Paired Kidney Donation and Listed Exchange

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Abstract

Deceased-donor and live-donor kidneys are the two sources for transplantation, and these sources are utilized via two different programs. One of these programs, a *paired kidney donation (PKD)*, involves two donor-patient pairs, for each of whom transplantation from donor to intended recipient is not possible due to medical incompatibilities, but such that the patient in each couple could receive a transplant from the donor in the other pair. These two pairs can then exchange donated kidneys. Another possibility is a *paired listed exchange (PLE)*: given two donor-patient pairs, the first donor provides a kidney to a candidate on the deceased-donor waiting list, the first patient receives the kidney of the second donor, and the second patient receives a priority on the waiting list. By allowing only PKD's and PLE's, we characterize the set of kidney exchanges, which include the maximum number of kidney transplants from the set of incompatible patient-donor pairs. This characterization generalizes the Gallai-Edmonds Decomposition Theorem, a well-known result from combinatorial optimization.

Keywords : Kidney exchange, Gallai-Edmonds Decomposition

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1 Introduction

Transplantation is the preferred treatment for the most serious forms of kidney disease. Unfortunately, there is a considerable shortage of deceased-donor kidneys: as of June 13, 2008, there are 76,313 patients waiting for kidney transplants in the U.S., with the median waiting time of over 3 years, and in 2007, there were only 10,587 transplants of deceased-donor kidneys. The cadaveric kidneys are not the only sources for transplantation. Since healthy people have two kidneys and can remain healthy on one, it is also possible for a kidney patient to receive a live-donor transplant. In 2007, there were 6,038 transplants of live-donor kidneys. Our goal is to characterize the exchanges utilizing these two sources of kidneys as much as possible subject to the constraint that an exchange includes two transplantations.

The two sources of kidneys enable the medical authorities to develop different programs to increase the number of transplantations. One of these programs is a *paired kidney donation*. A paired kidney donation involves two patient-donor pairs, for each of whom a transplant from donor to intended recipient is not possible due to medical incompatibilities, but such that the patient in each couple could receive a transplant from the donor in the other couple (Rapaport [8], Ross et al. [9, 10]). These two pairs can then exchange donated kidneys. A paired kidney donation is depicted in Figure 1.

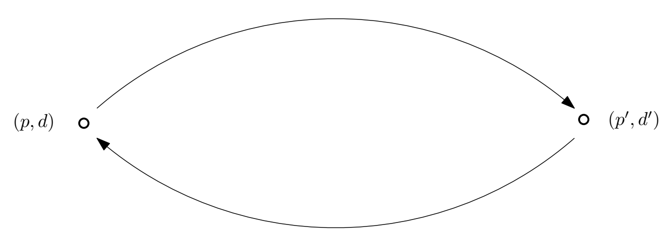


Figure 1: A pairwise kidney donation

Another possibility is a *paired listed exchange*. In a paired listed exchange, there are two donor-patient pairs: the first donor provides a kidney to a candidate on the deceased-donor waiting list, the first patient receives the kidney of the second donor, and the second patient receives a priority on the waiting list. This improves the welfare of the first patient, but also of the second patient, compared to having a long wait for a compatible cadaver kidney, and it benefits the recipient of the live kidney on the waiting list who benefit from the increase in the kidney supply due to an additional living donor. Through April 2006, 24 listed exchanges have been performed. A paired listed exchange is depicted in Figure 2.

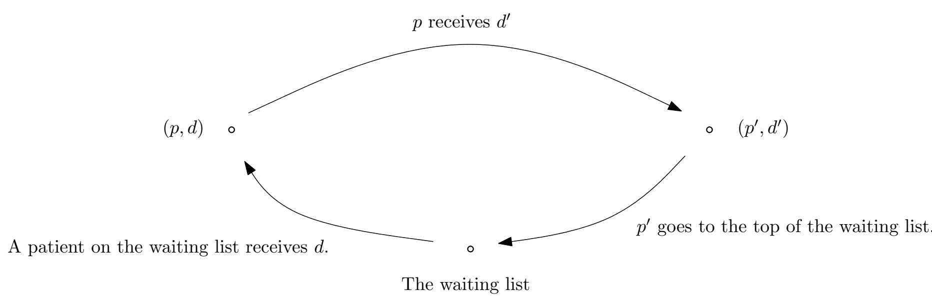


Figure 2: A pairwise listed exchange

Any type of exchange includes multiple transplantations, and these transplantations are carried out simultaneously to avoid conflicts which may arise when a donor gives up after her patient receives a kidney transplant from another donor. But this practice excludes the exchanges with three or more incompatible pairs. Thus, the only available procedures are paired kidney donations and paired listed exchanges and any exchange mechanism (constrained in this way) matches pairs. Our goal is to characterize the set of matchings with the maximum number of patients receiving a transplant.

Kidney exchange is matching problem, where the pairs are matched according to the medical compatibilities between them. There are two different interpretations when two pairs are matched: it is either a paired kidney donation, where the pairs exchange the live donors, or a paired listed exchange, where the first patient receives a live donor kidney and the second pair receives a priority on the waiting list. Since the second pair actually could have engaged in a listed exchange by itself, it is the first pair who benefited from the paired listed exchange; the second patient would have received a priority on the waiting list anyway. Thus, while two patients benefit from a paired kidney donation, only one patient benefits from a paired listed exchange and the kidney exchange problem is a weighted matching problem. We provide a characterization of the matchings with the maximum number of patients receiving a transplant; these are the maximum weight matchings. This result generalizes the result on the maximum cardinality matchings.[1]

2 Related Literature

While the transplantation community approved the use of the paired donations and listed exchanges to increase kidney donations, it has provided little guidance about how to organize such exchanges. Roth, Sönmez, and Ünver [11, 12, 13, 14] suggested that, by modeling kidney exchange as a mechanism design problem, integrating the paired donations and listed exchanges may benefit additional candidates. This approach turns out to be very successful and is supported by the medical community. Since then, a centralized mechanism for kidney exchange based on these two protocols has been used in the regional exchange program in New England (The United Network for Organ Sharing-UNOS-Region 1).

Roth, Sönmez, and Ünver [12] assumed dichotomous preferences and considered the constrained kidney exchange problem, in which only the paired kidney donations are allowed. They show that, in the constrained problem, efficient and strategy-proof mechanisms exist. These mechanisms include a deterministic mechanism based on the priority setting that organ banks currently use for the allocation of cadaver kidneys, and a stochastic mechanism motivated by the fairness considerations. The results of Roth, Sönmez, and Ünver [18] on the egalitarian mechanism generalize the corresponding results of Bogomolnaia and Moulin [5] to general (not necessarily bipartite) graphs.[2] This work is tied to the current one as follows: they construct a random mechanism on the set of maximum cardinality matchings in

[1]The maximum cardinality matching problem is well analyzed in the graph theory literature. More specifically, the Gallai [4, 5]-Edmonds [2, 3] Decomposition Lemma characterizes the set of maximum cardinality matchings. We make use of this result in constructing an efficient exchange.

[2]Roth, Sönmez, and Ünver [13] also explore that, for a specific preference profile of the patients (this profile is constructed according to the medical facts on the blood-type compatibilities), when multiple-way paired donations are feasible, three-way kidney donations as well as paired kidney donations will have a substantial effect (and larger than three-way kidney donations have less impact) on the number of transplants that can be arranged.

the corresponding graph; the characterization of the maximum cardinality matchings given by the Gallai-Edmonds Decomposition Lemma [2, 3, 4, 5, 7]. While maximum cardinality matchings are defined with respect to regular graphs with uniform edges, our contribution is to generalize this decomposition result to the graphs with weights 1 and 2.

Yılmaz [18] explores how to organize kidney exchange by integrating the multiple ways kidney donations and listed exchanges, under the assumptions of dichotomous preferences of the patients, and that the success rates of transplants from live donors are higher than those from cadavers. He characterizes the set of random matchings, which are Pareto efficient and fair.

Recently, Sönmez, and Ünver [16] extended the Decomposition Theorem by including the compatible pairs as well. They consider the paired kidney donations between the incompatible pairs or between an incompatible pair and a compatible pair. In terms of modeling kidney exchange, and their characterization result, this work is the closest to the current one.

3 The model

A **pair** consists of a patient and a donor such that the donor cannot medically donate her kidney to the patient of the pair. Let N be the set of all pairs. Given two pairs $n, n' \in N$, n **is compatible with** n' if the donor of n' can medically donate her kidney to the patient of n. For each pair, the donor has the same preferences with the patient; let $\succcurlyeq_n$ denote the preferences of the pair n over the set N. Let $\succ_n$ denote the strict preference relation and $\sim_n$ denote the indifference relation associated with $\succcurlyeq_n$. Let $\succcurlyeq = (\succcurlyeq_n)_{n \in N}$.

Each pair n has dichotomous preferences on N: it is indifferent between all compatible pairs, it indifferent between all incompatible pairs and it strictly prefers a compatible pair to remaining unmatched and remaining unmatched to an incompatible pair. A pair can be matched to a compatible pair or remain unmatched or receive a cadaveric kidney transplantation. A **paired kidney donation (PKD)** involves two mutually compatible pairs n, n' such that the patient of n receives the kidney of the donor of n' and the patient of n' receives the kidney of the donor of n. A **listed exchange** involves a pair such that the patient receives the top priority in the waiting list, and in exchange, his donor donates her kidney to a patient on the waiting list. A **paired listed exchange (PLE)** involves two pairs n, n', such that n is compatible with n', the patient of n receives the kidney of the donor of n', the donor of n donates to a patient in the waiting list, and the patient of n' receives the top priority in the waiting list. Due to a medical fact, that a live donor kidney has a substantially higher patient survival and graft survival rates than the cadaveric donor kidney, each pair strictly prefers a compatible pair to a cadaveric kidney.

There are two types of pairs: a **p-pair** prefers being unmatched to a cadaveric kidney transplantation and an **l-pair** prefers a cadaveric kidney transplantation to being unmatched. The sets of p-pairs and the l-pairs are denoted by N_p and N_l, respectively and they partition the set N.

For each $x, y \in N$ with $r_{x,y} = 2$, we refer to the pair (x, y) as a **feasible paired kidney donation**. For $x, y \in N$ with $r_{x,y} = 1$, we refer to the pair (x, y) as a **feasible paired listed exchange** where the patient of x receives the kidney from the donor of y. A **feasible exchange matrix** $R = [r_{x,y}]_{x,y \in N}$ identifies all feasible PKD's and PLE's where

$$r_{x,y} = \begin{cases} 2 & \text{if } y \in N \setminus \{x\}, \text{ and } x, y \text{ are mutually compatible} \\ 1 & \text{if } x \text{ is compatible with } y, \ y \text{ is not compatible with } x, \text{ and } y \in N_l \\ 0 & \text{otherwise.} \end{cases}$$

A **kidney exchange problem** (or simply a **problem**) (N, R) consists of a set of pairs and its feasible exchange matrix. Given a problem (N, R) and $N' \subseteq N$, the reduced problem is denoted by $(N', R|_{N'})$, where $R|_{N'}$ is the reduced matrix of R on N'.

A **matching** μ is a set of mutually feasible *paired kidney donations* and *paired listed exchanges*. For each matching μ, the set of PKD's in μ is denoted by μ_2 and the set of PLE's in μ is denoted by μ_1; thus, $\mu = \mu_1 \cup \mu_2$. For each matching μ, $(x, y) \in \mu_2$ means that the patient of each pair receives a kidney from the donor of the other pair; $(x, y) \in \mu_1$ means that the patient of x receives a kidney from the donor of y, the donor of x donates to a patient on the waiting list, the patient of y receives the top priority in the waiting list, and $y \in N_l$. For a problem (N, R), let $\mathcal{M}(N, R)$ denote the set of all matchings.

Observe that an l-pair can always receive the top priority in the waiting list by simply accepting to be in a listed exchange. Thus, comparing two different matchings, only the patients, who receive a transplant from a live donor matter. For each matching μ, let T^μ denote the set of all pairs who receive a transplant from a live donor. Formally,

$$T^\mu = \{x \in N : \mu_2(x) \neq x \text{ or } (x, y) \in \mu_1 \text{ where } x \neq y\}$$

For each $\mu, \mu' \in \mathcal{M}$, μ **Pareto-dominates** μ' if, for each $x \in N$, $\mu(x) \succcurlyeq_x \mu'(x)$, and for some $x \in N$, $\mu(x) \succ_x \mu'(x)$. A matching $\mu \in \mathcal{M}$ is **Pareto efficient** if no other matching Pareto dominates μ. For a problem (N, R), let $\mathcal{E}(N, R)$ denote the set of Pareto efficient matchings.

4 Maximal matchings

Our model relies on the existence of the sets N_p and N_l and the interpretation of this model is the following: There is a set of pairs N_p, each of whom expects to receive a transplant from a live donor; on the other hand, there is a set of pairs N_l, each of whom is accepted to be in a listed exchange to receive the top priority in the waiting list. These two sets are integrated so that both groups of pairs will benefit from the extended set of feasible paired kidney donations and feasible paired listed exchanges. For example, let $N_p = \{x\}$ and $N_l = \{y\}$ such that $r_{x,y} = 1$. If these two pairs are considered separately, then the pair x remains unmatched, and the pair y receives the top priority in the waiting list. On the other hand, if, as our model suggests, these two groups are considered together, then the pair x receives a transplant from a the donor of y and the pair y receives the top priority in the waiting list.

When there are no l-pairs, a well-known result, the Gallai [4,5]-Edmonds [2] Decomposition Theorem, characterizes the structure of Pareto efficient matchings, and the same number of pairs are matched at each Pareto efficient matching.[3] For a kidney exchange problem with p-pairs and l-pairs, what is critical is the set of pairs who receive a transplant from a live donor, not the set of pairs who are matched, and for this general case, this result does not extend and the number of pairs, who receive a transplant from a live donor, may be different in different Pareto efficient matchings.

Example 1: Let $N_p = \{x\}$ and $N_l = \{u, v\}$. The feasible exchange matrix R is such that $r_{x,u} = 1$ and $r_{u,v} = 2$. Observe that there are two efficient matchings, $\mu = \{(x, u)\}$ and $\mu' = \{u, v\}$, where $|T^{\mu}| = 1$, and $\left|T^{\mu'}\right| = 2$.

Observe that in Example 1, the number of p-pairs, who receive a transplant from a live donor, is also different in different Pareto efficient matchings. Thus, some Pareto efficient matchings can be improved in terms of the number of pairs who receive a transplant. A matching μ has the **maximum number of transplants** if there is no other matching μ' such that $\left|T^{\mu'}\right| > |T^{\mu}|$. Also, the transplantation centers' preferred exchange is the paired kidney donation. Thus, it is plausible to minimize the number paired listed exchanges, while maximizing the number of transplant. A matching is called **maximal** if it has the maximum number of transplants and it has the maximum number of paired kidney donations in the set of the matchings with the maximum number of transplants. From now on, we focus on this particular property. Given a problem (N, R), let $\mathcal{E}^m(N, R)$ denote the set of maximal matchings.

The existing exchange mechanism used in practice considers the pool of the p-pairs separately and matches the pairs in a Pareto efficient way. A particular interpretation of our model is the integration of the l-pairs to the pool of the p-pairs so that this enhances the number of transplants with respect to the existing mechanism. At this point, the foremost important question is whether the l-pairs may have a negative externality on the welfare of the p-pairs, if the transplantation center insists on the maximality property. More specifically, suppose a Pareto efficient matching is fixed for the group of p-pairs only. Let T be

[3] Observe that when there no l-pairs, only the PKD's are feasible, and there is no PLE in a matching.

the set of pairs who receive a transplant in this matching. Then, after the integration of the l-pairs to the pool of p-pairs, does there exist a maximal matching in this new problem, so that all the patients in T receive a transplant in this maximal matching? As our first result shows, the answer is yes.

Proposition 1 *Let $(N_p \cup N_l, R)$ be a problem. Let $\mu \in \mathcal{E}^m(N_p, R|_{N_p})$. Then, there exists a matching $\mu' \in \mathcal{E}^m(N_p \cup N_l, R)$ such that $T^{\mu'} \supseteq T^{\mu}$.*

5 The structure of maximal matchings

Our goal is to characterize the set of maximal matchings. Our model is built on integrating the groups of p-pairs and l-pairs. Thus, we focus on the static problem, in which the sets N_p and N_l are given; there are no strategic issues such as a patient revealing truthfully or not whether he is an $l-$patient.

For each problem (N, R), define

$$D(N, R) = \{x \in N : \exists \mu \in \mathcal{E}^m(N, R) \text{ s.t. } \mu(x) = x\}.$$

Let $A_1(N, R)$ be the set of pairs who are part of only paired listed exchanges in each maximal matching and have a compatibility with at least one pair in $D(N, R)$.

Theorem 1 *Let (N, R) be a problem. Then, in any maximal matching,*

1. *each pair in $A_1(N, R)$ is matched to a pair in $D(N, R)$;*
2. *if the subgraph induced by $G - A_1(N, R)$ contains a component N' in $D(N, R)$, then*
   (a) this component is such that, for each $x \in N'$, $N' - x$ has a perfect matching,
   (b) any maximal matching contains a near-perfect matching of this component,
   (c) any maximal matching matches at most one pair of this component to a pair in $A_1(N, R)$.

This result extends the GED Theorem. Given (N, R), when no two pairs are mutually compatible, thus, when for all $x, y \in N$, $r_{x,y} \in \{0, 1\}$, the result of Theorem 1 reduces to the Gallai [4,5]-Edmonds [2] Decomposition.

While the GED-type structure is not present anymore for the PKD's, this result has a positive aspect in terms of how to organize the PLE's under maximality.[4]

A similar result on the extension of the GED Theorem is by Sönmez, and Ünver [16], they extend the Decomposition Theorem by including the compatible pairs in a way that the PKD's occur between the incompatible pairs or between an incompatible pair and a compatible pair. This can be interpreted as a particular case of our model: Each compatible pair is matched only with an incompatible pair and if it is matched, it benefits the incompatible pair that it is matched but not itself, because its patient is compatible with its donor and would receive her kidney anyways. Thus, the compatible pair accepts to be matched with an incompatible pair purely for altruistic reasons. Thus, in such a match, only the patient of the incompatible pair benefits. Thus, N_l is the set of compatible pairs in this model. However, there is a restriction: no two compatible pairs can be matched. Thus, $x, y \in N_l$ implies $r_{x,y} = 0$.[5] For this particular class of problems, the GED-type structure is fully preserved. Thus, our result is a robustness check of this particular model: when we generalize this altruistic kidney exchange model by allowing altruistic pairs to be matched, only the structure of the PLE's is preserved and the structure of the PKD's is lost.

6 Conclusion

We characterize the set of maximal matchings when only paired kidney donations and listed exchanges are allowed. This generalizes the existing kidney exchange mechanism where the PKD's and PLE's are considered separately. There is a pool of incompatible pairs who only accept live donor kidney transplantations and they are matched in a way that the number

[4] The characterization result above is tight and there are examples which illustrate that there is no further GED-type structure than the one given in Theorem 1. These examples are available upon request.

[5] Note that in the general model that we consider, there is no such restriction for the l-pairs: an l-pair can be compatible with another l-pair, also two l-pairs can be mutually compatible.

of patients who receive live donor kidney transplantations, is maximized. Also, there is a pool of incompatible pairs who are ready to be in a PLE to receive the top priority in the deceased-donor waiting list. The existing mechanism handles these two pools separately.

The integration of these two pools clearly improves the existing mechanism in terms of the number of patients who receive a live donor kidney transplantation. Our first result shows that these efficiency gains are always at the full level, even if the set of matchings is restricted so that each pair in a subset of N_p is guaranteed to receive a live donor kidney transplantation. This result implies that, given a priority ordering on N_p and a priority mechanism based on this ordering, the property of maximality can be achieved under the integration without making a pair in N_p worse off with respect to the outcome of this priority mechanism.

Our second result characterizes the set of maximal matchings for a general problem where there are p-pairs and l-pairs. While the GED-type structure is not present anymore for the PKD's, this result has a positive aspect in terms of how to organize the PLE's under maximality.

Another question is related to the maximum number of kidney transplantation and the level of the efficiency gains under integration. While our results are not helpful in this direction, we present a simple result, which simplifies the problem of finding the maximum number of transplantations under integration (see Appendix 7.3).

7 Appendix

7.1 Preliminaries on graphs

A problem (N, R) can be represented by a weighted graph $G = (V, E, w)$, where V is the set of vertices, E is the set of edges, and w is a weight function, $w : E \to \{1, 2\}$. The graph representation of a problem (N, R) is obtained as follows: Each pair u is a vertex, thus $V = N$. Let $u, v \in V$ be two vertices. If $r_{u,v} = 2$, then the set E contains the edge uv, the weight of which is 2. If $r_{u,v} = 1$, then the set E contains the directed edge from u

to v, the weight of which is 1. An edge with weight 1 is called a **thin edge**; an edge with weight 2 is called a **thick edge**.[6] For a weighted graph $G = (V, E, w)$, and $E' \subseteq E$, let $w(E') = \sum_{uv \in E'} w(uv)$.

Let $G = (V, E, w)$ be a weighted graph. For a subgraph $H \subseteq G$, let H_2 be the induced graph of H on the thick edges. The set of thin and thick edges are denoted by $E_1(G)$ and $E_2(G)$, respectively. Note that a matching μ is a subset of the edges such that no two edges meet at a common vertex. Let $\mathcal{M}(G)$ denote the set of all matchings. Let $\nu(G) = \underset{\mu \in \mathcal{M}(G)}{Max}\, w(\mu)$. If the weights are uniform, then $\nu(G)$ is called the **matching number** of G.

Definition 1 *A vertex is **free** with respect to a matching μ if it is not incident with any edge in μ.*

Definition 2 *A path (or a cycle) is **alternating** with respect to a matching μ if its edges are alternately in μ and not in μ.*

Definition 3 *An **augmenting path** is an alternating path between free vertices. An augmenting path with respect to a matching μ is called an $\mu-$**augmenting path**.*

If the weights of the edges are uniform, then a well-known result in matching theory characterizes the condition for the maximality of a matching.

Theorem 2 *Given all the edges have the same weight, a matching μ is a maximal matching if and only if there does not exist an $\mu-$augmenting path.*

7.2 Proofs

First, we extend Theorem 2 to the graphs consisting of both thick and thin edges. Let $\oplus$ denote the set difference operator, for $E', E'' \subseteq E$, $E' \oplus E'' = (E' \setminus E'') \cup (E'' \setminus E')$.

Lemma 1 *A matching μ is maximal if and only if $|\mu_2| = \nu(G_2)$ and there does not exist an $\mu-$augmenting path P with $E_2(P \cap \mu) = E_2(P \setminus \mu)$.*

[6] For expositional purposes, thin and thick edges are depicted as dotted and solid, respectively, in the figures.

Proof. (Only if) Clearly, $|\mu_2| \leq \nu(G_2)$. Suppose the inequality is strict. Then, by Theorem 2, there is an μ_2-augmenting path in G_2, say P. If both of the end vertices in P are free in μ, then $w(\mu \oplus P) = w(\mu) + 2$, which contradicts with μ having the maximum weight. Suppose only one of the end vertices, say u, is covered in μ, say $ux \in \mu$. Since P is an μ_2-augmenting path in G_2, the edge ux is thin. Then, $w(\mu \oplus (P + ux)) = w(\mu) + 1$, which again contradicts with μ having the maximum weight. Suppose both end vertices in P, say u and v, are covered by the (thin) edges in G, say by ux and vy. Then, $w(\mu \oplus (P + ux + vy)) = w(\mu)$ and the matching $\mu \oplus (P + ux + vy)$ has one more thick edge than the matching μ, which contradicts with maximality of μ. Thus, $|\mu_2| = \nu(G_2)$. Now, suppose there exists an $\mu-$augmenting path P such that $E_2(P \cap \mu) = E_2(P \setminus \mu)$. Since the end vertices of the path P are free, the set $P \oplus \mu$ is another matching. Since the matching $P \oplus \mu$ contains one more edge than μ and both μ and $P \oplus \mu$ have equal number of thick edges, $w(P \oplus \mu) = w(\mu) + 1$, which contradicts with the matching μ having the maximum weight.

(If) Suppose μ is a non-maximum weight matching with $|\mu_2| = \nu(G_2)$. Let μ' be a maximal matching. Since, in a matching, no two edges meet at a common vertex, each vertex is incident with at most one vertex in μ and one vertex in μ'. Thus, the set $\mu \oplus \mu'$ contains connected components of the form of either an $\mu-$alternating path or an $\mu-$alternating cycle. Since $w(\mu') > w(\mu)$, one of these alternating paths or cycles, say H, is such that $w(H \cap \mu') > w(H \cap \mu)$. Suppose there exists such an alternating path, say P. Since μ contains the maximum possible number of thick edges, $E_2(P \cap \mu) \geq E_2(P \cap \mu')$. Otherwise, $\mu \oplus P$ contains more thick edges than μ, which is a contradiction. Now, suppose $|P \cap \mu'| = |P \cap \mu|$ or $|P \cap \mu'| = |P \cap \mu| - 1$. Since $E_2(P \cap \mu) \geq E_2(P \cap \mu')$, in either case, $w(P \cap \mu') \leq w(P \cap \mu)$. Thus, P is such that $|P \cap \mu'| = |P \cap \mu| + 1$ and $E_2(P \cap \mu) = E_2(P \cap \mu')$. Note that this implies that P is an $\mu-$augmenting path with $E_2(P \cap \mu) = E_2(P \setminus \mu)$. Now, suppose there exists an alternating cycle, say C, such that $w(C \cap \mu') > w(C \cap \mu)$. Since each vertex is incident to at most one vertex in μ and one vertex in μ', the cycle C is an even size cycle. But this is impossible since $E_2(C \cap \mu) \geq E_2(C \cap \mu')$ implies $w(C \cap \mu) \geq w(C \cap \mu')$. ■

PROOF of PROPOSITION 1:

Let $(N_p \cup N_l, R)$ be a problem. Let $\mu \in \mathcal{E}^m(N_p, R|_{N_p})$. Let $\mu' \in \mathcal{E}^m(N_p \cup N_l, R)$ such that $T^{\mu'} \not\supseteq T^{\mu}$. Then, there exists $x \in T^{\mu} \setminus T^{\mu'}$. Since x is covered by μ but not by μ', the set $\mu \oplus \mu'$ contains a path P starting at x. Let $P = x, x_1, x_2, \ldots, x_k, y$ be this path. Observe that since μ is a matching for the problem $(N_p, R|_{N_p})$ and all the vertices $x_1, x_2, \ldots, x_k$ are covered by μ and μ', these vertices are in N_p. Moreover, according to the definition of R, $r_{a,b} = 1$ implies that $b \in N_l$. Thus, all the edges $x_1x_2, x_2x_3, \ldots, x_{k-1}x_k$ in P are thick. If y is covered by μ, then P is an $\mu'-$augmenting path: the matching $P \oplus \mu'$ contains more thick edges and has a higher weight than μ', contradicting the maximality of μ'. Thus, y is not covered by μ but covered by μ'. Then, the matching $P \oplus \mu'$ has a weight at least as $w(\mu')$, has the number of thick edges at least as $|\mu'_2|$. Thus, by the maximality of μ' the edge x_ky is thick, and the matching $P \oplus \mu'$ is maximal as well. Observe that the matching $P \oplus \mu'$ covers all the vertices $x, x_1, x_2, \ldots, x_k$, but not y. Since y is not covered by μ, we obtained a maximal matching $P \oplus \mu'$ such that $T^{P \oplus \mu'} = (T^{\mu'} \cup \{x\}) \setminus \{y\}$. By applying the same argument recursively, a maximal matching is obtained, where it covers all the vertices in T^{μ}.

PROOF of THEOREM 1:

Let $D_1(N, R)$ and $D_2(N, R)$ be the sets of pairs who are in $D(N, R)$ and part of only paired listed exchanges and of only paired kidney donations, respectively, in any maximal matching. Let $D_{1,2}(N, R) = D(N, R) \setminus (D_1(N, R) \cup D_2(N, R))$. Let $C_1(N, R)$ be the set of pairs who are part of only paired listed exchanges in each maximal matching, which are not in $A_1(N, R)$. Let $A_{1,2}(N, R)$ be the set of pairs who are part of a paired listed exchange in some maximal matchings and part of a paired kidney donation in the remaining maximal matchings, who have a compatibility with at least one pair in $D_1(N, R) \cup D_{1,2}(N, R)$. Let $C_{1,2}(N, R)$ be the set of pairs who are part of a paired listed exchange in some maximal matchings and part of a paired kidney donation in the remaining maximal matchings, who are not in $A_{1,2}(N, R)$.

Lemma 2 *If $u \in D_1(G)$, then $\delta(u) \subseteq E_1(G)$.*

Proof. Let $u \in D_1(G)$. Suppose there exists a thick edge incident with u in G. Since $u \in V(G_2)$, either $u \in D(G_2)$ or $u \in V(G_2) \setminus D(G_2)$. If the latter, then by Lemma 1, $u \in C_2(G)$, which is a contradiction. Thus, $u \in D(G_2)$. Let M be a maximal matching missing u. By Lemma 1, each neighbor of u in $V(G_2)$ is covered by a thick edge in M. Let v be such an edge and let $vw \in M$. Then, clearly $(M \setminus \{vw\}) \cup \{uv\}$ is another maximal matching. But this contradicts with $u \in D_1(G)$. ■

Lemma 3 *Let G be any graph. Let $u \in A_1(G)$. Then, $D_1(G - u) = D_1(G)$.*

Proof. First, we show that $D_1(G) \subseteq D_1(G - u)$. Let $v \in D_1(G)$ and let M be a matching missing v. Consider $M - u$. that is the matching obtained by removing the edge incident with u in M. Since the weight of $M - u$ is $w(M) - 1$ and $u \in A_1(G)$, the matching $M - u$ is a maximal matching in $G - u$. Since $M - u$ misses v, $v \in D(G - u)$. Since, by Lemma 2, $v \in D_1(G)$ implies $\delta(u) \subseteq E_1(G)$, $v \notin D_{1,2}(G - u)$. Thus, $v \in D_1(G - u)$.

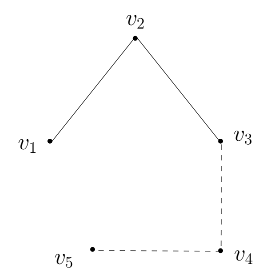
Now, we show that $D_1(G - u) \subseteq D_1(G)$. Let $v \in D_1(G - u)$. Suppose that there exists a maximal matching in G, which covers v by a thick edge. Since $u \in A_1(G)$, this maximal matching does not contain uv. By removing the edge that covers u, one obtains a maximal matching in $G - u$, which covers v by a thick edge. But, since $v \in D_1(G - u)$, it is a contradiction. Thus, $v \notin D_{1,2}(G) \cup D_2(G)$. Now, let M be a maximal matching in $G - u$ missing v. Let v' be a vertex in $D_1(G) \cup D_{1,2}(G)$ adjacent to u. Let M' be a maximal matching in G missing v'. If $v' = v$ or M' misses v, then $v \in D_1(G)$. So, suppose M' covers v. Consider the subgraph $M \cup M'$. Since both M and M' are matchings, $M \cup M'$ contains alternating paths and cycles. Since M misses v, $M \cup M'$ contains an alternating path, P, starting at v with an edge in M'. Suppose P ends with an edge of M. Then, since M is in $G - u$, the path P does not cover u. If $w(P \cap M) > w(P \cap M')$, then $M' \oplus P$ is another matching in G such that $w(M' \oplus P) > w(M')$, contradicting with the maximality of M'; similarly, if $w(P \cap M') > w(P \cap M)$, then it contradicts with the maximality of M. Thus, $w(P \cap M) = w(P \cap M')$. This, together with the fact that P is an even-length alternating path, implies that $P \cap M$ contains the same number of thick edges as $P \cap M'$. But then

$M' \oplus P$ is another maximal matching in G, which misses v. Thus, $v \in D_1(G)$.

Suppose P ends with an edge of M'. We claim P ends at u. Suppose not. Then, if $w(P \cap M) > w(P \cap M')$, then $M' \oplus P$ is another matching in G such that $w(M' \oplus P) > w(M')$, contradicting with the maximality of M'. Similarly, if $w(P \cap M') > w(P \cap M)$, then it contradicts with the maximality of M. Thus, $w(P \cap M) = w(P \cap M')$. Then, since $P \cap M'$ contains more edges than $P \cap M$, $P \cap M$ contains one more thick edge than $P \cap M'$. But then, $M' \oplus P$ has weight $\nu(G)$ but $|M_2'| < \nu(G_2)$, contradicting with the maximality of M'. Thus, P ends at u. Now, consider $P' = P \cup \{uv'\}$. If $w(P' \setminus M') > w(P' \cap M')$, then $w(P' \oplus M') > w(M')$, which contradicts with the maximality of M'. Thus, $w(P' \setminus M') \leq w(P' \cap M')$. Suppose $w(P' \setminus M') < w(P' \cap M')$. Let x be the vertex such that $ux \in M'$. (Note that, since $u \in A_1(G)$, $w(ux) = w(uv') = 1$.) Define $P'' = P' \setminus \{xu, uv'\}$. Since $w(P' \setminus M') < w(P' \cap M')$, we have $w(P'' \setminus M') < w(P'' \cap M')$. But then, $P'' \oplus M$ is a matching in $G - u$, and moreover $w(P'' \oplus M) > w(M)$, contradicting with the maximality of M. Thus, $w(P' \setminus M') = w(P' \cap M')$. But then, $M' \oplus P'$ is a matching in G. Moreover, since P' is an even-length path and $w(M' \oplus P') = w(M')$, the matchings $M' \oplus P'$ and M' have the same number of thick edges. Thus, $M' \oplus P'$ is a maximal matching in G. Since $M' \oplus P'$ does not cover v, $v \in D_1(G)$. ■

Lemma 4 *Let G be any graph. Let $u \in A_1(G)$. Then, (i) $D_{1,2}(G - u) \subseteq D_{1,2}(G)$, (ii) $D_{1,2}(G) \setminus D_{1,2}(G - u) \subseteq D_2(G - u)$. (iii) Moreover, for each $v \in D_{1,2}(G) \setminus D_{1,2}(G - u)$, the only vertex incident to v in $A_1(G)$ is u.*

Proof. (i) The second part of the proof of Lemma 3 implies that $D_{1,2}(G-u) \subseteq D(G)$. This, together with Lemma 3, implies that $D_{1,2}(G-u) \subseteq D_{1,2}(G) \cup D_2(G)$. Thus, in order to prove that $D_{1,2}(G-u) \subseteq D_{1,2}(G)$, it is sufficient to show that, for each $v \in D_{1,2}(G-u)$, there exists a maximal matching in G, which covers v by a thin edge. Let $v \in D_{1,2}(G - u)$ and M be a maximal matching in $G-u$, which covers v by a thin edge. Since $v \in D_{1,2}(G) \cup D_2(G)$, there exists a maximal matching, M' in G, which misses v. By definition of $A_1(G)$, u is covered by a thin edge, say ux, in G. Since $u \in A_1(G)$, $M' \setminus \{ux\}$ is a maximal matching in $G - u$. Moreover, the matching $M' \setminus \{ux\}$ misses v. Consider the set $M \cup (M' \setminus \{ux\})$. It contains an alternating path P, starting at v with an edge of M. By maximality of M and $M' \setminus \{ux\}$, this path is an even-length path containing the same number of thick edges of M and $M' \setminus \{ux\}$. Also, since the matching $M' \setminus \{ux\}$ misses x, and the path ends with an edge of $M' \setminus \{ux\}$, the path P does not cover the vertex x. But then, the matching $(M' \setminus \{ux\}) \oplus P$ is a maximal matching in $G - u$, which covers v by a thin edge. Since this last matching doe not cover the vertices u and x, the matching $M' \oplus P$ is a maximal matching in G and it covers v by a thin edge as well. Thus, $v \in D_{1,2}(G)$. (Note that in general, $D_{1,2}(G) \nsubseteq D_{1,2}(G - u)$. This can be easily seen via a simple example. Consider the following graph:



Here, $D_{1,2}(G) = \{v_3\}$, $A_1 = \{v_4\}$, $D_1 = \{v_5\}$, and $D_{1,2}(G - v_4) = \emptyset$.)

(ii) Let $v \in D_{1,2}(G) \setminus D_{1,2}(G-u)$. Let M and M' be two maximal matchings in G missing v and covering v by a thick edge, respectively. Since $u \in A_1(G)$, both $M - u$ and $M' - u$ are maximal in $G - u$. By Lemma 3, $v \notin D_1(G - u)$. Thus, $v \in D_2(G - u)$.

(iii) Suppose v is incident to $v' \in A_1(G)$ with $v' \neq u$. Suppose M contains uv'. But then, since $(M \setminus \{uv'\}) \cup \{uv\}$ is also maximal and misses v', this contradicts with $v' \in A_1$. Thus, M does not contain uv'. Thus, $M - u$ covers v' by a thin edge, say xv' and misses v. If the edge vv' is thick, then $M - xv' + vv'$ is a matching in $G - u$, and has a greater weight than $M - u$, contradicting with the maximality of $M - u$ in $G - u$. If the edge vv' is thin, then $M - xv' + vv'$ is also maximal in $G - u$, contradicting with $v \in D_2(G - u)$. ■

Lemma 5 *Let G be any graph. Let $u \in A_1(G)$. Then, $D(G - u) = D(G)$.*

Proof. Let M be a maximal matching in G missing v. Then, since $M - u$ is maximal in $G - u$ and misses v, $v \in D(G - u)$. Thus, $D(G) \subseteq D(G - u)$. The second part of the proof of

Lemma 3 implies that $D_2(G-u) \subseteq D(G)$. Then, by Lemma 3 and 4, $D(G-u) \subseteq D(G)$. ■

Lemma 6 *Let G be any graph. Let $u \in A_1(G)$. Then, $A_1(G) \setminus \{u\} = A_1(G-u)$.*

Proof. Let $v \in A_1(G) \setminus \{u\}$. Suppose that in all the maximal matchings in G, u is matched to v. Let v' be a vertex in $D_1(G) \cup D_{1,2}(G)$, which is incident to v. Consider a maximal matching M' missing v'. Since $uv \in M'$, the matching $M' - uv + vv'$ is also maximal in G and misses u, which contradicts with $u \in A_1(G)$. Thus, there exists at least one maximal matching in G such that u and v are not matched. Let M be such a maximal matching in G and consider the matching $M - u$ in $G - u$. Since $u \in A_1(G)$ and $w(M-u) = w(M) - 1$, the matching $M - u$ is maximal in $G - u$. Moreover, it covers v with a thin edge. Thus, $v \in D_1(G-u) \cup D_{1,2}(G-u) \cup A_1(G-u) \cup A_{1,2}(G-u) \cup C_1(G-u) \cup C_{1,2}(G-u)$. By Lemma 3 and 4, $v \notin D_1(G-u) \cup D_{1,2}(G-u)$. Suppose $v \in C_1(G-u) \cup C_{1,2}(G-u)$. By Lemma 3 and 4, this is possible only if the set of neighbors of v in $D_1(G) \cup D_{1,2}(G)$, say V', is in $D_{1,2}(G) \setminus D_{1,2}(G-u)$. But, Lemma 4 (iii) implies that, any vertex in V' has only one incident vertex in $A_1(G)$, which is u. Thus, $v \notin C_1(G-u) \cup C_{1,2}(G-u)$. Thus, $A_1(G) \setminus \{u\} \subseteq A_1(G-u) \cup A_{1,2}(G-u)$.[7] Now, suppose that $v \in A_{1,2}(G-u)$. Let M be a maximal matching in $G-u$ covering v by a thick edge. Let $y \in D_1(G) \cup D_{1,2}(G)$ be a neighbor of u and M' be a maximal matching in G, which is missing y. Also, let $ux \in M'$. Note that $x \neq v$, since otherwise, $M' + uy - uv$ has the same weight as M', thus it is maximal in G and missing v, which contradicts with $v \in A_1(G)$. The matching $M' - ux$ is maximal in $G - u$. If M misses y, then $M + uy$ is maximal and it covers v by a thick edge, which contradicts with $v \in A_1(G)$. Thus, M covers y. Then, $M \oplus (M' - uv)$ contains a path P starting at y with an edge of M. Suppose the path P covers x. Then, since $M' - ux$ misses v, P should end at x with an edge of M, which, by Lemma 1, contradicts with the maximality of $M' - ux$ in $G - u$. Thus, the path P does not cover x. But then, $M' \oplus P$ is maximal in G,

[7] This can be shown as follows as well: We claim that in $G-u$, each vertex in V' is missed by at least one maximal matching. Let $v'' \in V'$. Let M'' be a maximal matching in G missing v''. Since $M'' - u$ is maximal in $G - u$, v'' is missed by at least one maximal matching in $G-u$, in particular by M''. Now, since $v \in C_1(G-u)$, M'' covers v by a thin edge, say by ux. But then the matching $M'' - ux + uv''$ is also maximal in $G-u$, contradicting with $v'' \notin D_{1,2}(G-u)$.

covering v by a thick edge, which contradicts with $v \in A_1(G)$. Thus, $v \notin A_{1,2}(G-u)$. Thus, $A_1(G) - \{u\} \subseteq A_1(G-u)$.

Now, we show that $A_1(G-u) \subseteq A_1(G) \setminus \{u\}$. Suppose this is not true. Let $x \neq u$ such that $x \in A_1(G-u)$ but $x \notin A_1(G)$. Since for each maximal matching M missing a vertex $x \neq u$, $M-u$ is also missing x and maximal in $G-u$, each maximal matching in G covers x. Similarly, since for each maximal matching M covering a vertex $x \neq u$ by a thick edge, $M-u$ also covers x by a thick edge and maximal in $G-u$, no maximal matching in G covers x by a thick edge. Thus, $x \in A_1(G) \cup C_1(G)$. Suppose $x \in C_1(G)$. Then, by definition of $C_1(G)$, x is not incident to any vertex in $D_1(G) \cup D_{1,2}(G)$. But then, since by Lemma 3 and 4, $D_1(G-u) = D_1(G)$ and $D_{1,2}(G-u) \subseteq D_{1,2}(G)$, x is not incident to any vertex in $D_1(G-u) \cup D_{1,2}(G-u)$ neither. But this contradicts with $x \in A_1(G-u)$. Thus, $A_1(G-u) \subseteq A_1(G) \setminus \{u\}$. ■

Lemma 7 *Let $u \in A_1(G)$. Then, in each maximal matching, u is matched with a vertex in $D_1(G) \cup D_{1,2}(G)$.*

Proof. Let $u \in A_1(G)$ and M be a maximal matching. Suppose M contains uv where $v \in A_1(G)$. Since $u \in A_1(G)$, the matching $M - uv$ is maximal in $G-u$. By Lemma 6, $v \in A_1(G-u)$. But, this contradicts with $M-uv$ being maximal and missing v in $G-u$. Now, suppose M contains uv where $v \in C_1(G) \cup C_{1,2}(G) \cup A_{1,2}(G)$. Let $v' \in D_1(G) \cup D_{1,2}(G)$ be a vertex incident with u. The matching M covers v', since otherwise, $M - uv + uv'$ is maximal in G and misses v, contradicting with $v \in C_1(G)$. Since $v' \in D_1(G) \cup D_{1,2}(G)$, there exists a maximal matching missing v'; let M' be such a matching. The set $M \oplus M'$ contains alternating paths and cycles. Since v' is covered by M and missed by M', $M \oplus M'$ contains an alternating path, P, starting at v'. By Lemma 1, the maximality of M and M' implies that P is not an augmenting path. Thus, the matchings $M \oplus P$ and $M' \oplus P$ are maximal in G. There are two cases to consider. *Case 1:* The path P does not contain uv. Then, the matching $M \oplus P$ contains uv and misses the vertex v'. But then, since both uv and uv' are thin edges, the matching $(M \oplus P) - uv + uv'$ is maximal and missing v, which contradicts with $v \in C_1(G) \cup C_{1,2}(G) \cup A_{1,2}(G)$. *Case 2:* The path P contains uv. Then, the matching $M' \oplus P$ is maximal and contains uv. Since $u \in A_1(G)$, the matching $M \setminus \{uv\}$ is maximal in $G - u$. Also, it misses v. Thus, $v \in D(G - u)$. But, by Lemma 5, this contradicts with $v \in C_1(G) \cup C_{1,2}(G) \cup A_{1,2}(G)$. Thus, u is matched with a vertex in $D_1(G) \cup D_{1,2}(G)$. ■

Definition 4 *A graph is called **hypomatchable** if for each $v \in V$, $G - v$ has a perfect matching.*

Lemma 8 *Let $G - A_1(G)$ be the subgraph induced by removing the vertices in $A_1(G)$. The components of $G - A_1(G)$ in $D(G)$, if any, are hypomatchable, and each maximal matching of G contains a near-perfect matching of each such component.*[8]

Proof. By Lemma 6, the subgraph $G - A_1(G)$ is such that if we remove all the vertices in $A_1(G)$ one-by-one, $D(G - A_1(G)) = D(G)$. Let M be a maximal matching in G. Let H be a component of $G - A_1(G)$ in $D(G)$. By Lemma 7, the matching $M - A_1(G)$ is maximal in $G - A_1(G)$. Now, consider the graph $G - A_1(G)$. Suppose that there are two maximal matchings M' and M'' missing the vertices A and B respectively, such that $|A| < |B|$. Now, suppose that all the vertices in $B \setminus A$ are covered by M'. Let $u \in B \setminus A$. Since M'' misses u, the set $M' \oplus M''$ contains a path P starting at u with an edge of M'. Since both M' and M'' are maximal, Lemma 1 implies that the path P is even. Also, since $|A \setminus B| < |B \setminus A|$, without loss of generality, we can assume that P does not end in A. Note that P does not contain any vertex of A. Then, by Lemma 1, $M' \oplus P$ is maximal and misses the vertices in $A \cup \{u\}$. Thus, there exists a maximal matching which misses A and at least one vertex of $B \setminus A$. Now define the binary relation $\sim$ as follows: $u \sim v$ if and only if $u = v$ or no maximal matching misses both u and v. Suppose $u \sim v$ and $v \sim w$. Let M' be a maximal matchings missing v and M'' one missing u and w. But then, there is a maximal matching missing v and w, which is a contradiction. Thus, $u \sim w$ and $\sim$ is an equivalence relation. Now, since H is connected, any two vertices of H must be equivalent. Thus, any maximal matching misses at most one vertex

[8] This result derives from Gallai's Lemma (Lovasz and Plummer [7]). The proof is almost the same, but for the sake of completeness, we present it here.

of H. Also, since any vertex $u \in H$ is also in $D(G-A_1(G))$, $\nu(G-A_1(G)) = \nu(G-A_1(G)-u)$. Thus, the reduced submatching of M on H is a near-perfect matching of H. ■

7.3 The maximum number of kidney transplantations

Theorem 3 *Let $G = (V, E)$ be a graph and let $w \in \Re_+^E$ be a weight function. Then the maximum weight of a matching is equal to the minimum value of*

$$\sum_{v \in V} y_v + \sum_{S \in \mathcal{P}_{odd}(V)} z_S \lfloor \frac{1}{2} |S| \rfloor,$$

where $y \in \Re_+^E$ and $z \in \Re_+^{\mathcal{P}_{odd}(V)}$ satisfy

$$\sum_{v \in V} y_v \chi^{\delta(v)} + \sum_{S \in \mathcal{P}_{odd}(V)} z_S \chi^{E[S]} \geq w.$$

Definition 5 *A collection $\mathcal{F}$ of sets is called* laminar *if, for all $S, T \in \mathcal{F}$, either $S \cap T = \emptyset$ or $S \subseteq T$.*

Theorem 4 *(Cunningham-Marsh formula) In Theorem 1, if w is integer, we can take y and z integer. We can take z moreover such that the collection $\{S \in \mathcal{P}_{odd}(V) : z_S > 0\}$ is laminar.*

Proposition 2 *Let (y, z) be a solution to the minimization problem in Theorem 1. For $k = 1, 2$, let $U_k = \{v \in V : y_v = k\}$, and $W_k = \{S \in \mathcal{P}_{odd} : z_S = k\}$. Then, (i) the set U_2 and the sets in W_2 are disjoint, (ii) for each $S \in W_1$, $S \cap U_2 = \emptyset$, (iii) for each $S \in W_2$, $|S \cap U_1| \leq 1$, (iv) if $S \in W_1$ and $T \in W_2$, then $S \cap T = \emptyset$.*

Proof. By the Cunningham-Marsh formula, for all $v \in V$, either $y_v \in \{0, 1, 2\}$ or $z_S \in \{0, 1, 2\}$ for some odd set S containing v.

(i) Let $S, T \in W_2$. Suppose $S \cap T \neq \emptyset$. If $S \cup T$ is an odd set, then

$$\lfloor \frac{|S|}{2} \rfloor + \lfloor \frac{|T|}{2} \rfloor = \frac{|S| + |T|}{2} - 1 \geq \frac{|S \cup T| + 1}{2} - 1 = \lfloor \frac{|S \cup T|}{2} \rfloor.$$

Let $z'_S = z'_T = 0$, $z'_{S\cup T} = 2$, and for each $S' \in \mathcal{P}_{odd} \setminus \{S, T, S \cup T\}$, $z'_{S'} = z_{S'}$. Since $E(S \cup T) \supseteq E(S) \cup E(T)$, z' is feasible; it also reduces the value of the objective function. If $S \cup T$ is an even set, let $j \in S \cup T$ and set $z'_{S\cup T-\{j\}} = 2$ and $y'_j = 2$. Note that (z', y') is feasible. Since

$$\frac{|S| + |T|}{2} - 1 \geq \frac{|S \cup T| + 2}{2} - 1 = \frac{|S \cup T - \{j\}| - 1}{2} + 1,$$

(z', y') does not increase the cost function. Thus, at an optimal solution, the sets in W_2 are disjoint. If for some $S \in W_2$, $v \in U_2 \cap S$, then for some $u \in S$, by letting $y_u = 2$, $z'_S = 0$ and $z'_{S\setminus\{u,v\}} = 2$, we obtain another feasible solution, which has the same value of the objective function.

(ii) Suppose there exists $S \in W_1$ such that S contains v for some $v \in U_2$. Note that the case $U_2 \supseteq S$ is not possible, since then, setting $z'_S = 0$ reduces the value of the objective function. Thus, there exists $u \in S$ such that $y_u \in \{0, 1\}$. If $y_u = 1$, then setting $z'_S = 0$, $z'_{S\setminus\{u,v\}} = 1$ and $y'_u = 2$ is feasible and does not increase the value of the objective function. Similarly, if $y_u = 0$, then setting $z'_S = 0$, $z'_{S\setminus\{u,v\}} = 1$ and $y'_u = 1$ is feasible and does not increase the value of the objective function.

(iii) Suppose $u, v \in U_1 \cap S$ for some $S \in W_2$. Then, set $z'_{S\setminus\{u,v\}} = 2$ and $y'_u = y'_v = 2$. The value of the objective function under (y', z') does not change.

(iv) Let $S \in W_1$ and $T \in W_2$. By the Cunningham-Marsh Theorem, the collection W_1, W_2 is laminar. Thus, either $S \subseteq T$ or $S \supseteq T$ or $S \cap T = \emptyset$. The first two cases contradict with the optimality: if $S \subseteq T$, then set $z'_S = 0$; if $S \supseteq T$, then set $z'_T = 1$. In either case, the value of the objective function is decreased. Thus, $S \cap T = \emptyset$. ■

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