exists $c > 0$ such that for each firm f, $\Theta_f \supseteq [0, c]^n$, and for each worker w, $\Theta_w \supseteq [0, c]^m$. Then there exists no matching mechanism which satisfies dominant strategy incentive compatibility, efficiency and exact budget balance.*

The argument of the proof is as follows. Assume that (5) is satisfied. Suppose that $u \in \Theta$ is such that no linear relationships hold for any components of u, for example, $2u_{f_1}(w_2) \neq u_{w_1}(f_2) + u_{w_2}(f_1)$. Then each component of u must appear an even number of times in (5), matching the negative and positive appearances. However, because of the rich type space assumption, we can find $u' \in \Theta$ such that one component does not appear an even number of times, with the same property that no linear relationships hold for any components of u'. This gives a contradiction.

5 Ex Post Stability

In this section we study ex post stability. At the ex post stage preferences of agents are common knowledge. Therefore, each agent can identify partners such that pairing up with that agent can be mutually beneficial. If such a pair exists, we say that the pair blocks the matching. *Ex post no blocking* rules out existence of such pairs. The formal definition is as follows.

Definition 10. *A direct revelation mechanism (μ, t) satisfies **ex post no blocking** if, for all $(f, w) \in F \times W$,*

$$v_f(u) + v_w(u) \geq \phi(f, w)$$

for all $u \in \Theta$.

A matching mechanism is **ex post stable** if it satisfies ex post no blocking and individual rationality. A high correlation between ex post stability and the success of mechanisms has been documented: for example, in various regions of the British National Health Services (see Niederle et al. (2008)). This has also been observed in controlled lab experiments (Kagel and Roth (2000)). In light of this, central clearinghouses have been designed for various markets to deliver stable outcomes. Examples include New York City high schools, Boston public schools, kidney exchange

programs, and various medical specialties using the National Resident Matching Program (see Roth (2008)).

However, it turns out that this property cannot be satisfied together with dominant strategy incentive compatibility and budget balance.

Proposition 2. *Suppose that there are at least three agents in the matching with wages problem, and that there exists $c > 0$ such that for each firm f, $\Theta_f \supseteq [0, c]^n$ and for each worker w, $\Theta_w \supseteq [0, c]^m$. Then there exists no dominant strategy incentive compatible, ex post stable, and budget balanced mechanism.*

In fact, even a weaker condition than ex post stability is also impossible to satisfy.

Definition 11. *A direct revelation mechanism (μ, t) satisfies **no cross-subsidies** if, for any matched pair (for any (f, w) such that $\mu_f(u) = w$),*

$$t_f(u) + t_w(u) \geq 0$$

for all $u \in \Theta$.

No cross-subsidies requires that pairs who are matched do not subsidize the system.

Proposition 3. *Suppose that there are at least three agents in the matching with wages problem. Suppose also that there exists $c > 0$ such that for each firm f, $\Theta_f \supseteq [0, c]^n$ and for each worker w, $\Theta_w \supseteq [0, c]^m$. Then there exists no matching mechanism which satisfies dominant strategy incentive compatibility, individual rationality, efficiency, budget balance, and no cross-subsidies.*

Since ex post stability implies efficiency, Proposition 3 is a stronger result than Proposition 2. Before proving Proposition 3, we present a lemma.

Lemma 3. *If (μ, t) satisfies budget balance, individual rationality, and no cross-subsidies, then the following holds for all u:*

1. *If $\mu_i(u) = i$ then $t_i(u) = 0$,*

2. *If $\mu_f(u) = w$ (and $\mu_w(u) = f$) then $t_f(u) + t_w(u) = 0$.*

This lemma states that if a mechanism satisfies budget balance, individual rationality, and no cross-subsidies, then each single agent makes no transfer and that the sum of transfers for a pair is zero. Therefore, these conditions imply exact budget balance. On the other hand, by Theorem 4, we know that exact budget balance is not compatible with dominant strategy incentive compatibility, efficiency, and the additional assumption that the type space is rich. Since the same assumption is made in Proposition 3, the proof is complete. A more direct proof is in Appendix 2.

If there are only two agents in the market, then the matching which matches them without any transfer satisfies dominant strategy incentive compatibility, ex post stability, and budget balance. Similarly, if there are more than two agents and the type space is not rich enough it may be possible to construct a mechanism with all the properties. For such situations, see examples 1,2, and 4. Hence, the assumptions made in Proposition 3 are, in a sense, necessary.

6   Ex Ante Stability

In the previous section we show that no dominant strategy incentive compatible, ex post stable, and budget balanced mechanism exists. In this section we consider a weaker no blocking condition where agents can form blocking pairs only at the ex ante stage.

Definition 12. *A mechanism satisfies* ***ex ante no blocking*** *if, for all $(f, w) \in F \times W$,*

$$E_u[v_f(u)] + E_u[v_w(u)] \geq E_u[\phi(f, w)].$$

In words, ex ante no blocking means that the sum of expected utilities of any firm-worker pair from participating in the mechanism is greater than or equal to the expected production of the pair. If this condition holds, then no pair "can make a deal" at the ex ante stage which gives them both higher expected utilities than the expected utility they get from participation in the mechanism.

A matching mechanism is **ex ante stable** if it satisfies individual rationality and ex ante no blocking.

We can achieve ex ante stability when firms and workers are each ex ante symmetric.

Theorem 5. *Suppose that $u_w(f)$ is drawn from a distribution D_W (with density d_W and mean $\mu_W < \infty$) and $u_f(w)$ is drawn from a distribution D_F (with density d_F and mean $\mu_F < \infty$) for all f and w independently. Then there exists a mechanism which satisfies dominant strategy incentive*

compatibility, efficiency, ex ante stability, and ex ante exact budget balance for the matching with wages problem.

The idea of the proof is as follows. Consider the mechanism described in Proposition 1. This mechanism is dominant strategy incentive compatible, individually rational, efficient, and budget balanced. Moreover, it creates an expected surplus. To get ex ante no blocking, we distribute the surplus across the agents as a lump sum transfer. Suppose that the distribution is such that each agent gets the same extra payment. If the number of workers is different than the number of firms, ex ante no blocking may not be satisfied. The reason is that the sum of additional payments that a firm and a worker gets is small, so they may find it beneficial to ex ante block.

Since we assume that workers and firms are each ex ante identical, the expected utilities are the same except the extra transfer which comes from the distribution of the surplus. Therefore, ex ante no blocking is satisfied if and only if the worker and firm who get the minimum extra transfer do not form a blocking pair. The best distribution of the surplus, which maximizes the sum of transfers for these agents, is the one which distributes the total surplus equally among the agents on the short side of the market. In the proof we show that this construction works.

Proof. First, we show a weaker version of the claim where instead of ex ante exact budget balance we have that the sum of expected transfers is non-positive. Since we can increase the transfer of any agent by a lump-sum transfer, we can attain exact budget balance with the other properties.

Let $\mu^e(u)$ be an efficient matching. Let $t_f(u) = SS(u) - SS(0, u_{-f}) - u_f(\mu^e_f(u)) + k_F$ and $t_w(u) = SS(u) - SS(0, u_{-w}) - u_w(\mu^e_w(u)) + k_W$. If we apply Lemma 1 to the mechanism described in Proposition 1 and (μ^e, t), we get that (μ^e, t) is dominant strategy incentive compatible. Moreover it is efficient.

Now we analyze ex ante no blocking condition. Since firms and workers are ex ante identical, ex ante no blocking conditions for any pair of firms and workers are the same. This condition can be written as:

$$
\begin{aligned}
\mathrm{E}_u[v_f(u)] + \mathrm{E}_u[v_w(u)] = \quad & 2\mathrm{E}_u[SS(u)] - \mathrm{E}_{u_{-w}}[SS(0, u_{-w})] - \mathrm{E}_{u_{-f}}[SS(0, u_{-f})] + k_W + k_F \\
\geq \quad & \mu_F + \mu_W.
\end{aligned}
$$

Reordering gives:

$$k_W + k_F \geq \mu_F + \mu_W - 2\mathrm{E}_u[SS(u)] + \mathrm{E}_{u_{-w}}[SS(0, u_{-w})] + \mathrm{E}_{u_{-f}}[SS(0, u_{-f})].$$

If the RHS of the last inequality is negative, then we can set $k_F = k_W = 0$. In this case, (μ^e, t) coincides with the mechanism in Proposition 1. Therefore, it is budget balanced which implies that the sum of expected transfers to be non-positive. For this case the proof is complete. Otherwise, suppose without loss of generality that $m \geq n$. To minimize the sum of expected transfers, we set $k_F = 0$ and $k_W = \mu_F + \mu_W - 2\mathrm{E}_u[SS(u)] + \mathrm{E}_{u_{-w}}[SS(0, u_{-w})] + \mathrm{E}_{u_{-f}}[SS(0, u_{-f})].$

Therefore, (μ, t) satisfies dominant strategy incentive compatibility, individual rationality, efficiency, and ex ante no blocking. Now, we prove that the sum of expected transfers is nonnegative.

$$
\begin{aligned}
\sum_i \mathrm{E}_u[t_i(u)] &= (m - n - 1)\mathrm{E}_u[SS(u)] - (m - n)\mathrm{E}_{u_{-f}}[SS(0, u_{-f})] + n(\mu_F + \mu_W) \\
&= (m - n)\{\mathrm{E}_u[SS(u)] - \mathrm{E}_{u_{-f}}[SS(0, u_{-f})]\} - \mathrm{E}_u[SS(u)] + n(\mu_F + \mu_W).
\end{aligned}
$$

Hence, the necessary and sufficient condition is:

$$(m - n)\{\mathrm{E}_u[SS(u)] - \mathrm{E}_{u_{-f}}[SS(0, u_{-f})]\} \leq \mathrm{E}_u[SS(u)] - n(\mu_F + \mu_W). \tag{6}$$

If $m = n$, (6) reduces to $n(\mu_F + \mu_W) \leq E_u[SS(u)]$. This equation is satisfied because in the efficient matching there are n pairs with each pair's production exceeding $\mu_F + \mu_W$ on average.

Before proceeding with the general case, we introduce some notation. When there are l firms and j workers, let $\mathrm{E}_u[SS_{l,j}]$ be the expected social surplus. In (6), the social surplus is calculated for m firms and n workers.

Since $\mathrm{E}_{u_{-f}}[SS_{m,n}(0, u_{-f})] \geq \mathrm{E}_u[SS_{m-1,n}]$, we get

$$\mathrm{E}_u[SS_{m,n}(u)] - \mathrm{E}_{u_{-f}}[SS_{m,n}(0, u_{-f})] \leq \mathrm{E}_u[SS_{m,n}(u)] - \mathrm{E}_u[SS_{m-1,n}].$$

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Therefore, instead of (6), we can prove the following:

$$(m - n)\{\mathrm{E}_u[SS_{m,n}(u)] - \mathrm{E}_u[SS_{m-1,n}(u)]\} \leq \mathrm{E}_u[SS_{m,n}(u)] - n(\mu_F + \mu_W). \tag{7}$$

We prove the general case by mathematical induction on m. To be more clear, fix n. Since we have assumed $m \geq n + 1$, m takes values in $\{n + 1, n + 2, \ldots\}$. The base case corresponds to $m = n + 1$.

Base Case (m=n+1): When $m = n + 1$ (7) reduces to $n(\mu_f + \mu_W) \leq \mathrm{E}_u[SS_{n+1,n}(u)]$ which we argue to be true above.

Inductive Step: Suppose that (7) holds for $m = n + 1, n + 2, \ldots, l$. Now we prove it for $m = l + 1$.

Consider the difference $\mathrm{E}_u[SS_{n+i,n}(u)] - \mathrm{E}_u[SS_{n+i-1,n}(u)]$ for $i \geq 1$. This difference is the extra surplus that an additional firm contributes when there are more firms than workers in the economy. Since all the workers are matched in the efficient matching, this contribution is smaller when there are more firms. Therefore, we get:

$$\mathrm{E}_u[SS_{l+1,n}(u)] - \mathrm{E}_u[SS_{l,n}(u)] \leq \mathrm{E}_u[SS_{l,n}(u)] - \mathrm{E}_u[SS_{l-1,n}(u)]. \tag{8}$$

Since (7) holds for l, we have the following:

$$(l - n)\{\mathrm{E}_u[SS_{l,n}(u)] - \mathrm{E}_u[SS_{l-1,n}(u)]\} \leq \mathrm{E}_u[SS_{l,n}(u)] - n(\mu_F + \mu_W).$$

Multiply both sides of (8) by $l - n$ and use the inequality above to get:

$$(l - n)\{\mathrm{E}_u[SS_{l+1,n}(u)] - \mathrm{E}_u[SS_{l,n}(u)]\} \leq \mathrm{E}_u[SS_{l,n}(u)] - n(\mu_F + \mu_W)$$

Adding $\mathrm{E}_u[SS_{l+1,n}(u)] - \mathrm{E}_u[SS_{l,n}(u)]$ to both sides of this equation gives:

$$(l + 1 - n)\{\mathrm{E}_u[SS_{l+1,n}(u)] - \mathrm{E}_u[SS_{l,n}(u)]\} \leq \mathrm{E}_u[SS_{l+1,n}(u)] - n(\mu_F + \mu_W).$$

Therefore, (7) holds and the proof is complete. ∎

7 Interim Stability

Agents may be able to form blocking coalitions before they agree to participate in the mechanism but after they learn their values in some markets. This can happen if participation in the mechanism

is a binding commitment and agents know their values before participation. For example, in the National Resident Matching Program, which matches medical school graduates to residency programs, the outcome of the matching mechanism is enforced. Hence, ex ante and ex post no blocking conditions may not be the appropriate notions to study for such markets. In this section we introduce an interim no blocking notion to solve this problem.

Interim no blocking notion is harder to define than ex ante and ex post no blocking. The main reason is that agents have asymmetric information at the interim stage. Therefore, if a firm and a worker agree to make a side deal, then this gives information about the firm's valuation to the worker and similarly, information about the worker's valuation to the firm. With this new information they presumably update their estimates of expected utilities from participating in the mechanism. Therefore, it may no longer be beneficial for them to do the side deal.[9]

In our definition of interim no blocking, agents have belief domains about each other's valuation. Moreover, if there are two agents who interim block a mechanism, they are able to make a side deal without information updating.

Another reason for the difficulty is that there is also a bargaining issue since agents' willingness to pay depend on their private information. We resolve the bargaining problem by noticing that in a one firm-one worker case, there is a focal incentive compatible, individually rational, efficient, and budget balanced mechanism. In this mechanism, the firm and the worker get matched without making any transfers. Therefore, we assume that this is the only deal that they make.

Definition 13. *A direct revelation mechanism satisfies **interim no blocking** if there exists no pair (f, w) with non-empty neighborhoods N_f and N_w such that*

- *$E_{u_{-f}}[v_f(u)|u_w \in N_w] < u_f(w)$ if and only if $u_f \in N_f$,*

- *$E_{u_{-w}}[v_w(u)|u_f \in N_f] < u_w(f)$ if and only if $u_w \in N_w$.*

In this definition, N_w can be thought of as the belief domain of the firm about the worker's utility function. Similarly, N_f is the belief domain of the worker about the firm's utility function. If they can block the mechanism, then their belief domains are accurate: the firm wants to make

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[9]Similar difficulties arise in auctions with collusion literature as well. See for example Laffont and Martimort (1997), Laffont and Martimort (2000) and Pavlov (2008).

this deal if and only if its valuation vector is in N_f and the worker does the deal if and only if her valuation vector is in N_w.

A mechanism is **interim stable** if it satisfies individual rationality and interim no blocking. We need the following assumption, which puts a bound on how fast density functions increase.

Definition 14. *A random variable x is* **regular** *if*

$$E_x[x|x \geq c] \leq E_x[x] + c$$

for every c in the support.

If the probability density function of a random variable is non-decreasing (like the uniform distribution), then this property is satisfied. Apart from regularity, we also assume that the mean of the utility functions are the same for all workers which is another restrictive assumption. Moreover, we can only show this result when there is one firm or one worker. However, in auctions there is only one worker, so our result is applicable to this important case.

Proposition 4. *Suppose that there is only one firm in the matching with wages problem and that values are independent. Let $u_w(f)$ be regular and have a bounded domain with expected value μ for every w. Then there exists a mechanism which satisfies dominant strategy incentive compatibility, interim stability, efficiency, and ex ante exact budget balance.*

The idea of the proof is as follows. Consider the mechanism described in Proposition 1. This mechanism is dominant strategy incentive compatible, individually rational, efficient, and budget balanced. Moreover, it creates an expected surplus. To get interim no blocking, we give the extra surplus to the firm. Although workers are willing to block the mechanism, we show in the proof that the extra surplus given to the firm is high enough so that the firm never wants to block the mechanism. It is important to give all the surplus to the firm, since the firm is in any potential blocking pair.

8   Other Applications

There are many potential applications of our general theory to specific markets. We do not include the following applications in our discussion: many-to-many matching markets with transfers,

networks, coalition formation problems, and partnership dissolution problems. Nevertheless, we consider the following applications: seller-buyer markets with discrete heterogeneous goods, housing markets, and roommate problems with transfers. We hope these applications convince the reader that our methods can be applied to the others as well.

In the following applications, we assume that agents have private values.

8.1 Seller-Buyer Markets with General Preferences

Consider a market with many sellers and buyers where each seller can own multiple units of heterogenous goods and buyers can demand more than one unit. Buyers' valuation functions allow goods to be substitutes or complements.

In a **seller-buyer market**, m of the agents are buyers and n of them are sellers. Let B and S denote the set of buyers and sellers, respectively. Seller i's endowment, e_i, denotes the available vector of goods to sell for k types of goods. Buyers do not have any endowments. Each seller has constant marginal cost of production for each type of good. Moreover, each seller's type has a degenerate distribution which gives information about the marginal cost of production. Thus, if seller i with type $\theta_i \in \mathbb{R}_+^k$ produces $x_i \in \mathbb{Z}_+^k$, then the cost of production is $\theta_i \cdot x_i$. Buyer i's type θ_i gives information about the value of all possible bundles. Therefore, buyer i with type θ_i has value $u_i(x_i, \theta_i)$ for bundle $x_i \in \mathbb{Z}_+^k$. The value for owning the empty bundle is zero for all buyers, that is, for all $i \in N$ and $\theta_i \in \Theta_i$, $u_i(0, \theta_i) = 0$. All buyers value more goods to less, i.e., for all $i \in N$, $\theta_i \in \Theta_i$ and $x_i, y_i \in \mathbb{Z}_+^k$ such that $x_i \geq y_i$, $u_i(x_i, \theta_i) \geq u_i(y_i, \theta_i)$.

A feasible allocation is a vector of goods for each agent. For a seller it specifies the vector of goods to sell, and for a buyer it denotes the vector of goods to purchase. More formally,

$$\mathcal{A}^f = \{x | \forall i \in B \ x_i \geq 0, \forall i \in S \ 0 \leq x_i \leq e_i \text{ and, } \sum_{i \in S} x_i = \sum_{i \in B} e_i\}.$$

Proposition 5. *There exists a dominant strategy incentive compatible, individually rational, efficient, and exact budget balanced mechanism for the seller-buyer markets.*

We prove that the condition in Corollary 2 holds for seller-buyer markets in Appendix 2. It is crucial that sellers have constant marginal costs that are commonly known. This implies that in the efficient allocation each extra unit which gets traded has a higher cost of production than

the previous units. Corollary 2 implies that there exists a dominant strategy incentive compatible, individually rational, and efficient mechanism which creates a surplus. Distribute the surplus to the sellers in any way. Since the sellers do not face incentive issues, the new mechanism will satisfy all the properties and in addition it will be exact budget balanced.

8.2   Housing Markets

In a **housing market**, there are n agents with unit demand. Each agent owns a separate house. $\mathcal{A}^f$ is such that each agent gets one house. Suppose that the type of each agent is a vector with n coordinates, with each coordinate representing the value of a house. We show that there does not exist a mechanism which satisfies dominant strategy incentive compatibility, individual rationality, efficiency, and budget balance if the type space is rich enough.

Corollary 5. *If $n \geq 2$ and there exists $c > 0$ such that $\Theta_i \supseteq [0,c]^n$ for all $i \in N$, then there does not exist a dominant strategy incentive compatible, individually rational, efficient, and budget balanced mechanism for the housing market.*

The proof in Appendix 2 provides an example where the condition in Corollary 2 fails.

Similarly, there does not exist a dominant strategy incentive compatible, efficient, and exact budget balanced mechanism for the housing market.

Corollary 6. *Suppose that $n \geq 2$ and that there exists $c > 0$ such that $\Theta_i \supseteq [0,c]^n$ for all $i \in N$. Then there does not exist a dominant strategy incentive compatible, efficient, and exact budget balanced mechanism for the housing market.*

There exists such a mechanism if and only if (3) holds. Since the type space is rich enough we can provide a counter example as in the proof of Theorem 4. We omit the proof.

8.3   Roommates Problem with Transfers

The roommates problem of Gale and Shapley (1962) with the possibility of making transfers is a special case of our model. In a **roommates problem with transfers**, there are n agents, each of which owns a unique good. $\mathcal{A}^f$ is such that if agent i gets good j, then agent j gets good i. The interpretation is that all agents are paired up as roommates or left alone. Suppose that agent i's

type is $u_i = (u_{i1}, \ldots, u_{in})$ where u_{ij} is the value that agent i has for good j. Moreover, $u_{ii} = 0$ for all $i \in N$ and $u_{ij} \geq 0$ for all $i, j \in N$. Therefore, we assume that pairing up with anyone is as good as staying alone. Suppose also that $0 \in \Theta_i$.[10]

Corollary 7. *There exists a dominant strategy incentive compatible, individually rational, efficient, and budget balanced mechanism for the roommates problem with transfers.*

The deriving force for this corollary is the assumption that each agent prefers to be paired up rather than staying single, which might be a plausible assumption for some markets. It turns out that we cannot require exact budget balance in this mechanism, even if individual rationality is not required if the type space is rich enough.

Corollary 8. *Suppose that there are at least three agents and there exists $c > 0$ such that $\Theta_i \supseteq [0, c]^{i-1} \times \{0\} \times [0, c]^{n-i}$. Then there exists no dominant strategy incentive compatible, efficient, and exact budget balanced mechanism for the roommates problem with transfers.*

There exists such a mechanism if and only if (3) holds for the roommates problem with transfers. Since the type space is rich enough we can provide a counter example as in the proof of Theorem 4. We omit the proof.

9   Conclusion

We provide a characterization of ex post incentive compatible direct revelation mechanisms in combination with other properties that allow interdependent values and correlated types. To be specific, we give necessary and sufficient conditions for the existence of mechanisms with the following properties:

- Ex post incentive compatibility, individual rationality, and budget balance.

- Ex post incentive compatibility and exact budget balance.

Our model is quite general and includes several market design problems as applications (with the addition of monetary transfers) by restricting the set of feasible allocations, $\mathcal{A}^f$. However, depending on the restriction, the answers that we get for the existence of efficient mechanisms

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[10]This can also be viewed as a model of partnership formation.

with the above two lists of properties may change. Our main example is the one-to-one matching problem with wages. For this problem we show that there exists a mechanism with the properties in the first list. On the other hand, there exists no mechanism with the properties in the second list. Later, we consider several other applications where the results change. However, we do not know if the results depend on some certain structural property of $\mathcal{A}^f$. It might be worthwhile to investigate this in the future.

It has often been noted that results for matching markets with transfers and without transfers have parallels. To quote a few, Roth et al. (1993) note that

> ... one of the oldest puzzles arising out of the game theoretic analysis of two-sided matching concerns the fact that virtually identical conclusions about the core arise from two apparently quite different models, namely the marriage model of Gale and Shapley (1962) and the linear assignment model of Shapley and Shubik (1972).

Similarly, Balinski and Gale (1990) state the following:

> There is, by now, a substantial literature on these problems and one is struck by the fact that almost all results proved for the ordinal case have analogues in the cardinal case...

Given the above consensus, one might wonder *why* we study one-to-one matching markets with transfers under incomplete information. There are two main reasons for this.

The first reason is that, although the evidence thus far agrees with the above observation for models with complete information (see Roth and Sotomayor (1990)), results do not *always* carry over between these two models in the presence of incomplete information. For example, Yenmez (2009) provides such a result.[11]

The second reason is that, allowing the possibility of making endogenous transfers allows us to ask questions specific to the markets with transfers. For example, whether it is possible to have budget balanced or exact budget balanced mechanisms, which is a meaningless question in the absence of transfers. For example, in Theorem 3 we show that a mechanism with budget balance

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[11]Roth and Rothblum (1999) show that in a *symmetric* environment the best response of firms for the worker-proposing deferred acceptance procedure is to use *truncation strategies*. Truncation strategies have been defined for markets with transfers by Day and Milgrom (2008). Yenmez (2009) shows that in a one-to-one matching environment with transfers truncation strategies are not best response strategies.

and other properties is possible for a labor market. On the other hand, in Theorem 4 we prove that the previous result cannot be strengthened by requiring exact budget balance.

An interesting future research question is which of our results for the matching problem carry over to a setup with interdependent values or to many-to-one and many-to-many matching markets with transfers. As our work has shown, new questions, which were not present without the possibility of making endogenous transfers, can arise.

In the seller-buyer application, we have shown that there exists a dominant strategy incentive compatible, individually rational, efficient, and budget balanced mechanism. We can construct a direct revelation mechanism that has these properties. However, when it comes to applications this game might be hard to implement as each agent submits a bid for each possible combination of goods whose number increases exponentially with the number of goods. Thus, it might be desirable to find an indirect game, perhaps a simultaneous ascending auction, that implements this allocation. This remains an open question for future research.

10 Appendix 1: The Monopoly Case

In this Appendix, we consider the case when there is only one agent on one side of the market. In this case, there are several issues to be investigated. First we characterize the revenue maximizing mechanism for the firm when the firm has preferences over the workers. Then we invent a new ascending auction to implement this mechanism. We show that this auction is inefficient. Next we concentrate on the efficient direct revelation mechanism introduced in Proposition 1. Finally, we introduce another ascending auction to implement this mechanism.

This special case includes auctions when the auctioneer has preferences over potential buyers. For example, in a government procurement auction the government may prefer entrants over incumbents to increase competition, or domestic contractors over foreign ones.

To simplify notation, we denote w_i by i, $u_{w_i}(f)$ by u_i where u_i is the type of worker i. Suppose that u_i is distributed according to the distribution D_i with density d_i. Instead of using the direct mechanism (μ, t), we can equivalently use a direct mechanism (x, t) where $x_i(u)$ is the probability that i is matched to the firm and $t_i(u)$ is the transfer that i makes.

10.1 The Revenue Maximizing Mechanism for the Firm

We maximize the revenue of the firm subject to interim incentive compatibility, interim individual rationality, and ex ante budget balance.

Definition 15. *A direct revelation mechanism (x, t) satisfies **interim incentive compatibility** if, for all $i \in N$ and $u_i \in \Theta$,*

$$E_{u_{-i}} v_i(u) \geq E_{u_{-i}}[u_i(\mu_i(u_i', u_{-i})) + t_i(u_i', u_{-i})]$$

for all $u_i' \in \Theta_i$.

Interim incentive compatibility states that each agent prefers truth-telling, assuming other agents report truthfully. It is weaker than ex post incentive compatibility.

Definition 16. *A direct revelation mechanism (x, t) satisfies **interim individual rationality** if, for all $i \in N$,*

$$E_{u_{-i}} v_i(u) \geq 0$$

for all $u_i \in \Theta_i$.

Interim individual rationality states that each agent is better off participating in the mechanism than not participating on average. The individual rationality that we use in the main body of the paper requires agents to be better off on every state of the game, not just on average.

Let $J_i(u) \equiv u_f(i) + u_i - \frac{1 - D_i(u_i)}{d_i(u_i)}$ be the virtual valuation of worker i. The first term is the value that the firm has for the worker.

Proposition 6. *Suppose that the virtual valuation of i is increasing in u_i. Then the revenue maximizing mechanism matches the firm to the worker with the highest virtual valuation provided that the virtual valuation is non-negative.*

The proof parallels that of Myerson (1981). We provide a sketch here.

The firm's revenue is $R \equiv E_u[v_f(u)]$. This is equal to $E_u\{\sum_i [u_f(i)x_i(u)] + t_f(u)\}$. Because of ex ante budget balance we can write $E_u[\sum_i - t_i(u)]$ instead of $E_u[t_f(u)]$. Therefore, we get $R = E_u\{\sum_i [u_f(i)x_i(u) - t_i(u)]\}$. Now substitute $E_u\{u_i x_i(u) - v_i(u)\}$ for $E_u\{-t_i(u)\}$. Hence, $R = E_u\{\sum_i [(u_f(i) + u_i)x_i(u) - v_i(u)]\}$.

Because of interim incentive compatibility $\mathrm{E}_{u_{-i}}[v_i(u)] = \mathrm{E}_{u_{-i}}[v_i(0, u_{-i})] + \int\limits_{\eta=0}^{u_i} \mathrm{E}_{u_{-i}}\{x_i(\eta, u_{-i})\}d\eta$. Since we are maximizing the revenue of the firm set $\mathrm{E}_{u_{-i}}[v_i(0, u_{-i})] = 0$. Plug this in the expression for R and do integration by parts to get:

$$
\begin{aligned}
R &= \quad \mathrm{E}_u\{\sum_i [(u_f(i) + u_i)x_i(u) - v_i(u)]\} \\
&= \quad \mathrm{E}_u\{\sum_i [(u_f(i) + u_i - \tfrac{1 - D_i(u_i)}{d_i(u_i)})x_i(u)]\} \\
&= \quad \mathrm{E}_u\{\sum_i [J_i(u)x_i(u)].
\end{aligned}
$$

Therefore, to maximize the revenue of the firm, we should match the firm to the worker with the highest virtual valuation. To be explicit, $x_i(u) = 1$ if $i \in \arg\max_i J_i(u)$, otherwise $x_i(u) = 0$ (in case of ties choose one worker randomly).

Of course this mechanism, which maximizes the revenue of the firm, can be implemented as a direct revelation mechanism. In reality, agents may be unwilling to report their types to the firm. Therefore, we introduce an auction which implements this mechanism.

We use an ascending auction with personal prices (so there is price discrimination). In the beginning of the auction, the firm reports values. Using this report, we calculate the virtual valuation functions of workers. At time 0, we set the price of worker i to $p_i(0)$, so that $J_i(p_i(0)) = 0$. At time τ we set the price of worker i to $p_i(\tau)$ so that $J_i(p_i(\tau)) = \tau$. Increase the prices until there is one worker remaining in the auction. The last worker gets matched to the firm and pays the corresponding personal price.

Auction A1:

Step -1. The firm reports values $u_f(i)$. Using this report calculate J_i.

Step 0. Set $p_i(0) = J_i^{-1}(0)$. Allow each bidder to drop out. If all workers drop out, do not match the firm to anyone.

Step τ. Set $p_i(\tau) = J_i^{-1}(\tau)$. Allow each remaining bidder to drop out. If there is only worker remaining, then match the worker to the firm and make the worker pay the corresponding price.

Corollary 9. *In **A1** it is an equilibrium for workers to remain in the auction until the price is equal to their value and for the firm to report u_f truthfully. Moreover, this auction is payoff equivalent to the mechanism which maximizes the revenue of the firm.*

Proof. Each worker remains in the auction until the price hits their value for the firm. Additionally, it is better for the firm to report its values truthfully.

The last worker is the one with the highest virtual valuation provided that the virtual valuation is non-negative. Moreover the losing workers do not have to make any payments. Therefore, by the Revenue Equivalence Theorem, A1 is payoff equivalent to the mechanism which maximizes the revenue of the firm. ∎

Now, let us consider the following example to see how this auction works.

Example 6. There are two workers 1 and 2. They have the same distribution function $D_1(x) = D_2(x) = 1 - \sqrt{1 - x/100}$ where $x \in [0, 100]$. Suppose that 1's value for working at the firm u_1 is 70 and 2's value for working at the firm is 30. Moreover, the firm's value for hiring 1 is 20 and for hiring 2 is 60.

At $\tau = -1$ the firm reports $u_f(1) = 20$ and $u_f(2) = 60$. Using this report we calculate the virtual valuation functions: $J_1(\tilde{u}_1) = 20 + 2\tilde{u}_1 - 100 = 2\tilde{u}_1 - 80$ and $J_2(\tilde{u}_2) = 60 + 2\tilde{u}_2 - 100 = 2\tilde{u}_2 - 40$.

At $\tau = 0$ set $p_1(0) = 40$ and $p_2(0) = 20$. Both workers remain in the auction. We increase the prices according to the given rule. At time τ, $p_1(\tau) = 40 + \tau/2$ and $p_2(\tau) = 20 + \tau/2$ such that $J_i(p_i(\tau)) = \tau$. Therefore, the first worker to drop out is the second worker at $\tau = 20$ when the corresponding price is equal to her value 30. Therefore, the first worker gets matched to the firm and pays 50. Her profit is $70 - 50 = 20$.

In this example, it is equally efficient to match the firm to 1 or 2 since $\phi(f, 1) = \phi(f, 2) = 90$. Their virtual valuations are $J_1(u_1) = 60$ and $J_2(u_2) = 20$. Therefore, the first worker is the winner. However, if we decrease u_1 by a small amount, then it is more efficient to match the firm to the second worker but the auction still matches the first worker to the firm. Therefore, this auction is inefficient.

As we have seen the inefficiency can rise from mis-hiring. It can also happen because of no hiring since prices in the beginning can be higher than values.

10.2   The Efficient Auction

Let us now consider an auction which implements the efficient direct mechanism described in Proposition 1.

We use an ascending auction with hidden reserve prices. The price starts from zero. At every time, the workers only see the current price but not the set of active players. The auction stops when everybody has dropped out of the auction. Suppose that worker i has dropped out at x_i.

In the beginning the firm reports its vector of values, u_f, to the mechanism. Without loss generality assume that $u_f(1) \geq u_f(2) \geq \ldots \geq u_f(n)$. Then the hidden reserve price for worker 1 is set to $r_1 = 0$, the reserve price for the second worker is set to $r_2 = u_f(1) - u_f(2)$, and so on. It is equally efficient to match any of the workers to the firm if the worker's value is equal to the reserve price.

The winner is the worker with the highest difference of drop out time and reserve price, $x_i - r_i$. In case of ties, the winner is determined randomly. The winner pays the minimum price which would make her win the auction provided that the payment is non-negative. However, the payment does not go to the firm. Unlike most of the auctions, the firm also makes a payment. The payment of the firm is equal to the change in the sum of allocation utilities of workers it causes by introducing reserve prices. If all reserve prices were zero, then the allocation utility of workers would be $\max\{x_1, x_2, \ldots, x_n\}$. Now the allocation utility of the workers is x_i. Therefore, the firm pays the difference which is $\max\{x_1, x_2, \ldots, x_n\} - x_i$.

To sum up, **A2** can be described as follows:

Step -1. The firm reports values u_f. Using this report calculate reserve prices $r_i = u_f(i) - \max\{u_f(1), \ldots, u_f(n)\}$.

Step 0. Set $p(0) = 0$. Start increasing the price with time.

Step τ. Set $p(\tau) = \tau$. Allow each remaining bidder to drop out. If there is no worker remaining, then match the worker with the highest $x_i - r_i$ to the firm.

Proposition 7. *It is an equilibrium for the firm to report its values truthfully and for workers to drop out at their values. Moreover, the equilibrium allocation is always efficient. In particular, **A2** is equivalent to the mechanism of Proposition 1.*

Proof. Since the payment does not depend on the actual drop out time of the winner, dropping out later or sooner is not beneficial. Therefore, it is a weakly dominant strategy for the workers to drop out at their values. Then the firm's payoff is equal to $u_f(i) - [\max\{x_1, \ldots, x_n\} - x_i] = u_f(i) + x_i - \max\{x_1, \ldots, x_n\}$. Since $x_i = u_i$ this payoff is equal to $SS(u) - SS(0, u_{-f})$ which is the payoff of the

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firm in Proposition 1. Therefore, it is also optimal for the firm to report truthfully. Moreover since the smallest reserve price is zero, there is always a winner. Moreover, the winner is determined by the highest $x_i - r_i = u_i - (\max\{u_f(1), \ldots, u_f(n)\} - u_f(i)) = u_i + u_f(i) - \max\{u_f(1), \ldots, u_f(n)\}$. Therefore, the allocation is efficient.

We just have to show that workers receive the same payoff as in Proposition 1. Because unmatched agents get 0, we just have to compare the payoff of the winning worker.

The winning worker's payment is such that $t_i - r_i = \max\{-r_i, u_j - r_j\}$ where $u_j - r_j$ is the second highest difference between drop out times and reserve prices. Therefore, $t_i = r_i + \max\{-r_i, u_j - r_j\} = \max\{0, u_j - r_j + r_i\} = \max\{0, u_j + u_f(j) - u_f(i)\} = \max\{u_f(i), u_j + u_f(j)\} - u_f(i)$. Since this is the payment that the worker makes, her net utility is $u_i - t_i = u_i + u_f(i) - \max\{u_f(i), u_j + u_f(j)\} = SS(u) - SS(0, u_{-i})$ which is equal to her utility in the mechanism described in Proposition 1. ∎

11 Appendix 2: The Omitted Proofs

In this Appendix we provide the omitted proofs.

Proof of Lemma 1.

"Only if" part: Theorem 3 of Chung and Ely (2002) holds because Θ_i is connected, u_i is continuous in θ_i, and the set of feasible allocations is finite.

"If" part: Suppose (μ, t) is ex post incentive compatible and t' is such that $t_i(\theta) = t'_i(\theta) + g_i(\theta_{-i})$. Since (μ, t) is ex post incentive compatible, $v_i(\theta) \geq [u_i(\mu_i(\theta'_i, \theta_{-i}), \theta) - u_i(e_i, \theta)] + t_i(\theta'_i, \theta_{-i})$ $\forall i, \forall \theta$ and $\forall \theta'_i$. Which is equivalent to $[u_i(\mu_i(\theta), \theta) - u_i(e_i, \theta)] + t_i(\theta) \geq [u_i(\mu_i(\theta'_i, \theta_{-i}), \theta) - u_i(e_i, \theta)] + t_i(\theta'_i, \theta_{-i})$. Substitute $t'_i(\theta) + g_i(\theta_{-i})$ for $t_i(\theta)$ and $t'_i(\theta'_i, \theta_{-i}) + g_i(\theta_{-i})$ for $t_i(\theta'_i, \theta_{-i})$ and cancel $g_i(\theta_{-i})$ to get $v'_i(\theta) \equiv [u_i(\mu_i(\theta), \theta) - u_i(e_i, \theta)] + t'_i(\theta) \geq [u_i(\mu_i(\theta'_i, \theta_{-i}), \theta) - u_i(e_i, \theta)] + t'_i(\theta'_i, \theta_{-i})$. Hence, (μ, t') satisfies ex post incentive compatibility. ∎

Proof of Lemma 2.

By ex post incentive compatibility, $v_i(\theta) = \sup_{\theta'_i}[u_i(\mu_i(\theta'_i, \theta_{-i}), \theta) - u_i(e_i, \theta) + t_i(\theta'_i, \theta_{-i})]$. Since u_i is continuous in θ_i, $v_i(\theta)$ is the supremum of continuous functions (in θ_i). Therefore, $v_i(\theta)$ is lower semi-continuous in θ_i. Because Θ_i is compact and v_i is lower semi-continuous in θ_i, there exists a worst-off type. ∎

Proof of Corollary 1.

Suppose without loss of generality that agent 1's type has a degenerate distribution. Define t' as follows: $t'_1(\theta) = t_1(\theta) - \sum_{j \neq 1} t_j(\theta)$ and for all $i \neq 1$, $t'_i(\theta) = t_i(\theta)$. Since $t_j(\theta)$ is a function of θ_{-1}, (μ, t') is ex post incentive compatible by Lemma 1. It is also exact budget balanced. ■

Proof of Corollary 3.

The first part follows from Theorem 2.

Part 2: For each agent i, we show that u_i terms appearing in (5) cancel out. The terms with the endowment, $u_i(e_i, \theta_i)$, cancel out by counting. Now consider the terms with the efficient allocation, $u_i(\mu_i^e(\theta), \theta_i)$. Take any subset of agents $j \notin \{i_1, \ldots, i_s\}$. Consider $SS(\underline{\theta}_{i_1}, \ldots, \underline{\theta}_{i_j}, \theta_{-i_1, \ldots, i_s})$ and $SS(\underline{\theta}_{i_1}, \ldots, \underline{\theta}_{i_j}, \underline{\theta}_j, \theta_{-j, i_1, \ldots, i_s})$. Since agent i's efficient allocation is the same when agent j's type change, u_i's in these two expressions have the same argument, and therefore, they are the same. In (5), they have different signs since one of them has one more fixed type. Since we can pair up any subset S which does not include j with $S \cup \{j\}$ in the same manner, all the u_i terms cancel out and (5) holds. ■

Proof of Theorem 3. Since values are private, the statement is true if and only if (4) holds.

We need to identify the worst-off types to check the inequality. Since everybody gets the social surplus, the worst-off type for any agent is the type which assigns the lowest possible value to each agent on the other side. By assumption $u_i^*(u_{-i}) \geq 0$.[12] Therefore, the left hand side of (4) is at least as much as $\sum_i SS(0, u_{-i})$. On the other hand, the right hand side is $(m + n - 1)SS(u)$. Thus, we need to show that $\sum_i SS(0, u_{-i}) \geq (m + n - 1)\, SS(u)$ for all u.

Let N_1 be the set of agents that remain single in the efficient matching, $\mu^e(u)$, and let N_2 be the set of agents that are matched. For $i \in N_1$, $SS(0, u_{-i}) = SS(u)$ so there is no surplus loss. However, for $i \in N_2$, if we match agents according to $\mu^e(u)$ when utility profile is $(0, u_{-i})$ the sum of match utilities is $SS(u) - u_i(\mu_i^e(u))$. Therefore, we have $SS(0, u_{-i}) \geq SS(u) - u_i(\mu_i^e(u))$ for

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[12]Throughout the paper we use 0 to denote a vector consisting of zeros, of the appropriate length.

$i \in N_2$. Hence, if we sum all these inequalities we get

$$\sum_{i \in N} SS(0, u_{-i}) \geq (m+n)SS(u) - \sum_{i \in N_2} u_i(\mu_i^e(u))$$
$$= (m+n-1)SS(u).$$

∎

Proof of Proposition 1. Since (μ^e, t) is a VCG mechanism, it is dominant strategy incentive compatible and efficient. It is individually rational since $v_i(u) = SS(u) - SS(0, u_{-i})$ is non-negative. Moreover, $\sum_i t_i(u) = (m+n-1)SS(u) - \sum_i SS(0, u_{-i})$ which we argued to be non-positive in the proof of Theorem 3. Therefore, (μ^e, t) is also budget balanced. Next we prove that each agent is making a non-positive transfer.

To see this, note that $[SS(u) - u_i(\mu_i^e(u))]$ is the sum of match utilities for all agents except i when the matching function is $\mu^e(u)$. Whereas $SS(0, u_{-i})$ is the sum of match utilities for all agents except i when the matching function is $\mu^e(0, u_{-i})$. The first matching function takes agent i's preference into account but the second matching function discards this information resulting a higher match utility for the rest of the agents. Therefore, $t_i(u) = [SS(u) - u_i(\mu_i^e(u))] - SS(0, u_{-i}) \leq 0$. ∎

Proof of Theorem 4. Assume without loss of generality that the number of workers is at least 2.

By Corollary 3 such a mechanism exists if and only if

$$\sum_{j=0}^{n+m} (-1)^j \sum_{\{i_1, \ldots, i_j\} \subseteq N} SS(\underline{\theta}_{i_1}, \ldots, \underline{\theta}_{i_j}, \theta_{-i_1, \ldots, i_j}) = 0, \quad \forall \theta.$$

Consider a utility vector profile u' such that $u'_f \in [0, c]^n$ for all f, $u'_w \in [0, c]^m$ for all w such that no integer multiples of utility values form a linear relationship (for example $2u'_{f_1}(w_2) \neq u'_{f_2}(w_2) + u'_{w_1}(f_1)$). Hence, $SS(0, u'_{-i_1, \ldots, i_j})$ is uniquely determined for every $\{i_1, \ldots, i_j\} \subseteq N$. Every $u_i(i')$ appearing in the above sum must appear an even number of times in total, since the coefficient of $u_i(i')$ must be zero at the end (matching the negative and positive appearances).

Among $u'_{f_1}(w_1), \ldots, u'_{f_1}(w_n)$ take the smallest one, say $u'_{f_1}(w_1)$. In (5) there is a term which sets

utilities of agents other than f_1 to zero. This term is equal to the maximum of $\{u'_{f_1}(w_1), \ldots, u'_{f_1}(w_n)\}$ which is different from $u'_{f_1}(w_1)$.

Increase $u'_{f_1}(w_1)$ continuously until there is a tie in one of the summands (if it does not happen earlier this happens for the term which equals the maximum of $\{u'_{f_1}(w_1), \ldots, u'_{f_1}(w_n)\}$ by the assumption that $u'_{f_1} \in [0, c]^n$). If we keep increasing $u'_{f_1}(w_1)$ by a small amount, then the number of times $u'_{f_1}(w_1)$ appears in positive and negative terms differ by one. Moreover, we can change $u'_{f_1}(w_1)$ by a small amount such that no integer multiples of utility values form a linear relationship. This is a contradiction since the coefficient of $u'_{f_1}(w_1)$ must be zero for (5) to hold. ∎

Proof of Lemma 3. Let N_1 be the set of single agents in the matching $\mu(u)$ and N_2 be the set of matched agents. First, by individual rationality every single agent should get a non-negative transfer. Therefore,

$$\sum_{i \in N_1} t_i(u) \geq 0. \tag{9}$$

Additionally, for any matched pair agent (f, w) we have $t_f(u) + t_w(u) \geq 0$ by no cross-subsidies. If we sum this inequality over all paired agents we get:

$$\sum_{i \in N_2} t_i(u) \geq 0. \tag{10}$$

Summing inequalities (9) and (10) yields $\sum_{i \in N} t_i(u) \geq 0$. On the other hand, budget balance requires $\sum_{i \in N} t_i(u) \leq 0$. Hence, we must have $\sum_{i \in N} t_i(u) = 0$. This implies that inequalities (9) and (10) must hold as equalities as well as the inequalities which lead to them. Therefore, $t_i(u) = 0$ for all single agents and $t_f(u) + t_w(u) = 0$ for all matched pairs. ∎

Proof of Proposition 3. We provide an example for which there exists no such mechanism. Suppose there are two workers w_1, w_2, and a firm f. If the economy is larger, then we can embed this example into that economy with the new agents' types fixed at 0. The impossibility result carries over.

Assume that there exists such a mechanism (μ, t). In particular, (μ, t) is dominant strategy incentive compatible and efficient. If we apply Lemma 1 to (μ, t) and the mechanism described in

Proposition 1 we get $v_a(u) = SS(u) - SS(0, u_{-a}) + g_a(u_{-a})$.

Claim: If there exists u_i such that i is single in $\mu(u)$, then $g_i(u_{-i}) = 0$.

Suppose $\mu_i(u) = i$. Then, by Lemma 3, $t_i(u) = 0$ which implies that $v_i(u) = 0 = SS(u) - SS(0, u_{-i}) + g_i(u_{-i}) = g_i(u_{-i})$.

Now, consider the following utility state u^1, where $c/4 \geq x > 0$:[13]

	f
w_1	$(2x,\ 0)$
w_2	$(4x,\ 0)$

Since w_2 is single in $(0, u^1_{-w_2})$, $g_{w_2}(u^1_{-w_2}) = 0$ by the previous claim. Therefore, $v_{w_2}(u^1) = SS(u^1) - SS(0, u^1_{-w_2}) + g_{w_2}(u^1_{-w_2}) = 4x - 2x + 0 = 2x$. Moreover, by Lemma 3, $t_{w_2}(u^1) + t_f(u^1) = 0$ since f is matched with w_2 in the efficient matching. This implies $v_{w_2}(u^1) + v_f(u^1) = 4x$. Therefore, $v_f(u^1) = 2x = SS(u^1) - SS(0, u^1_{-f}) + g_f(u^1_{-f}) = 4x - 4x + g_f(u^1_{-f})$. We get that $g_f(u^1_{-f}) = 2x$.

Now, consider the following utility state u^2:

	f
w_1	$(2x,\ 3x)$
w_2	$(4x,\ 0)$

In this utility state, f is matched to w_1 in the efficient matching. If the utility state is $(0, u^2_{-w_1})$, then w_1 is single. By the previous claim $g_{w_1}(u^2_{-w_1}) = 0$. Hence, $v_{w_1}(u^2) = SS(u^2) - SS(0, u^2_{-w_1}) + g_{w_1}(u^2_{-w_1}) = 5x - 4x + 0 = x$.

On the other hand, $u^2_{-f} = u^1_{-f}$, so $g_f(u^2_{-f}) = g_f(u^1_{-f}) = 2x$. Using this we get, $v_f(u^2) = SS(u^2) - SS(0, u^2_{-f}) + g_f(u^2_{-f}) = 5x - 4x + 2x = 3x$. Therefore, $v_f(u^2) + v_{w_1}(u^2) = 4x$ which implies that $t_f(u^2) + t_{w_1}(u^2) = -x$. This contradicts Lemma 3. ∎

Proof of Proposition 4. Assume without loss of generality that the density function of $u_w(f)$, d_w, has support over $[0, 1]$.

Consider the mechanism described in Proposition 1. The allocation function is an efficient allocation function and transfers are determined such that $v_i(u) = SS(u) - SS(0, u_{-i})$ for every agent i. This mechanism is dominant strategy incentive compatible, individually rational, efficient,

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[13]The cell in the intersection of row w with column f has two values. The first value is $u_w(f)$ and the second value is $u_f(w)$.

and budget balanced. But there is an expected surplus. We increase the transfer of the firm so that the budget balances exactly ex ante. Therefore, $v_f(u) = SS(u) - SS(0, u_{-f}) + l$ where $l \geq 0$ is determined by:

$$\mathrm{E}_u[SS(u)] = \sum_i \mathrm{E}_u[v_i(u)]$$

$$= (n+1) \mathrm{E}_u[SS(u)] - \sum_i \mathrm{E}_u[SS(0, u_{-i})] + l.$$

Therefore, we get $l = \sum_i \mathrm{E}_u[SS(0, u_{-i})] - n \mathrm{E}_u[SS(u)]$.

We show in the proof of Theorem 3 that $\sum_i [SS(0, u_{-i})] - n[SS(u)] \leq 0$, $\forall u$. Therefore, $l \geq 0$, so the mechanism individually rational. Moreover, it is dominant strategy incentive compatible, efficient, and ex ante exact budget balanced. To finish the proof, we need to show that there does not exist a blocking pair (f, w_0) with neighborhoods N_f and N_{w_0}. Assume to the contrary that there exists such a blocking pair with associated neighborhoods.

By definition, N_f and N_{w_0} are non-empty open sets. $0 \leq v_{w_0}(u) = SS(u) - SS(0, u_{-w_0}) \leq u_{w_0}(f)$ for all u. Let $\hat{u}_f$ be such that $\hat{u}_f(w_0) = 1$ and $\hat{u}_f(w) = 0$ for all $w \neq w_0$. Since N_f is a neighborhood, $N_f \neq \{\hat{u}_f\}$. Therefore, if $u_{w_0}(f) \neq 0$, then $\mathrm{E}_{u_{-w_0}}[v_{w_0}(u)|u_f \in N_f] < u_{w_0}(f)$. Hence, the worker always wants to block the mechanism provided that her value for the firm is not zero. Therefore, $N_{w_0} = (0, 1]$.

We claim that $\mathrm{E}_{u_{-f}}[v_f(u)|u_w \in N_w] \geq u_f(w_0)$ for all u_f. If this inequality holds for $\hat{u}_f$, then it holds for all u_f and we get that $N_f = \emptyset$ which is a contradiction.

For $u_f \neq \hat{u}_f$, this inequality simplifies to

$$\mu \geq n \mathrm{E}_u[SS(u)] - \sum_w \mathrm{E}_u[SS(0, u_{-w})] = \mathrm{E}_u[\min\{u_{w_1}(f), u_{w_1}(f) + u_f(w_1) - (u_{w_2}(f) + u_f(w_2))\}]$$

where $w_1, w_2, \ldots, w_n$ is a renumbering of the workers for every u such that $\phi(f, w_1) \geq \phi(f, w_2) \geq \ldots \geq \phi(f, w_n)$. Let $y = -u_f(w_1) + (u_{w_2}(f) + u_f(w_2))$. Then we want to show that $\mu \geq \mathrm{E}_u[\min\{u_{w_1}(f), u_{w_1}(f) - y\}|\phi(f, w_1) \geq \phi(f, w_2) \geq \ldots \geq \phi(f, w_n)]$. Note that y is a random variable independent of $u_{w_1}(f)$. We claim that the inequality holds for every realization of y when the expectation is taken over $u_{w_1}(f) \geq y$.

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If $y \leq 0$, then $\min\{u_{w_1}(f), u_{w_1}(f) - y\} = u_{w_1}(f)$. Hence, $\mathrm{E}_{u_{w_1}}[\min\{u_{w_1}(f), u_{w_1}(f) - y\}|u_{w_1}(f) \geq y] = \mathrm{E}_{u_{w_1}}[u_{w_1}(f)] = \mu$.

If $y \geq 0$, then $\min\{u_{w_1}(f), u_{w_1}(f) - y\} = u_{w_1}(f) - y$. Therefore, $\mathrm{E}_{u_{w_1}}[\min\{u_{w_1}(f), u_{w_1}(f) - y\}|u_{w_1}(f) \geq y] = \mathrm{E}_{u_{w_1}}[u_{w_1}(f) - y|u_{w_1}(f) \geq y] = \mathrm{E}_{u_{w_1}}[u_{w_1}(f)|u_{w_1}(f) \geq y] - y$. Since u_{w_1} is regular we have, $\mathrm{E}_{u_{w_1}}[u_{w_1}(f)|u_{w_1}(f) \geq y] \leq \mu + y$. Hence, we conclude $\mathrm{E}_{u_{w_1}}[\min\{u_{w_1}(f), u_{w_1}(f) - y\}|u_{w_1}(f) \geq y] \leq \mu$.

We have shown that $\mathrm{E}_{u_{w_1}}[\min\{u_{w_1}(f), u_{w_1}(f) - y\}|u_{w_1}(f) \geq y] \leq \mu$ for every value of y. We conclude that $\mathrm{E}_u[\min\{u_{w_1}(f), u_{w_1}(f) - y\}|\phi(f, w_1) \geq \phi(f, w_2) \geq \ldots \geq \phi(f, w_n)] \leq \mu$. $\blacksquare$

Proof of Proposition 5. By Corollary 2, there exists a mechanism which is dominant strategy incentive compatible, individually rational, efficient, and budget balanced if and only if

$$\sum_i SS(\theta_i^*(\theta_{-i}), \theta_{-i}) \geq (|N| - 1)SS(\theta) \, \forall \theta.$$

For $x \in \mathbb{Z}_+^k$ such that $x \leq \sum_{i \in S} e_i$, define

$$C(x|\theta) = \min_{\substack{y_1 + \ldots + y_n = x \\ y_1 \leq e_1, \ldots, y_n \leq e_n}} \sum_{i \in S} y_i \cdot \theta_i.$$

$C(x|\theta)$ is the cost of providing vector x when the type profile is θ. With this notation we can rewrite $SS(\theta)$ as $\sum_{i \in B} u_i(\mu_i^e(\theta), \theta_i) - C(\sum_{i \in B} \mu_i^e(\theta)|\theta)$.

Claim: $(m - 1)C(\sum_{i \in B} \mu_i^e(\theta)|\theta) \geq \sum_{i \in B} C(\sum_{j \neq i, j \in B} \mu_j^e(\theta)|\theta)$.

Proof: Fix a type of good. Let r be the total number of times this good is counted in $\sum_{i \in B} \mu_i^e(\theta)$ and r_i be the number of times it is counted in $\sum_{j \neq i, j \in B} \mu_j^e(\theta)$. Note that $r_i \leq r \, \forall i \in B$ and $\sum_{i \in B} r_i = (m - 1)r$. Without loss of generality assume that r_i is increasing. Let x_i be the cost of supplying the i-th copy of the good, so $0 \leq x_1 \leq \ldots \leq x_r$. Thus, $(m - 1)C(\sum_{i \in B} \mu_i^e(\theta)|\theta) = (m - 1)(x_1 + \ldots + x_r)$ and $C(\sum_{j \neq i, j \in B} \mu_j^e(\theta)|\theta) = x_1 + \ldots + x_{r_i}$. We want to show $(m - 1)(x_1 + \ldots + x_r) \geq \sum_{i=1}^m (x_1 + \ldots + x_{r_i})$ to prove the claim. Let us count the sum of coefficients of x_i where $j + 1 \leq i \leq n$. For the left hand side it is $(m - 1)(r - j)$. For the right hand side if $j + 1 \leq r_1$, then the sum of the coefficients is $(r_i - j) + \ldots + (r_m - j) = (m - 1)r - mj$, less than the sum for the left hand side. Otherwise, if $r_i < j + 1 \leq r_{i+1}$ for some i, then the sum of the coefficients is $(r_{i+1} - j) + \ldots + (r_m - j) =$

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$(m-1)r - (r_1 + \ldots + r_i) - (m-i)j$ which is less than or equal to the sum of the coefficients for the right hand side if and only if $r_1 + \ldots + r_i \leq ij$, which holds. Therefore, for any $0 \leq x_1 \leq \ldots \leq x_r$, $(m-1)(x_1 + \ldots + x_r) \geq \sum_{i=1}^{m}(x_1 + \ldots + x_{r_i})$, so the claim holds.

For $i \in S$, $SS_i(\theta_i^*(\theta_{-i}), \theta_{-i}) = SS(\theta)$ since θ_i can take only one value (since it has a degenerate distribution). Thus,

$$\sum_{i \in S} SS_i(\theta_i^*(\theta_{-i}), \theta_{-i}) = nSS(\theta). \tag{11}$$

For $i \in B$, when the type profile is $(\theta_i^*(\theta_{-i}), \theta_{-i})$ we can allocate every other buyer according to $\mu^e(\theta)$. Hence, $v_i(\theta_i^*(\theta_{-i}), \theta_{-i}) = SS(\theta_i^*(\theta_{-i}), \theta_{-i}) \geq \sum_{j \neq i, j \in B} u_j(\mu_j^e(\theta), \theta_j) - C(\sum_{j \neq i, j \in B} \mu_j^e(\theta)|\theta)$. If we sum this up over all $i \in B$ we get:

$$\sum_{i \in B} SS_i(\theta_i^*(\theta_{-i}), \theta_{-i}) \geq (m-1)\sum_{i \in B} u_j(\mu_j^e(\theta), \theta_j) - \sum_{i \in B} C(\sum_{j \neq i, j \in B} \mu_j^e(\theta)|\theta). \tag{12}$$

If we add up (11) and (12) we get

$$
\begin{aligned}
\sum_{i \in N} SS_i(\theta_i^*(\theta_{-i}), \theta-i) &\geq nSS(\theta) + (m-1)\sum_{i \in B} u_j(\mu_j^e(\theta), \theta_j) - \sum_{i \in B} C(\sum_{j \neq i, j \in B} \mu_j^e(\theta)|\theta) \\
&\geq nSS(\theta) + (m-1)\sum_{i \in B} u_j(\mu_j^e(\theta), \theta_j) - (m-1)C(\sum_{i \in B} \mu_i^e(\theta)|\theta) \\
&= nSS(\theta) + (m-1)SS(\theta) = (n+m-1)SS(\theta).
\end{aligned}
$$

The second inequality follows from the claim. Hence, we have shown that a dominant strategy incentive compatible, individually rational, efficient, and budget balanced mechanism exists. To get the exact budge balance distribute the surplus to sellers in any way. Since they have degenerate distributions, they do not face any incentive issues. Hence, exact budget balance is also satisfied in addition to other properties. ∎

Proof of Corollary 5. There exists such a mechanism if and only if (4) holds. We show that this inequality is violated for an example with 2 agents. For any bigger economy this example can be embedded when the type profile of the other agents is fixed at 0.

Let d be a constant such that $0 < d \leq c$. The first agent's type is $(0, d)$- his value for the house he has is 0, his value for the other house is d. The second agent's type is $(d, 0)$- her value

for the house she owns is 0 and her value for the other one is d. According to the VCG mechanism each agents gets the social surplus ex post which is $2d$, so each agent gets a transfer of d. The sum of transfers is $2d$. The worst-off type for the first agent is $(d, 0)$ in which case the social surplus is 0 which is equal to his utility. Similarly, the worst-off type for the second agent is $(0, d)$ in which case the social surplus is 0 which is equal to her utility. The sum of these utilities is 0 which is smaller than the sum of transfers that is $2d$. Therefore, (1) fails. The conclusion follows. ■

Proof of Corollary 7. The statement is true if and only if (4) holds. Since each agent gets the social surplus ex post, $u_i^*(u_{-i}) = 0$. The inequality reduces to $\sum_{i \in N} SS(0, u_{-i}) \geq (n-1)SS(u)$. When utility profile is $(0, u_{-i})$, we can match all agents according to $\mu^e(u)$. Therefore, $SS(0, u_{-i}) \geq \sum_{j \neq i} u_j(\mu_j^e(u))$. If we sum this over all agents we get the desired inequality. ■

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