Trade Rules for Uncleared Markets with a Variable Population*

İpek Gürsel Tapkı†

Sabancı University

November 6, 2009

Preliminary and Incomplete–Please Do Not Quote

Abstract

We analyze markets in which the price of a traded commodity is such that the supply and the demand are unequal and the population is variable. The agents have single peaked preferences on their consumption and production choices. For such markets, we analyze the implications of *consistency* and *population monotonicity* properties. We first show that a subclass of "Uniform trade rules "uniquely satisfies *Pareto optimality, no-envy, peak-only, consistency*, and an informational simplicity property. Next, we show that there is no trade rule that satisfies *strong population monotonicity* together with *Pareto optimality, no-envy*, and *peak-only.* Finally, we show that a particular subclass of "Uniform trade rules "uniquely satisfies *Pareto optimality, no-envy, peak-only*, and *population monotonicity* as well as *Pareto optimality, no-envy, peak-only*, and *one-sided population monotonicity.*

JEL Classification numbers: D5, D6, D7

Keywords: market disequilibrium, trade rule, variable population, efficiency, consistency, population monotonicity

*I am indebted to Özgür Kıbrıs for his guidance, support and encouragement.

†Faculty of Arts and Social Sciences, Sabanci University, 34956, Istanbul, Turkey. E-mail: gursel@su.sabanciuniv.edu

1 Introduction

We analyze markets with a variable population in which the price of a traded commodity is fixed at a level where the supply and the demand are possibly unequal. This stickiness of prices is observed in many markets, either because the price adjustment process is slow, such as in labor and housing markets or because the price is controlled from the outside of the market, such as in health and education. There is a wide literature about this phenomenon. For a detailed discussion, please see Benassy (1993, 1982, 2002).

In our model, there are exogenously differentiated sets of buyers and sellers. Sellers face demand from buyers who might be individuals or producers that use the traded commodity as input. We assume that the buyers have strictly convex preferences on the consumption bundles. Thus, they have single-peaked preferences on the boundary on their budget sets, and therefore, on their consumption of the commodity. Similarly, we assume that the sellers have strictly convex production sets. Thus, their profits are single-peaked in their input.

We analyze open economies and thus consider possible transfers to/from outside the economy.

A trade rule in our model is made up of two components: a trade-volume rule and an allocation rule. The trade-volume rule takes the preferences of the buyers and the sellers and a possible transfer and determines the trade-volume that will be carried out in the economy and thus, the total consumption and the total production. Then, the allocation rule allocates the total consumption among the buyers and the total production among the sellers.

The trade-volume rule is related to Moulin (1980) who analyzes the determination of a one-dimensional policy issue among agents with single-peaked preferences. Particularly, when there is only one buyer and one seller, the trade-volume is exactly like a public good for these two agents. However, this is no more true when there are multiple buyers and sellers.

The problem faced by the allocation rule is to allocate a social endowment (that is, the total consumption) among agents with single peaked preferences. This problem is extensively analyzed by the literature following Sprumont (1991). However, our domain is an extension of his domain. We also analyze *just-buyer markets* (when there is no seller in the market) and *just-seller markets* (when there is no buyer in the market). These markets coincide with the allocation problem analyzed by Sprumont. He proposed and analyzed a “Uniform rule ”which later becomes a central rule of that literature (for example, see Dagan (1996), Ching (1992, 1994), Thomson (1994 a, b)). This rule will also play an important role in this paper.

Let us note that our model is not a simple conjunction of Moulin (1980) and Sprumont (1991). The interaction between the determination of the agent's shares and the trade-volume makes the model much richer. For example, the agents can manipulate their shares also by manipulating the trade-volume. Also, the requirements like *Pareto optimality*, or “fairness ”become much more demanding as what is to be allocated becomes endogenous. Another important difference is the existence of two types of agents (buyers and sellers) in our model. This duality limits the implications of requirements like *anonymity* or *no-envy.*

Our model is closely related to those of Kıbrıs and Küçükşenel (2009). They, however, analyze closed markets with a fixed population. That is, they do not consider transfers. These authors analyze a class of trade rules each of which is a composition of the Uniform rule with a trade-volume rule that picks the median of total demand, total supply and an exogenous constant. They show that this class uniquely satisfies *Pareto optimality, strategy proofness, no-envy,* and an informational simplicity axiom *independence of trade-volume.*

Our model is also related to Thomson (1995) and Klaus, Peters, and Storcken (1997, 1998). They analyze the reallocation of an infinitely divisible commodity among agents with single peaked preferences and individual endowments. The agents whose endowments are greater than their peaks are considered as suppliers and those whose endowments are less than their peaks as demanders. These authors also characterize a *Uniform reallocation rule* with respect to some properties. Note that, in their model the suppliers and the demanders are not differentiated. The identities of the agents depend on the relation between their peaks and endowments. In our model, however, producers and consumers are exogenously distinct identities. This difference has implications over the properties. For example, fairness properties in our model compare agents on the same side of the market. Also, in our model the agents do not have exogenously given endowments.

We introduce a class of *Uniform trade rules* each of which is a composition of the *Uniform rule* and a *trade-volume rule.* We axiomatically analyze *Uniform trade rules* on the basis of some central properties concerning the variation of the population: *consistency* and *popula-*

tion monotonicity. We also analyze the implications of standard properties such as *Pareto optimality, strategy-proofness*, and *no-envy*, and some informational simplicity properties such as *peak-only* and *independence of trade volume.*

Consistency has been analyzed in many contexts such as bargaining, coalitional form games, and taxation (for a detailed discussion of this, please see Section 2). Loosely speaking, a rule is *consistent* if a recommendation it makes for an economy always agrees with its recommendations for the associated reduced economies obtained by the departure of some of the agents with their promised shares. An issue in our model is how to define a reduced economy. Without a transfer from outside, the recommendation for an economy may not be feasible for its reduced economies. Thus, we use an updated transfer for the reduced economies to solve this problem. This practice is similar to the analysis of *consistency* in economies with individual endowments. Thomson (1992) introduced a "generalized economy "that consists of a preference profile of the agents, an endowment profile and a trade vector that is updated in the reduced economies. The trade vector in that model corresponds in our model to the transfer. We show in Theorem 1 that a particular subclass of *Uniform trade rules* uniquely satisfies *consistency* together with *Pareto optimality, no-envy*, and *peak-only.*

Population monotonicity has also been widely analyzed in many different contexts such as in classical economies, single-peaked preferences, and public goods (for a detailed discussion, please see Section 2). Loosely speaking, it requires that for a given economy, upon the departure of some agents, the welfare level of the remaining agents should be affected in the same direction. In our model, we analyze three types of population monotonicity properties. First, we analyze a *strong population monotonicity* property which compares agents without differentiating them as buyers or sellers. We show in theorems 2 and 3 that no trade rule satisfies this property together with *Pareto optimality, no-envy*, and *peak only* as well as *Pareto optimality, no-envy*, and *strategy-proofness.*

We also analyze a weaker monotonicity property, *population monotonicity* that only compares agents on the same side of the market. This is an adaptation of *population monotonicity* introduced by Thomson (1995). We first note that there are trade rules that simultaneously satisfy three properties, which on Sprumont's domain are incompatible: *Pareto optimality, no-envy*, and *population monotonicity.* In Theorem 4, we characterize trade rules that satisfy *population monotonicity* together with *Pareto optimality, no-envy*, and *peak-only*. We next characterize in Theorem 5 the class of trade rules that satisfies *population monotonicity* together with *Pareto optimality, no-envy*, and *strategy-proofness*. We next analyze the implications of *independence of trade volume* and *population monotonicity*. We show in Theorem 6 that the Uniform trade rule that always clears the long side uniquely satisfies *population monotonicity* and *independence of trade volume* together with *Pareto optimality,no-envy* and *peak-only* as well as *Pareto optimality,no-envy* and *strategy-proofness*.

We next note that the given monotonicity properties allow very radical population changes. For an economy in which the short side is the sellers, the departure of sufficiently many agents may turn it into one in which the short side is the buyers. This is very demanding and thus we also analyze a much weaker monotonicity property, *one-sided population monotonicity*. This property is an adaptation of *one-sided population monotonicity* introduced by Thomson (1995). We characterize in Theorem 7 the class of trade rules that satisfies *one-sided population monotonicity* together with *Pareto optimality, no-envy*, and *peak-only*. We next characterize in Theorem 8 the class of trade rules satisfying *one-sided population monotonicity* together with *Pareto optimality, no-envy*, and *strategy-proofness*.

The paper is organized as follows. In Section 2, we introduce the model. In Section 3, we analyze the implications of *consistency* and in Section 4, the implications of *population monotonicity* properties.

2 Model

There are countably infinite universal sets, $\mathcal{B}$ of potential buyers and $\mathcal{S}$ of potential sellers. Let $\mathcal{B} \cap \mathcal{S} = \emptyset$. Let $\mathbb{R}_+$ be the consumption/ production space for each agent. Let R be a preference relation and P be the strict preference relation associated with R. The preference relation R is **single-peaked** if there is $p(R) \in \mathbb{R}_+$ called the peak of R, such that for all $x, y \in \mathbb{R}_+$, $x < y \leq p(R)$ or $x > y \geq p(R)$ implies $y\ P\ x$. Each $i \in \mathcal{B} \cup \mathcal{S}$ is endowed with a continuous single-peaked preference relation R_i over $\mathbb{R}_+$. Let $\mathcal{R}$ denote the set of all continuous and single-peaked preference relations on $\mathbb{R}_+$.

Given a finite set $B \subset \mathcal{B}$ of buyers and a finite set $S \subset \mathcal{S}$ of sellers such that either $B \neq \emptyset$

or $S \neq \emptyset$, let $N = B \cup S$ be a **society**. Let $\mathcal{N}$ be the set of all societies. A preference profile R_N for a society N is a list $(R_i)_{i \in N}$ such that for each $i \in N$, $R_i \in \mathcal{R}$. Let $\mathcal{R}^N$ denote the set of all profiles for the society N. Given $N' \subset N$ and $R_N \in \mathcal{R}^N$, let $R_{N'} = (R_i)_{i \in N'}$ denote the restriction of R_N to N'.

A **market for society $\boldsymbol{N}$** $= B \cup S$ is a list $\boldsymbol{(R_B, R_S, T)}$ where $(R_B, R_S) \in \mathcal{R}^N$ is a profile of preferences for buyers and sellers and $T \in \mathbb{R}$ is a **transfer**. Note that T can both be positive and negative. A positive T represents a transfer made from outside. Thus, it is added to the production of the sellers and together they form the total supply. On the other hand, a negative T represents a transfer that must be made from the economy to the outside. Thus, it is considered as an addition to the total demand.

Given a market (R_B, R_S, T) for a society $N = (B \cup S)$, a **(feasible) trade** is a vector $z \in \mathbb{R}_+^{B \cup S}$ such that $\sum_B z_b = \sum_S z_s + T$. Let $Z(R_B, R_S, T)$ denote the set of all trades for (R_B, R_S, T).

There are two special subclasses of markets. A market (R_B, R_S, T) is a **just-buyer market** if $B \neq \emptyset$ and $S = \emptyset$. For such markets, the feasible trades are as follows. If $T \geq 0$, $Z(R_B, R_S, T) = \{z \in \mathbb{R}_+^B : \sum_B z_b = T\}$. If $T < 0$, then $Z(R_B, R_S, T) = \emptyset$. (This is trivial because if there are no seller, all the agents are demanders, and thus, the supply is zero. Thus, if the outside transfer is positive, it would be equal to the total supply and it is divided among the buyers. However, if there is a negative transfer (that is, a transfer must be made to outside), since there is no seller, the transfer cannot be realized. Thus, in that case there is no trade.) A market (R_B, R_S, T) is a **just-seller market** if $B = \emptyset$ and $S \neq \emptyset$. For such markets, the feasible trades are as follows. If $T \leq 0$, $Z(R_B, R_S, T) = \{z \in \mathbb{R}_+^S : \sum_S z_s + T = 0\}$. If $T > 0$, then $Z(R_B, R_S, T) = \emptyset$. (The explanation is similar to above.) Note that *just-buyer markets* and *just-seller markets* mathematically coincide with the allocation problems analyzed by Sprumont (1991). Thus, his domain is a restriction of ours.

Since the markets with no feasible trade are trivial, we restrict ourselves to the set of markets for which the set of trades is nonempty. Let $\mathcal{M}^N = \{(R_B, R_S, T) : (R_B, R_S) \in \mathcal{R}^N, T \in \mathbb{R}, \text{ and } Z(R_B, R_S, T) \neq \emptyset\}$ be the set all markets for society $N = B \cup S$ and let

$$\mathcal{M} = \bigcup_{N \in \mathcal{N}} \mathcal{M}^N$$

be the set of all markets. Let $\mathcal{M}_{\mathcal{B}} = \{(R_B, R_S, T) \in \mathcal{M} : \; B \neq \emptyset, \; S = \emptyset, \text{ and } T \geq 0\}$ be the set of *just-buyer markets* and $\mathcal{M}_{\mathcal{S}} = \{(R_B, R_S, T) \in \mathcal{M} : \; B = \emptyset, \; S \neq \emptyset, \text{ and } T \leq 0\}$ be the set of *just-seller markets*.

Let $h(R_B, R_S, T)$ denote the **short side of the market $(\boldsymbol{R_B, R_S, T})$**, that is,

$$h(R_B, R_S, T) = \begin{cases} B & \text{if } \sum_B p(R_b) < \sum_S p(R_s) + T, \\ S & \text{if } \sum_S p(R_s) + T < \sum_B p(R_b). \end{cases}$$

A trade $z \in Z(R_B, R_S, T)$ is ***Pareto optimal with respect to* $(\boldsymbol{R_B, R_S, T})$** if there is no $z' \in Z(R_B, R_S, T)$ such that for all $i \in B \cup S$, $z'_i R_i z_i$ and for some $j \in B \cup S$, $z'_j P_j z_j$. *Pareto optimal* trades possess the following *same-sidedness* property. It requires that no two buyers (sellers) receive shares at opposite sides of their peaks (since then, a transfer from one to the other leads to a *Pareto* improvement). On Sprumont's domain, this is equivalent to *Pareto optimality*. It is formally stated in the following lemma. We omit its simple proof.

Lemma 1 For each $N = (B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^N$, *if* a trade $z \in Z(R_B, R_S, T)$ is *Pareto optimal with respect to* (R_B, R_S, T), *there is no* $K \in \{B, S\}$ *and* $k, k' \in K$ *such that* $z_k > p(R_k)$ *and* $z_{k'} < p(R_{k'})$.

In our framework, in addition to *sames-sidedness*, *Pareto optimality* requires the total consumption to be between the total supply and the total demand. It is formally stated in the following lemma. Its proof is also simple (please see Kıbrıs and Küçükşenel (2009)).

Lemma 2 For each $N = (B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^N$, if a trade $z \in Z(R_B, R_S, T)$ is *Pareto optimal with respect to* (R_B, R_S, T), then

(i) $h(R_B, R_S, T) = B$ implies $\sum_B p(R_b) \leq \sum_B z_b \leq \sum_S p(R_s) + T$ and

(ii) $h(R_B, R_S, T) = S$ implies $\sum_S p(R_s) + T \leq \sum_B z_b \leq \sum_B p(R_b)$.

A trade rule first determines the volume of trade that will be carried out in the economy and therefore, the total production and the total consumption. Then, it allocates the total production among the sellers and the total consumption among the buyers. Before defining a trade rule, we will first define a trade-volume rule.

A **trade-volume rule** $\Omega : \mathcal{M} \to \mathbb{R}^2_+$ associates each market (R_B, R_S, T) with a vector $\Omega(R_B, R_S, T) = (\Omega^B(R_B, R_S, T), \Omega^S(R_B, R_S, T))$ whose first coordinate, $\Omega^B(R_B, R_S, T)$ is the total consumption of the buyers and the second coordinate, $\Omega^S(R_B, R_S, T)$ is the total production of the sellers. Note that, for each market (R_B, R_S, T) and a *trade-volume rule* Ω, $\Omega^B(R_B, R_S, T) = \Omega^S(R_B, R_S, T) + T$. Thus, the volume of Ω^B determines the volume of Ω^S. Therefore, with an abuse of notation, we will sometimes call Ω^B a *trade-volume rule*.

In a *just-buyer market*, the transfer is divided among the buyers. Thus, the total consumption is equal to the transfer. In a *just-seller market*, however, the sellers produce an amount that corresponds to the transfer. Thus, in that case, the total production is equal to the transfer. Therefore, each *trade-volume rule* Ω satisfies the following:

$$\Omega(R_B, R_S, T) = \begin{cases} (T, 0) & \text{if } (R_B, R_S, T) \in \mathcal{M}_{\mathcal{B}} \\ (0, T) & \text{if } (R_B, R_S, T) \in \mathcal{M}_{\mathcal{S}} \\ (\Omega^B(R_B, R_S, T), \Omega^S(R_B, R_S, T)) & \text{otherwise} \end{cases}$$

Note that, the trade-volume is fixed for the *just-buyer* and the *just-seller markets*. Thus, for simplicity, we will define a *trade-volume rule* only by the volume it chooses for the other markets.

Let $\mathcal{V}$ be the set of all *trade-volume rules*. Let $\mathcal{V}^{[short,long]}$ be the set of *trade-volume rules*, Ω each of which chooses a trade-volume between the total demand and supply of the market, that is, for each market (R_B, R_S, T),

$$\Omega(R_B, R_S, T) \in [\sum_B p(R_b), \sum_S p(R_s) + T]^1.$$

The following subclass of $\mathcal{V}^{[short,long]}$ will be used extensively in rest of the paper. Let $\mathcal{V}^{\{short,long\}}$ be the set of *trade-volume rules*, Ω each of which alternates between picking the total demand/supply of the short and the long side of the market, that is, for each market

[1] If $\sum_B p(R_b) > \sum_S p(R_s) + T$, then consider $[\sum_S p(R_s) + T, \sum_B p(R_b)]$.

(R_B, R_S, T),

$$\Omega(R_B, R_S, T) \in \{\sum_B p(R_b), \sum_S p(R_s) + T\}.$$

Particularly, the following two members of $\mathcal{V}^{\{short,long\}}$ will be important in our analysis:
(i) the *trade-volume rule* that always coincides with the total demand/supply of the short side (Ω^{short}), that is,

$$\Omega^{short}(R_B, R_S, T) = \begin{cases} \sum_B p(R_b) & \text{if } h(R_B, R_S, T) = B \\ \sum_S p(R_s) + T & \text{if } h(R_B, R_S, T) = S \end{cases}$$

(ii) the *trade-volume rule* that always coincides with the total demand/supply of the long side (Ω^{long}), that is,

$$\Omega^{long}(R_B, R_S, T) = \begin{cases} \sum_S p(R_s) + T & \text{if } h(R_B, R_S, T) = B \\ \sum_B p(R_b) & \text{if } h(R_B, R_S, T) = S \end{cases}$$

An **allocation rule** $f : \mathcal{M}_{\mathcal{B}} \cup \mathcal{M}_{\mathcal{S}} \to \cup_{M \in \mathcal{M}_{\mathcal{B}} \cup \mathcal{M}_{\mathcal{S}}} Z(M)$ associates each *just-buyer* and *just-seller* market (R_K, T), for $K \in \{B, S\}$, with a trade $z \in Z(R_K, T)$. For example, Uniform rule, U, introduced by Sprumont (1991) is very central in the literature. In our paper, also, it will be used extensively. Formally, it is defined as follows: for each $K \in \{B, S\}$, $(R_K, T) \in \mathcal{M}_{\mathcal{K}}$, and $k \in K$,

$$U_k(R_K, \Omega^K) = \begin{cases} \min\{p(R_k), \lambda\} & \text{if } \sum_K p(R_k) \geq T \\ \max\{p(R_k), \mu\} & \text{if } \sum_K p(R_k) \leq T \end{cases}$$

where λ and μ is uniquely determined by the equations, $\sum_K \min\{p(R_k), \lambda\} = T$ and $\sum_K \max\{p(R_k), \mu\} = T$.

A **trade rule** $F : \mathcal{M} \to \cup_{M \in \mathcal{M}} Z(M)$ is a composition of a *trade-volume rule* Ω and an allocation rule f: $F = f \circ \Omega$. More precisely, for each market (R_B, R_S, T) and $K \in \{B, S\}$, $F_K(R_B, R_S, T) = f(R_K, \Omega^K(R_B, R_S, T))$. A *trade rule*, $F = U \circ \Omega$, that coincides the *uniform rule* with some *trade-volume rule* Ω is called the **uniform trade rule with respect to Ω.**

In our analysis, the trade rules $F = U \circ \Omega_{short}$ and $F = U \circ \Omega_{long}$ turn out to be central. Kıbrıs and Küçükşenel (2009) characterize a particular class of *uniform trade rules* for which Ω is the median of total demand, total supply, and an exogenous constant.

Now, we introduce properties of a trade rule. We start with efficiency. A trade rule F is **Pareto optimal** if for each $(R_B, R_S, T) \in \mathcal{M}$, the trade $F(R_B, R_S, T)$ is *Pareto optimal with respect to* (R_B, R_S, T).

Now, we present a fairness property. A trade is *envy free* if each buyer (respectively, seller) prefers his own consumption (respectively, production) to that of every other buyer (respectively, seller). A trade rule satisfies **no-envy** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, $K \in \{B, S\}$, and $i, j \in K$, $F_i(R_B, R_S, T) R_i F_j(R_B, R_S, T)$. Since in our model the agents on different sides of the market are exogenously differentiated, this property only compares agents on the same side of the market.

The following is a property on nonmanipulability. It requires that regardless of the others' preferences, an agent is best-off with the trade associated with his true preferences. Formally, a trade rule F is **strategy proof** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, $i \in N$, and $R'_i \in \mathcal{R}$, $F_i(R_i, R_{N\setminus i}, T)\ R_i\ F_i(R'_i, R_{N\setminus i}, T)$.

Next, we present some properties concerning possible variations in the number of agents. The first one is an adaptation of the standard *consistency* property to our domain. This property has been analyzed extensively in the context of bargaining by Lensberg (1987), single-peaked preferences by Thomson (1994), coalitional form games by Peleg (1985) and Hart and Mas-Colell (1989), taxation by Aumann and Maschler (1985) and Young (1987), cost allocation by Moulin (1985), fair allocation in classical economics by Thomson (1988), and matching by Sasaki and Toda (1992). To explain *consistency*, consider a trade rule F and a market (R_B, R_S, T). Suppose that F chooses the trade z. Imagine that some buyers and sellers leave with their shares they have been already assigned. This leads to a reduced problem in which the remaining agents, $(B' \cup S')$ are now facing an updated transfer from T to $T - \sum_{B\setminus B'} z_b + \sum_{S\setminus S'} z_s$. Consistency is about how the remaining agents' shares should be affected in this reduced problem. If F is consistent, it should assign to them the same shares as in the initial market. Formally, given a trade rule F, for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, and $N' = (B' \cup S') \subseteq N$, a **reduced problem of (R_B, R_S, T)**

for N' at $z \equiv F(R_B, R_S, T)$ is $r^z_{N'}(R_B, R_S, T) = (R_{B'}, R_{S'}, T - \sum_{B \setminus B'} z_b + \sum_{S \setminus S'} z_s)$. A trade rule F is **consistent** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, and $N' = (B' \cup S') \subseteq N$, if $z = F(R_B, R_S, T)$, then $z_{N'} = F(r^z_{N'}(R_B, R_S, T))$. *Consistency* of a *trade rule*, $F = f \circ \Omega$ has some implications over its associated *trade-volume rule* Ω. We state this implication as a *consistency* property of Ω. A *trade-volume rule* Ω is *consistent* if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, $N' = (B' \cup S') \subseteq N$, and $z \in Z(R_B, R_S, T)$, $\Omega(r^z_{N'}(R_B, R_S, T)) = \sum_{B'} z_{b'}$.

Next is a standard *population monotonicity* property. It has been extensively analyzed in classical economies by Chichilnisky and Thomson (1987), Thomson (1987), Chun and Thomson (1988), Moulin (1992), and Chun (1986), on domains of economies with indivisible goods by Alkan (1989), Tadenuma and Thomson (1990, 1993), Moulin (1990), Bevia (1992), and Fleurbaey (1993), on domains of economies with both private and public goods by Thomson (1987), Moulin (1990), in single-peaked preferences by Thomson (1995), and Klaus (2001). *Strong population monotonicity* requires that upon the departure of some agents, the welfare levels of all remaining agents should be affected in the same direction. Formally, a trade rule F is **strong population monotonic** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, $N' = (B' \cup S') \subseteq N$, either (i) for each $i \in N'$, $F_i(R_{B'}, R_{S'}, T)\ R_i\ F_i(R_B, R_S, T)$, or (ii) for each $i \in N'$, $F_i(R_B, R_S, T)\ R_i\ F_i(R_{B'}, R_{S'}, T)$. Note that this is a strong property because it requires all the remaining agents, without differentiating them as buyers and sellers, to be affected in the same direction. In our model, however, agents on different sides of the market are exogenously differentiated. Thus, we also analyze a weaker property that only compares agents on the same side of the market.

A trade rule F is **population monotonic** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, $N' = (B' \cup S') \subseteq N$, $K \in \{B', S'\}$, either (i) for each $i \in K$, $F_i(R_{B'}, R_{S'}, T)\ R_i\ F_i(R_B, R_S, T)$, or (ii) for each $i \in K$, $F_i(R_B, R_S, T)\ R_i\ F_i(R_{B'}, R_{S'}, T)$.

The above two monotonicity properties allow very "radical "population changes. For example, for an economy in which the short side is the buyers, the departure of sufficiently many sellers may turn it into one in which the short side is the sellers. In theorems 2, 3, and 6, we find that this is too demanding: only a very small class of rules satisfy *population monotonicity* and no rule satisfies *strong population monotonicity* together with some standard properties. Thus, we weaken the monotonicity requirement by applying it only when the change in the population is not too disruptive, that is, when the short side does not change. This is an adaptation of *one-sided population monotonicty* introduced by Thomson (1995). A trade rule F is **one-sided population monotonic** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, $N' = (B' \cup S') \subseteq N$ with $h(R_B, R_S, T) = h(R_{B'}, R_{S'}, T)$, for $K \in \{B', S'\}$, either (i) for each $i \in K$, $F_i(R_{B'}, R_{S'}, T)\ R_i\ F_i(R_B, R_S, T)$, or (ii) for each $i \in K$, $F_i(R_B, R_S, T)\ R_i\ F_i(R_{B'}, R_{S'}, T)$.

Lastly, we present the following informational simplicity properties. *Peak-onliness* requires the trade only to depend on agents' peaks but not on the whole preference relation. According to this property, each agent may change his preference relation without changing his peak and this does not change the trade. Since obtaining the agents' whole preference relation is very difficult in real life, it is a requirement of informational simplicity. It is thus closely related to *strategy-proofness*. Formally, a trade rule F satisfies **peak-only** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T), (R'_B, R'_S, T) \in \mathcal{M}^N$, $p(R) = p(R')$ implies $F(R_B, R_S, T) = F(R'_B, R'_S, T)$.

Peak-onliness can also be defined for the *trade-volume rules*. *Peak-only-volume* requires the trade-volume only to depend on agents' peaks but not on the whole preference relation. Formally, a *trade-volume rule*, Ω satisfies **peak-only-volume** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T), (R'_B, R'_S, T) \in \mathcal{M}^N$, $p(R) = p(R')$ implies $\Omega(R_B, R_S, T) = \Omega(R'_B, R'_S, T)$.

The following is also a simplicity property concerning the *trade-volume rule*. *Independence of trade-volume* introduced by Kıbrıs et al (2009) requires the *trade-volume rule* only to depend on the total demand and supply but not on their individual components. A *trade-volume rule*, Ω satisfies **independence of trade volume** if for each $N = (B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T), (R'_B, R'_S, T) \in \mathcal{M}^N$, $\sum_{b \in B} p(R_b) = \sum_{b \in B} p(R'_b)$ and $\sum_{s \in S} p(R_s) = \sum_{s \in S} p(R'_s)$ implies $\Omega(R_B, R_S, T) = \Omega(R'_B, R'_S, T)$.

3 Results

3.1 Consistency

The following lemma shows that for *Pareto optimal* rules, the reduction of a market does not change the short side (for its proof, please see the Appendix).

Lemma 3 Let F be a *Pareto optimal* trade rule. Then, for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, and $N' = (B' \cup S') \subseteq N$ with $B' \neq \emptyset$ and $S' \neq \emptyset$, if $z = F(R_B, R_S, T)$, then we have

(*i*) $h(R_B, R_S, T) = B$ implies $h(r^z_{B' \cup S'}(R_B, R_S, T)) = B'$, and

(*ii*) $h(R_B, R_S, T) = S$ implies $h(r^z_{B' \cup S'}(R_B, R_S, T)) = S'$.

The following lemma analyzes the relationship between the properties satisfied by a trade rule $F = f \circ \Omega$, and its component f. It shows that *Pareto optimality, no-envy, peak-only, strategy proofness*, and *consistency* satisfied by F passes on to f (for its proof, please see the Appendix).

Lemma 4 If a trade rule $F = f \circ \Omega$ satisfies one of the following properties, then f also satisfies that property: *Pareto optimality, no-envy, peak-only, strategy proofness* and *consistency.*

Theorem 1 says that under the assumption of *independence of trade volume*, the subclass of *Uniform trade rules* $F = U \circ \Omega$ where Ω is consistent and $\Omega \in \mathcal{V}^{\{short, long\}}$ uniquely satisfies *Pareto optimality, no-envy* and *consistency* (for its proof, please see the Appendix):

Theorem 1 Let $\Omega \in \mathcal{V}$ satisfy *independence of trade-volume.* A trade rule $F = f \circ \Omega$ satisfies *Pareto optimality, no-envy*, and *consistency* if and only if $f = U$ and Ω is *consistent* and for $|B| \geq 3$, and $|S| \geq 3$, $\Omega \in \mathcal{V}^{\{short, long\}}$.

To prove Theorem 1, we use the following results. Dagan (1996) proves that the *Uniform rule* is the only *allocation rule* satisfying *Pareto optimality*, *no-envy*, and *bilateral consistency*, a weaker property than *consistency.* For its proof, please see Dagan (1996).

Lemma 5 *(Dagan, 1996)* If the potential number of agents is at least 4 and if an economy consists of at least 2 agents, then f satisfies *Pareto optimality, no-envy,* and *bilateral-consistency* if and only if $f = U$.

3.2 Population Monotonicity

First, we analyze the implications of *strong population monotonicity.* The following two theorems state that there is no trade rule satisfying *strong population monotonicity* together with *Pareto optimality, no-envy*, and *peak-only* as well as *Pareto optimality, no-envy*, and *strategy-proofness*(for their proof, please see the Appendix).

Theorem 2 There is no trade rule that satisfies *Pareto optimality, no-envy, peak-only*, and *strong population monotonicity.*

Theorem 3 There is no trade rule that satisfies *Pareto optimality, no-envy, strategy proofness*, and *strong population monotonicity.*

Strong population monotonicity is a too demanding property. It compares agents regardless of their identities such as buyers and sellers although in our model they are exogenously different identities. Thus, we also analyze a weaker monotonicity property, *population monotonicity.*

Our first observation is that there are *trade rules* that simultaneously satisfy three properties which, on Sprumont's domain, are incompatible: *Pareto optimality, no-envy,* and *population monotonicity* (see Thomson (1995) for a discussion). It is easy to show that the *Uniform trade rule*, $U \circ \Omega_{long}$ satisfies *Pareto optimality* and *no-envy.* The following lemma shows that it also satisfies *population monotonicity.* The lemma also shows that *population monotonicity* that satisfied by a *trade rule* $F = f \circ \Omega$ may not pass on to f (for its proof, please see the Appendix).

Lemma 6 The *Uniform trade rule,* $F = U \circ \Omega_{long}$ is *population monotonic.*

In the following theorem, we show that when there are at least two buyers and two sellers with different peaks, a particular subclass of Uniform trade rules each of which is a composition of the Uniform rule and a trade-volume rule $\Omega \in \mathcal{V}^{short,long}$ that satisfies the following conditions uniquely satisfies *Pareto optimality, no-envy, peak-only*, and *population monotonicity*. To understand this subclass, let the short side be the buyers. Suppose that some buyers and sellers leave the market and the short side of the new market becomes the sellers. Then, if $\Omega = \Omega_{short}$ ($\Omega = \Omega_{long}$) in the initial economy and if the new market contains a seller (a buyer, respectively) whose initial share is different than his peak, then $\Omega = \Omega_{short}$ ($\Omega = \Omega_{long}$) in the new market. Similar condition holds when the short side of the initial market is the sellers (for its proof, please see the Appendix).

Theorem 4 A trade rule $F = f \circ \Omega$ satisfies *Pareto optimality, no-envy, peak-only*, and *population monotonicity* if and only if $f = U$ and $\Omega \in \mathcal{V}^{short,long}$ is *peak-only* and satisfies the following conditions: for each $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^{B\cup S}$ such that there are $b_i, b_j \in B$ and $s_k, s_l \in S$ with $p(R_{b_i}) \neq p(R_{b_j})$ and $p(R_{s_k}) \neq p(R_{s_l})$,

(i) if $h(R_B, R_S, T) = B$, then for each $(B' \cup S') \subseteq (B \cup S)$ with $h(R_{B'}, R_{S'}, T) = S'$, we have

- if $\Omega(R_B, R_S, T) = \sum_B p(R_b)$ and there is $s' \in S'$ with $F_{s'}(R_B, R_S, T) \neq p(R_{s'})$, then $\Omega(R_{B'}, R_{S'}, T) = \sum_{S'} p(R_{s'}) + T$.
- if $\Omega(R_B, R_S, T) = \sum_S p(R_s) + T$ and there is $b' \in B'$ with $F_{b'}(R_B, R_S, T) \neq p(R_{b'})$, then $\Omega(R_{B'}, R_{S'}, T) = \sum_{B'} p(R_{b'})$.

(ii) the similar condition holds for the case $h(R_B, R_S, T) = S$, just replace S with B in (i).

To prove Theorem 4, we need the following lemma. Ching (1992) proves that the Uniform rule uniquely satisfies *Pareto optimality*, *strategy proofness*, and *no-envy.* To prove this result, he first shows that if an allocation rule is *Pareto optimal* and *strategy-proof*, then it satisfies *own-peak only* which is weaker than *peak-only.* Then, he shows that if an allocation rule satisfies *Pareto optimality*, *own-peak only*, and *no-envy*, then it coincides with the Uniform rule. Thus, we have the following lemma. For its proof, please see Ching (1992).

Lemma 7 *(Ching, 1992)*

(i) An allocation rule f satisfies *Pareto optimality, no-envy,* and *strategy proofness* if and only if $f = U$.

(ii) If an allocation rule f satisfies *Pareto optimality, no-envy,* and *peak-only,* then $f = U$.

Next, we replace *peak-only* with *strategy proofness.* We show that *population monotonicity* together with *Pareto optimality, no-envy*, and *strategy proofness* are uniquely satisfied by a subclass of *Uniform trade rules* each of which is of the form $U \circ \Omega$ for some $\Omega \in \mathcal{V}^{short,long}$ that satisfies the following conditions: let the short side of the market be the buyers and b be a buyer whose share z_b is greater than his peak. Then the trade-volume is invariant under the preference relation of b, but the only restriction is that his peak should be up to z_b. Similarly, let s be a seller whose share z_s is less than his peak. Then, the trade-volume is invariant under the preference relation of s, but the only restriction is that his peak should be greater than z_s. A similar condition holds if the short side is the sellers. Also Ω satisfies (i) and (ii) of Theorem 4 (for its proof, please see the appendix).

Theorem 5 A trade rule $F = f \circ \Omega$ satisfies *Pareto optimality, no-envy, strategy-proofness*, and *population monotonicity* if and only if $f = U$, $\Omega \in \mathcal{V}^{short,long}$ and satisfies the following: for each $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^{B \cup S}$, and for $z \equiv F(R_B, R_S, T)$,

(I) if there are $b, b' \in B$ and $s, s' \in S$ with $p(R_b) \neq p(R_{b'})$ and $p(R_s) \neq p(R_{s'})$, then Ω satisfies (i) and (ii) in Theorem 4.

(II) if $h(R_B, R_S, T) = B$, then for each $b \in B$ with $z_b > p(R_b)$, $s \in S$ with $z_s < p(R_s)$, $R'_b \in \mathcal{R}$ with $p(R'_b) < z_b$, and $R'_s \in \mathcal{R}$ with $z_s < p(R'_s)$, we have $\Omega(R'_b, R_{B\setminus\{b\}}, R_S, T) = \Omega(R_B, R'_s, R_{S\setminus\{s\}}, T) = \Omega(R_B, R_S, T)$.

(III) if $h(R_B, R_S, T) = B$, the similar condition holds, just replace B with S, b with s, and vice versa.

Next, we analyze the implications of *independence of trade volume* and *population monotonicity.* The following theorem shows that under the assumption of *independence of trade-volume*, the *Uniform trade rule*, $F = U \circ \Omega_{long}$ uniquely satisfies *Pareto optimality, no-envy, peak only*, and *population monotonicity* as well as *Pareto optimality, no-envy, strategy proofness*, and *population monotonicity*(for its proof, please see the Appendix).

Theorem 6 Let $\Omega \in \mathcal{V}$ satisfy *independence of trade-volume* and let $F = f \circ \Omega$. Then,

(i) F satisfies *Pareto optimality, no-envy, peak-only*, and *population monotonicity* if and only if $f = U$ and $\Omega = \Omega_{long}$.

(ii) F satisfies *Pareto optimality, no-envy, strategy-proofness*, and *population monotonicity* if and only if $f = U$ and $\Omega = \Omega_{long}$

Theorem 6 also shows that *independence of trade-volume* is a very demanding property. While a very large subclass of Uniform trade rules satisfies *population monotonicity* together with the other properties, once we have *independence of trade-volume*, only a unique trade rule, $U \circ \Omega_{long}$, satisfies *population monotonicity* together with the same properties.

Finally, we analyze the implications of *one-sided population monotonicity.* The following theorem states that *one-sided population monotonicity* together with *Pareto optimality, no-envy*, and *peak-only* are uniquely satisfied by a subclass of Uniform trade rules each of which is a composition of the Uniform rule and a *peak-only trade-volume*, $\Omega \in \mathcal{V}^{[short,long]}$ (for its proof, please see the Appendix).

Theorem 7 A trade rule $F = f \circ \Omega$ satisfies *Pareto optimality, no-envy, peak-only*, and *one-sided population monotonicity* if and only if $f = U$, $\Omega \in \mathcal{V}^{[short,long]}$ and it satisfies *peak-only.*

Next, we replace *peak-only* in Theorem 7 with *strategy proofness.* We show that a subclass of *Uniform trade rules* each of which is a composition of the Uniform rule and a trade-volume rule, $\Omega \in \mathcal{V}^{[short,long]}$ uniquely satisfies *one-sided population monotonicity* together with *Pareto optimality, no-envy*, and *strategy proofness* (for its proof, please see the Appendix).

Theorem 8 A trade rule $F = f \circ \Omega$ satisfies *Pareto optimality, no-envy, strategy proofness*, and *one-sided population monotonicity* if and only if $f = U$ and $\Omega \in \mathcal{V}^{[short,long]}$ and satisfies (II) and (III) of Theorem 5.

Theorem 7 and 8 show how weak *one-sided population monotonicity* is: there are a huge number of trade rules satisfying it.

References

[1] Benassy, JP. (1982) “The economics of market equilibrium ”, *Academic Press, New York.*

[2] Benassy, JP. (1993) “Nonclearing markets: microeconomic concepts and macroeconomic applications ”, *Journal of Economic Literature*, 31, 732-761.

[3] Benassy, JP. (2002) “The macroeconomics of imperfect competition and nonclearing markets: a dynamic general equilibrium approach ”, *MIT Press.*

[4] Ching, S. (1992) “A simple characterization of the Uniform rule ”, *Economics Letters*, 40, 57-60.

[5] Chun, Y. (1986) “A solidarity axiom for quasi-linear social choice problems ”, *Social Choice and Welfare*, 3, 297-320.

[6] Dagan, N. (1996) “A note on Thomson’s characterizations of the Uniform rule ”, *Journal of Economic Theory*, 69, 255-261.

[7] Klaus, B. (1997) “Fair allocation and reallocation: an axiomatic study ”, *University of Maastricht.*

[8] Kıbrıs, Ö. and Küçükşenel, S. (2009) “Uniform trade rules for uncleared markets”, *Social Choice and Welfare*, 32, 1, 101-121.

[9] Maschler, M. and Owen, G. (1989) “The consistent Shapley value for hyperplane games ”, *International Journal of Game Theory*, 18, 390407.

[10] Peleg, B. (1986) “On the reduced game property and its convers ”, *International Journal of Game Theory*, 15, 187200. “A correction ”, *International Journal of Game Theory*, 16, (1987).

[11] Sprumont, Y. (1991) “The division problem with single-peaked preferences: A characterization of the uniform rule.”, *Econometrica* 59, 509-519.

[12] Tadenuma, K. (1992) "Reduced games, consistency, and the core ", *International Journal of Game Theory*, 20, 325334.

[13] Thomson, W. (1983a) "The fair division of a fixed supply among a growing population ", *Mathematics of Operations Research*, 8, 319326.

[14] Thomson, W. (1983b) "Problems of fair division and the egalitarian solution ", *Journal of Economic Theory*, 31, 211226.

[15] Thomson, W. (1994) "Consistent solutions to the problem of fair division when preferences are single-peaked", *Journal of Economic Theory*, 63, 219245.

[16] Thomson, W. (1995) "Population monotonic solutions to the problem of fair division when preferences are single-peaked", *Economic Theory*, 5, 2, 229-246.

[17] Young, P. (1987) "On dividing an amount according to individual claims or liabilities ", *Mathematics of Operations Research*, 12, 398414.

4 Appendix

Proof. (Lemma 3) Let $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, and $(B' \cup S') \subseteq N$ be such that $B' \neq \emptyset$ and $S' \neq \emptyset$. Let $z \equiv F(R_B, R_S, T)$. First, suppose $h(R_B, R_S, T) = B$. Since F is *Pareto optimal*, z is *Pareto optimal with respect to* (R_B, R_S, T). Then, by Lemma 1 and 2, for each $b \in B$, $p(R_b) \leq z_b$ and for each $s \in S$, $z_s \leq p(R_s)$. Then,

$$\begin{aligned}\sum_{B\setminus B'} z_b + \sum_{B'} p(R_b) &\leq \sum_{B} z_b \\ &= \sum_{S} z_s + T \\ &\leq \sum_{S'} p(R_s) + \sum_{S\setminus S'} z_s + T.\end{aligned}$$

That is $\sum_{B'} p(R_b) \leq \sum_{S'} p(R_s) + T - \sum_{B\setminus B'} z_b + \sum_{S\setminus S'} z_s$. Note that $r^z_{B'\cup S'}(R_B, R_S, T) = (R_{B'}, R_{S'}, T')$ for $T' = T - \sum_{B\setminus B'} z_b + \sum_{S\setminus S'} z_s$. Thus, $h(r^z_{B'\cup S'}(R_B, R_S, T)) = B'$. This proves (i). The proof of (ii) is similar. ■

Proof. (Lemma 4) First, suppose for a contradiction $F = f \circ \Omega$ satisfies *Pareto optimality* whereas f does not. Then, there is $K \in \{B, S\}$ and $(R_K, T) \in \mathcal{M}_{\mathcal{K}}$ such that $f(R_K, T)$ is not *Pareto optimal* with respect to (R_K, T). Then, since $(R_K, T) \in \mathcal{M}$ and $F(R_K, T) = f(R_K, T)$, $F(R_K, T)$ is not *Pareto optimal* with respect to (R_K, T), a contradiction to F being *Pareto optimal*. The other properties can be proved similarly.

■

Proof. (Lemma 6) Let $F = f \circ \Omega$ satisfy *Pareto optimality* and let Ω satisfy *independence of trade volume*. Let $N = (B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^N$ be such that $h(R_B, R_S, T) = B$. **Claim:** $\Omega(R_B, R_S, T) \not< \sum_S p(R_s) + T$. Suppose for a contradiction $\Omega(R_B, R_S, T) < \sum_S p(R_s) + T$. Let $(R'_B, R'_S, T') \in \mathcal{M}^N$ be such that for each $b \in B$, $p(R'_b) = p(R_b)$, for each $s \in S$, $p(R'_s) = \frac{\sum_S p(R_s)+T-\Omega(R_B,R_S,T)}{2|S|}$, and $T' = \frac{\sum_S p(R_s)+T+\Omega(R_B,R_S,T)}{2}$. Note that $\sum_B p(R'_b) = \sum_B p(R_b)$ and $\sum_S p(R'_s) + T' = \sum_S p(R_s) + T$. Then, by *independence of trade volume*, $\Omega(R'_B, R'_S, T') = \Omega(R_B, R_S, T)$. Then, $\sum_S F_s(R'_B, R'_S, T') = \Omega(R_B, R_S, T) - T' = \frac{\Omega(R_B,R_S,T)-\sum_S p(R_s)-T}{2} < 0$, a contradiction to F being a trade rule.

Thus, $\Omega(R_B, R_S, T) \geq \sum_S p(R_s) + T$. By *Pareto optimality*, Lemma 2 implies $\Omega(R_B, R_S, T) = \sum_S p(R_s) + T$.

■

Proof. (Theorem 1) The if part is straightforward and thus, omitted. The only if part is as follows. Since F satisfies *Pareto optimality, no-envy*, and *consistency*, by Lemma 4, f also satisfies those properties. Then, by Lemma 5, $f = U$.

Now, let $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$ and $(B' \cup S') \in \mathcal{N}$ be such that $(B' \cup S') \subseteq (B \cup S)$. Let $z \equiv F(R_B, R_S, T)$ and $z' \equiv F(r^z_{B' \cup S'}(R_B, R_S, T))$. Since F is *consistent*, for each $i \in (B' \cup S')$, $z'_i = z_i$. Then, by the definition of Ω, $\Omega(r^z_{B' \cup S'}(R_B, R_S, T)) = \sum_{B'} z'_{b'} = \sum_{B'} z_{b'}$. Thus, Ω is *consistent*.

First, let $h(R_B, R_S, T) = S$. Suppose for a contradiction $\Omega(R_B, R_S, T) \notin \{\sum_B p(R_b), \sum_S p(R_s) + T\}$. Let $\sum_B p(R_b) = a$, $\sum_S p(R_s) + T = d$, and $\Omega(R_B, R_S, T) = c$. Since F is *Pareto optimal*, by Lemma 1, $c \in (d, a)$. Let $\varepsilon \in \mathbb{R}_+$ be such that $\varepsilon < \min\{\frac{c}{n}, \frac{2(a-c)}{(n-2)}, \frac{2(n-1)(c-d)}{(m-1)(n-2)}\}$. Also let

$m, n \in \mathbb{N}$ be such that $n \geq 3$ and $m > \max\{3, \frac{c-T}{d-T}\}$.

Let $(R_{B'}, R_{S'}, T) \in \mathcal{M}^{B' \cup S'}$ be such that

$p(R_{b'_1}) = \frac{c}{n} - \varepsilon$, $p(R_{b'_2}) = \cdots = p(R_{b'_n}) = \frac{a}{n-1} - \frac{c}{n(n-1)} + \frac{\varepsilon}{n-1}$,

$p(R_{s'_1}) = \frac{c}{m} - \frac{T}{m} + \frac{\varepsilon(m-1)(n-2)}{2(m-2)(n-1)}$, $p(R_{s'_2}) = \frac{d}{m-1} - \frac{T}{m} - \frac{c}{m(m-1)} + \frac{(n-2)(m-3)\varepsilon}{2(n-1)(m-2)}$,

$p(R_{s'_3}) = \cdots = p(R_{s'_m}) = \frac{d}{m-1} - \frac{T}{m} - \frac{c}{m(m-1)} - \frac{(n-2)\varepsilon}{(n-1)(m-2)}$,

and let $(R'_{B'}, R'_{S'}, T) \in \mathcal{M}^{B' \cup S'}$ be such that

$p(R'_{b'_1}) = \frac{c}{n} - \frac{\varepsilon}{2}$, $p(R'_{b'_2}) = \frac{a}{n-1} - \frac{c}{n(n-1)} - \frac{(n-3)\varepsilon}{2(n-1)}$, $p(R'_{b'_3}) = \cdots = p(R'_{b'_n}) = \frac{a}{n-1} - \frac{c}{n(n-1)} + \frac{\varepsilon}{n-1}$,

$p(R'_{s'_1}) = \frac{c}{m} - \frac{T}{m} + \frac{\varepsilon(m-1)(n-2)}{(m-2)(n-1)}$, $p(R'_{s'_2}) = \cdots = p(R'_{s'_m}) = \frac{d}{m-1} - \frac{T}{m} - \frac{c}{m(m-1)} - \frac{(n-2)\varepsilon}{(n-1)(m-2)}$.

Note that by the choice of ε and m, for each $k \in (B' \cup S')$, $p(R_{k'}) \geq 0$ and $p(R'_{k'}) \geq 0$. Also, $\sum_{B'} p(R_{b'}) = \sum_{B'} p(R'_{b'}) = a$ and $\sum_{S'} p(R_{s'}) = \sum_{S'} p(R'_{s'}) = d - T$. Then, by *independence of trade volume*, $\Omega(R_{B'}, R_{S'}, T) = \Omega(R'_{B'}, R'_{S'}, T) = c$.

For each $K \in \{B', S'\}$, let $z_K \equiv F_K(R_{B'}, R_{S'}, T) = U(R_K, c)$ and $z'_K \equiv F_K(R'_{B'}, R'_{S'}, T) = U(R'_K, c)$. Since for each $i = 2, \cdots, n$, $p(R_{b'_1}) < \frac{c}{n} < p(R_{b'_i})$, $p(R'_{b'_1}) < \frac{c}{n} < p(R'_{b'_i})$, and $\frac{1}{(n-1)}(c - p(R'_{b'_1})) < p(R'_{b'_i})$, we have $z_{b'_1} = p(R_{b'_1}) = \frac{c}{n} - \varepsilon$, $z_{b'_i} = \frac{1}{n-1}(c - p(R_{b'_1})) = \frac{c}{n} + \frac{\varepsilon}{n-1}$, $z'_{b'_1} = p(R'_{b'_1}) = \frac{c}{n} - \frac{\varepsilon}{2}$, and $z'_{b'_i} = \frac{1}{n-1}(c - p(R'_{b'_1})) = \frac{c}{n} + \frac{\varepsilon}{2(n-1)}$.

Since for each $i = 2, \cdots, m$, $p(R_{s'_i}) < \frac{c-T}{m} < p(R_{b'_1})$, $p(R'_{s'_i}) < \frac{c-T}{m} < p(R'_{s'_1})$, and $\frac{1}{(m-1)}(c - T - p(R_{s'_1})) > p(R'_{s'_i})$, we have $z_{s'_1} = p(R_{s'_1}) = \frac{c}{m} - \frac{T}{m} + \frac{\varepsilon(m-1)(n-2)}{2(m-2)(n-1)}$, $z_{s'_i} = \frac{1}{m-1}(c - T - p(R_{s'_1})) = \frac{c}{m} + \frac{T}{m} - \frac{\varepsilon(n-2)}{2(m-2)(n-1)}$, $z'_{s'_1} = p(R'_{s'_1}) = \frac{c}{m} - \frac{T}{m} + \frac{\varepsilon(m-1)(n-2)}{(m-2)(n-1)}$, and $z'_{s'_i} = \frac{1}{m-1}(c - T - p(R'_{s'_1})) = \frac{c}{m} + \frac{T}{m} - \frac{\varepsilon(n-2)}{(m-2)(n-1)}$.

Now, let $T' = \frac{2T}{m} - \frac{2(m-n)c}{mn} - \frac{3(n-2)\varepsilon}{2(n-1)}$ and consider the following two reduced problems:

(i) $r^z_{\{b'_1, b'_2, s'_1, s'_2\}}(R_{B'}, R_{S'}, T) = (R_{b'_1}, R_{b'_2}, R_{s'_1}, R_{s'_2}, T')$,

(ii) $r^{z'}_{\{b'_1, b'_2, s'_1, s'_2\}}(R'_{B'}, R'_{S'}, T) = (R'_{b'_1}, R'_{b'_2}, R'_{s'_1}, R'_{s'_2}, T')$.

Note that, $p(R_{b'_1}) + p(R_{b'_2}) = p(R'_{b'_1}) + p(R'_{b'_2})$ and $p(R_{s'_1}) + p(R_{s'_2}) = p(R'_{s'_1}) + p(R'_{s'_2})$. Then, by *independence of trade volume*, $\Omega(r^z_{\{b'_1, b'_2, s'_1, s'_2\}}(R_{B'}, R_{S'}, T)) = \Omega(r^{z'}_{\{b'_1, b'_2, s'_1, s'_2\}}(R'_{B'}, R'_{S'}, T))$. By *consistency*, for $i = 1, 2$, $F_{b'_i}(r^z_{b'_1, b'_2, s'_1, s'_2}(R_{B'}, R_{S'}, T)) = z_{b'_i}$ and $F_{b'_i}(r^{z'}_{b'_1, b'_2, s'_1, s'_2}(R'_{B'}, R'_{S'}, T)) = z'_{b'_i}$. Then,

$\Omega(r^z_{\{b'_1, b'_2, s'_1, s'_2\}}(R_{B'}, R_{S'}, T)) = z_{b'_1} + z_{b'_2} = \frac{2c}{n} + \frac{(2-n)\varepsilon}{n-1}$ and

$\Omega(r^{z'}_{\{b'_1, b'_2, s'_1, s'_2\}}(R'_{B'}, R'_{S'}, T)) = z'_{b'_1} + z'_{b'_2} = \frac{2c}{n} + \frac{(2-n)\varepsilon}{2(n-1)}$.

Then, $\Omega(r^z_{\{b'_1, b'_2, s'_1, s'_m\}}(R_{B'}, R_{S'}, T)) \neq \Omega(r^{z'}_{\{b'_1, b'_2, s'_1, s'_2\}}(R'_{B'}, R'_{S'}, T))$, a contradiction. Thus, $\Omega(R_B, R_S, T) \in \{\sum_B p(b), \sum_S p(s) + T\}$. ■

Proof. (Theorem 2) The only if part is as follows. Since F satisfies *Pareto optimality, no-envy*, and *peak-only*, Lemma 3 implies that f also satisfies those properties. Then, by Lemma 8, $f = U$. Since F is *Pareto optimal*, by Lemma 1, for each $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^{B \cup S}$, $\Omega(R_B, R_S, T) \in [\sum_B p(R_b), \sum_S p(R_s) + T]$. Since F is *peak-only*, Ω is *peak-only*. The proof of if part is as follows. It is easy to show that $F = U \circ \Omega$, where Ω satisfies the given properties, satisfies *Pareto optimality, no-envy*, and *peak-only*. To show that it also satisfies *one-sided population monotonicity*, let $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^{B \cup S}$. Without loss of generality, let $h(R_B, R_S, T) = B$. Let $(B' \cup S') \subseteq (B \cup S)$ be such that $h(R_{B'}, R_{S'}, T) = B'$. Let $z \equiv U \circ \Omega(R_B, R_S, T)$ and $z' \equiv U \circ \Omega(R_{B'}, R_{S'}, T)$. By the conditions on Ω, $\Omega(R_B, R_S, T) \in [\sum_B p(R_b), \sum_S p(R_s) + T]$ and $\Omega(R_{B'}, R_{S'}, T) \in [\sum_{B'} p(R_{b'}), \sum_{S'} p(R_{s'}) + T]$. Then, by definition, for each $b \in B$ and $s \in S$, $z_b = \max\{\lambda, p(R_b)\}$ and $z_s = \min\{\lambda, p(R_s)\}$ where λ satisfies $\sum_B z_b = \Omega(R_B, R_S, T)$. Also, for each $b' \in B'$ and $s' \in S'$, $z_{b'} = \max\{\lambda', p(R_{b'})\}$ and $z_{s'} = \min\{\lambda', p(R_{s'})\}$ where λ' satisfies $\sum_{B'} z'_{b'} = \Omega(R_{B'}, R_{S'}, T)$.

Claim: There are no $b', b'' \in B'$ such that $z_{b'} < z'_{b'}$ and $z_{b''} > z'_{b''}$. Similarly, there are no $s', s'' \in S''$ such that $z_{s'} < z'_{s'}$ and $z_{s''} > z'_{s''}$.

Proof of Claim: Suppose for a contradiction there are $b', b'' \in B'$ such that $z_{b'} < z'_{b'}$ and $z_{b''} > z'_{b''}$. Then, since $z_{b'} < z'_{b'}$, by definition, $\lambda' > \lambda$. But then, for each $b \in B'$, $z_b \leq z'_b$, a contradiction to $z_{b''} > z'_{b''}$. The other case is similar.

Now, let $K \in \{B', S'\}$. By Claim, we have either (i) for each $k \in K$, $z_k \leq z'_k$ or (ii) for each $k \in K$, $z_k \geq z'_k$. Also, by *Pareto optimality*, we have either (i) for each $k \in K$, $z_k \leq p(R_k)$ and $z'_k \leq p(R_k)$ or (ii) for each $k \in K$, $z_k \geq p(R_k)$ and $z'_k \geq p(R_k)$. Thus, we have either (i) for each $k \in K$, $z_k \; R_k \; z'_k$ or (ii) for each $k \in K$, $z'_k \; R_k \; z_k$. Therefore, $U \circ \Omega$ satisfies *one-sided population monotonicity*.

■

Proof. (Lemma 8) Let $F = U \circ \Omega$ satisfy *Pareto optimality* and *strategy proofness*. Let $N = (B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^N$. Without loss of generality, let $h(R_B, R_S, T) = B$ and $b_i \in B$ be such that $F_{b_i}(R_B, R_S, T) \neq p(R_{b_i})$. Let $R'_{b_i} \in \mathcal{R} \setminus \{R_{b_i}\}$ be such that $p(R_{b_i}) = p(R'_{b_i})$. Let $z \equiv F(R_B, R_S, T)$ and $z' \equiv F(R'_{b_i}, R_{B \setminus b_i}, R_S, T)$.

Claim 1. $(z'_{b_i} = z_{b_i})$ Suppose not. Since $h(R_B, R_S, T) = h(R'_{b_i}, R_{B \setminus b_i}, R_S, T) = B$ and

$z_{b_i} \neq p(R_{b_i})$, *Pareto optimality* implies $z_{b_i} > p(R_{b_i})$ and $z'_{b_i} \geq p(R_{b_i})$. Note that $z'_{b_i} \not> z_{b_i}$, because otherwise when the preference relation of b_i is R'_{b_i}, he pretends as if he has R_{b_i} as the preference relation, a contradiction to *strategy proofness*. Similarly, $z'_{b_i} \not< z_{b_i}$. Thus, $z'_{b_i} = z_{b_i}$.

Claim 2. $(\Omega(R_B, R_S, T) = \Omega(R'_{b_i}, R_{B\setminus b_i}, R_S, T))$ Since $F = U \circ \Omega$, $h(R_B, R_S, T) = B$, and $z_{b_i} \neq p(R_{b_i})$, $z_{b_i} = \max\{\lambda, p(R_{b_i})\} = \lambda$ where λ satisfies $\sum_B \max\{\lambda, p(R_b)\} = \Omega(R_B, R_S, T)$. Similarly, $z'_{b_i} = \max\{\lambda', p(R_{b_i})\}$ where λ' satisfies $\sum_B \max\{\lambda', p(R_b)\} = \Omega(R'_{b_i}, R_{B\setminus b_i}, R_S, T)$. By Claim 1, $z'_{b_i} = \lambda' = \lambda$. Thus, $\Omega(R_B, R_S, T) = \Omega(R'_{b_i}, R_{B\setminus b_i}, R_S, T)$.

Claim 3. (for each $b \in B\setminus\{b_i\}$ and $s \in S$, $z'_b = z_{b_i}$ and $z'_s = z_s$) It follows from $F = U \circ \Omega$ and Claim 2.

By Claim 1 and 3, $z' = z$, that is F is *restricted peak-only*.

■

Proof. (Theorem 3) The proof of the only if part is as follows. Since F satisfies *Pareto optimality, no-envy*, and *strategy proofness*, Lemma 3 implies that f also satisfies those properties. Then, by Lemma 8, $f = U$. Now, let $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^{B\cup S}$. Since F is *Pareto optimal*, by Lemma 1, $\Omega(R_B, R_S, T) \in [\sum_B p(R_b), \sum_S p(R_s) + T]$. To prove property (i), suppose $h(R_B, R_S, T) = B$ and let $b_i \in B$ be such that $z_{b_i} > p(R_{b_i})$ and $R'_{b_i} \in \mathcal{R}$ be such that $p(R'_{b_i}) < z_{b_i}$. Let $z' \equiv F_{b_i}(R'_{b_i}, R_{B\setminus b_i}, R_S, T)$. Suppose $z'_{b_i} < p(R_{b_i})$. Then, consider $R''_{b_i} \in \mathcal{R}$ such that $p(R''_{b_i}) = p(R_{b_i})$ and $z'_{b_i}\ P_{b_i}\ z_{b_i}$. By Lemma 10, $F_{b_i}(R''_{b_i}, R_{B\setminus b_i}, R_S, T) = z_{b_i}$. Then, when the preference relation of b_i is R''_{b_i}, he pretends as if he has R'_{b_i} as the preference relation, a contradiction to *strategy proofness*. Now, suppose $p(R_{b_i}) \leq z'_{b_i} < z_{b_i}$. Then, when the preference relation of b_i is R_{b_i}, he pretends as if he has R'_{b_i} as the preference relation, a contradiction to *strategy proofness*. If $z'_{b_i} > z_{b_i}$, then when the preference relation of b_i is R'_{b_i}, he pretends as if he has R_{b_i} as the preference relation, a contradiction to *strategy proofness*. Thus, $z'_{b_i} = z_{b_i}$. The proofs of the other case of (i) and (ii) are very similar.

The proof of the if part is as follows. The proof of *Pareto optimality* and *no-envy* is trivial. The proof of *one-sided population monotonicity* is the same as the proof of Theorem 7. To prove *strategy proofness*, let $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^N$. Without loss of generality, let $h(R_B, R_S, T) = B$ and $b_i \in B$. Let $R'_{b_i} \in \mathcal{R} \setminus \{R_{b_i}\}$, $z \equiv$

$F(R_B, R_S, T)$, and $z' \equiv F(R'_{b_i}, R_{B\setminus b_i}, R_S, T)$. Since z is *Pareto optimal* with respect to (R_B, R_S, T), $z_{b_i} \geq p(R_{b_i})$. If $z_{b_i} = p(R_{b_i})$, then trivially there is no profitable deviation for b_i. Thus, let $z_{b_i} > p(R_{b_i})$. If $p(R'_{b_i}) < z_{b_i}$, then (i) implies $z'_{b_i} = z_{b_i}$. Now, let $p(R'_{b_i}) \geq z_{b_i}$ and $h(R'_{b_i}, R_{B\setminus b_i}, R_S, T) = B$. Since F is *Pareto optimal*, Lemma 1 implies $z'_{b_i} \geq p(R'_{b_i}) \geq z_{b_i}$. Thus, R'_{b_i} is not a profitable deviation for b_i. Now, let $p(R'_{b_i}) \geq z_{b_i}$ and $h(R'_{b_i}, R_{B\setminus b_i}, R_S, T) = S$. Note that, $z_{b_i} = \max\{\lambda, p(R_{b_i})\} = \lambda$ where λ satisfies $\Sigma_B \max\{\lambda, p(R_b)\} = \Omega(R_B, R_S, T)$. Also, $z'_{b_i} = \min\{\lambda', p(R'_{b_i})\}$ where λ' satisfies $\min\{\lambda', p(R'_{b_i})\} + \Sigma_{B\setminus b_i} \min\{\lambda', p(R_{b_i})\} = \Omega(R'_{b_i}, R_{B\setminus b_i}, R_S, T)$. If $z'_{b_i} < z_{b_i}$, then $z'_{b_i} = \lambda'$ and so $\lambda' < \lambda$. Then, for each $b \in B$, $z'_b \leq z_b$. Thus, $\Omega(R'_{b_i}, R_{B\setminus b_i}, R_S, T) < \Omega(R_B, R_S, T)$. But, we have also $\sum_B p(R_b) \leq \Omega(R_B, R_S, T) \leq \sum_S p(R_s) + T$ and $\sum_S p(R_s) + T \leq \Omega(R'_{b_i}, R_{B\setminus b_i}, R_S, T) \leq p(R'_{b_i}) + \sum_{B\setminus b_i} p(R_b)$. Thus, $\Omega(R_B, R_S, T) \leq \Omega(R'_{b_i}, R_{B\setminus b_i}, R_S, T)$, a contradiction. Thus, $z'_{b_i} \geq z_{b_i}$ and so R'_{b_i} is not a profitable deviation for b_i. Therefore, F is *strategy proof.*

■

Proof. (Lemma 9) Let $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^{B\cup S}$. Without loss of generality, let $\sum_B p(R_b) \leq \sum_S p(R_s) + T$. Then, $\Omega_{long}(R_B, R_S, T) = \sum_S p(R_s) + T$. Let $z \equiv F(R_B, R_S, T)$. Then, for each $b \in B$, $z_b \geq p(R_b)$ and for each $s \in S$, $z_s = p(R_s)$. Let $(B' \cup S') \subseteq (B \cup S)$. Suppose first, $\sum_{B'} p(R_{b'}) \leq \sum_{S'} p(R_{s'}) + T$. Then, $\Omega_{long}(R_{B'}, R_{S'}, T) = \sum_{S'} p(R_{s'}) + T$. Let $z' \equiv F(R_{B'}, R_{S'}, T)$. Then, for each $s' \in S'$, $z'_{s'} = p(R_{s'})$. Thus, for each $s' \in S'$, $z'_{s'}\ I_{s'}\ z_{s'}$.

Claim: We have either (i) for each $b' \in B'$, $z'_{b'} \leq z_{b'}$ or (ii) for each $b' \in B'$, $z'_{b'} \geq z_{b'}$.

Proof of Claim: Suppose for a contradiction there are $\tilde{b}, \check{b} \in B'$ such that $z'_{\tilde{b}} > z_{\tilde{b}}$ and $z'_{\check{b}} < z_{\check{b}}$. By definition, $z'_{\tilde{b}} = \max\{\lambda', p(R_{\tilde{b}})\}$ where λ' is such that $\sum_{B'} \max\{\lambda', p(R_{b'})\} = \sum_{B'} p(R_{b'})$ and $z_{\tilde{b}} = \max\{\lambda, p(R_{\tilde{b}})\}$ where λ is such that $\sum_B \max\{\lambda, p(R_b)\} = \sum_B p(R_b)$. Since $z'_{\tilde{b}} > z_{\tilde{b}}$, $\lambda' > \lambda$. Then, $z'_{\check{b}} = \max\{\lambda', p(R_{\check{b}})\} \geq \max\{\lambda, p(R_{\check{b}})\} = z_{\check{b}}$, a contradiction to $z'_{\check{b}} < z_{\check{b}}$.

Thus, we have either (i) for each $b' \in B'$, $z'_{b'}\ R_{b'}\ z_{b'}$ or (ii) for each $b' \in B'$, $z_{b'}\ R_{b'}\ z'_{b'}$.

Now, suppose $\sum_{S'} p(R_{s'}) + T \leq \sum_{B'} p(R_{b'})$. Then, $\Omega_{long}(R_{B'}, R_{S'}, T) = \sum_{B'} p(R_{b'})$. Then, for each $b' \in B'$, $z'_{b'} = p(R_{b'})$. Thus, for each $b' \in B'$, $z'_{b'}\ R_{b'}\ z_{b'}$. Also, for each $s' \in S'$, $z'_{s'} \geq p(R_{s'})$, that is $z_{s'}\ R_{s'}\ z'_{s'}$. Therefore, $F = U \circ \Omega_{long}$ is *population monotonic.*

■

Proof. (Theorem 4) The if part is easy to prove. The only if part is as follows. Since F satisfies *Pareto optimality, no-envy*, and *peak-only*, Lemma 3 implies that f also satisfies those properties. Then, by Lemma 8, $f = U$. To prove that $\Omega = \Omega_{long}$, let $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^{B \cup S}$. First suppose $\sum_B \leq \sum_S + T$. Then, by Lemma 6, $\Omega(R_B, R_S, T) = \sum_S p(R_s) + T$. Now, let $\sum_S p(R_s) + T \leq \sum_B p(R_b)$. Suppose for a contradiction $\Omega(R_B, R_S, T) < \sum_B p(R_b)$. Let $|B| = n$ and $|S| = m$. Also, for simplicity, let $\sum_B p(R_b) = a$, $\sum_S p(R_s) + T = d$, and $\Omega(R_B, R_S, T) = c$. Consider $(R_{B'}, R_{S'}, T) \in \mathcal{M}^{B \cup S}$ such that $B' = \{b'_1, b'_2, \cdots, b'_n\}$, $S' = \{s'_1, s'_2, \cdots, s'_m\}$, and $p(b'_1) = \cdots = p(b'_{n-1}) = \frac{c}{n} - \varepsilon$, $p(b'_n) = a - \frac{n-1}{n}c + (n-1)\varepsilon$, $p(s'_1) = \cdots = p(s'_m) = \frac{d}{m}$, and $T' = 0$. Note that $\sum_{B'} p(b') = a$ and $\sum_{S'} p(s') + T' = d$. Then, by *independence of trade volume*, $\Omega(R_{B'}, R_{S'}, T') = c$. Let $z' \equiv F(R_{B'}, R_{S'}, T')$. Then, by the definition of U, for each $b'_i \in \{b'_1, \cdots, b'_{n-1}\}$, $z'_{b'_i} = p(b'_i)$ and $z'_{b'_n} = c - (n-1)z_{b'_1} < p(b'_n)$. Now, let $S'' = S' \cup \{s'_{m+1}\}$. Consider $(R_{B'}, R_{S''}, T'') \in \mathcal{M}^{B' \cup S''}$ such that $p(s'_{m+1}) = a - d + \varepsilon$ and $T'' = 0$. Then, $\sum_{B'} p(b') = a < \sum_{S''} p(s'') + T'' = a + \varepsilon$. Then, $\Omega(R_{B'}, R_{S''}, T'') = a + \varepsilon$. Let $z'' \equiv F(R_{B'}, R_{S''}, T'')$. Then, for each $b'_i \in \{b'_1, \cdots, b'_{n-1}\}$, $z''_{b'_i} = \frac{a+\varepsilon-p(b'_n)}{n-1} > p(b'_i)$ and $z''_{b'_n} = p(b'_n)$. Thus, for each b'_i for $i \in \{1, \cdots, n-1\}$, $z'_{b'_i} \; P_{b'_i} \; z''_{b'_i}$ and $z''_{b'_n} \; P_{b'_n} \; z'_{b'_n}$, a contradiction to F satisfying *population monotonicity.*

■

Proof. (Theorem 5) The if part is easy to prove. The only if part is as follows. Since F satisfies *Pareto optimality, no-envy*, and *strategy proofness*, Lemma 3 implies that f also satisfies those properties. Then, by Lemma 8, $f = U$. The proof of $\Omega = \Omega_{long}$ is the same as the proof of Theorem 4.

■

Proof. (Theorem 6) The if part is easy to prove. The only if part is as follows. Since F satisfies *Pareto optimality, no-envy*, and *peak-only*, Lemma 3 implies that f also satisfies those properties. Then, by Lemma 8, $f = U$. Since F is *peak-only*, Ω is also *peak-only.* To prove that $\Omega = \Omega_{long,short}$ and satisfies (i) and (ii), let $(B \cup S) \in \mathcal{N}$ be such that $|B| = n$ and $|S| = m$, $(R_B, R_S, T) \in \mathcal{M}^{B \cup S}$ be such that there are $b_i, b_j \in B$ and $s_k, s_l \in S$ with $p(R_{b_i}) \neq p(R_{b_j})$ and $p(R_{s_k}) \neq p(R_{s_l})$. First, suppose $h(R_B, R_S, T) = S$. Suppose for a contradiction, $\Omega(R_B, R_S, T) \in (\sum_S p(R_s) + T, \sum_B p(R_b))$. Without loss of generality, let

$B = \{b_1, \cdots, b_n\}$, $S = \{s_1, \cdots, s_m\}$, and $b_1 \leq b_2 \leq \cdots \leq b_n$ and $s_1 \leq s_2 \leq \cdots \leq s_m$. Let $z \equiv F(R_B, R_S, T)$. Then, by definition of Uniform rule, $z_{b_1} \leq p(b_1)$, $z_{b_n} < p(b_n)$, $z_{s_1} > p(s_1)$, and $z_{s_m} \geq p(s_m)$. Now, let $S' = S \cup \{s_{m+1}, \cdots, s_{m+l}\}$ such that $p(s_{m+1}) = \cdots = p(s_{m+l})$, $p(s_1) < p(s_{m+1}) < p(s_m)$, and l is chosen such that $h(R_B, R_{S'}, T) = B$. Let $z' \equiv F(R_B, R_{S'}, T)$. First, suppose $\Omega(R_B, R_{S'}, T) = \sum_{S'} p(R_{s'}) + T$. Then, by definition of Uniform rule, we have the following cases:

Case 1: Let $z_{b_1} = p(b_1)$, $z_{b_n} < p(b_n)$, and $z'_{b_1} = z'_{b_n} > p(b_n)$. Then, consider R'_{b_n} such that $p(R'_{b_n}) = p(b_n)$ and $z'_{b_n}\ P'_{b_n}\ z_{b_n}$. By *peak-only*, $F(R_{B\setminus\{b_n\}}, R'_{b_n}, R_{S'}, T) = z'$ and $F(R_{B\setminus\{b_n\}}, R'_{b_n}, R_S, T) = z$. Then, we have $z_{b_1}\ P_{b_1}\ z'_{b_1}$ and $z'_{b_n}\ P'_{b_n}\ z_{b_n}$, a contradiction to F satisfying *population monotonicity.*

Case 2: Let $z_{b_1} = p(b_1)$, $z_{b_n} < p(b_n)$, $z'_{b_1} > p(b_1)$, and $z'_{b_n} = p(b_n)$. Then, we have $z_{b_1}\ P_{b_1}\ z'_{b_1}$ and $z'_{b_n}\ P_{b_n}\ z_{b_n}$, a contradiction to F satisfying *population monotonicity.*

Case 3: Let $z_{b_1} = z_{b_n} < p(b_1)$, and $z'_{b_1} = z'_{b_n} > p(b_n)$. Then, consider R'_{b_1} and R'_{b_n} such that $p(R'_{b_1}) = p(b_1)$, $p(R'_{b_n}) = p(b_n)$, $z_{b_1}\ P'_{b_1}\ z'_{b_1}$, and $z'_{b_n}\ P'_{b_n}\ z_{b_n}$. By *peak-only,* $F(R_{B\setminus\{b_1,b_n\}}, R'_{b_1}, R'_{b_n}, R_{S'}, T) = z'$ and $F(R_{B\setminus\{b_1,b_n\}}, R'_{b_1}, R'_{b_n}, R_S, T) = z$. Then, we have $z_{b_1}\ P'_{b_1}\ z'_{b_1}$ and $z'_{b_n}\ P'_{b_n}\ z_{b_n}$, a contradiction to F satisfying *population monotonicity.*

Case 4: Let $z_{b_1} = z_{b_n} < p(b_1)$, $z'_{b_1} > p(b_1)$, and $z'_{b_n} = p(b_n)$. Then, consider R'_{b_1} such that $p(R'_{b_1}) = p(b_1)$ and $z_{b_1}\ P'_{b_1}\ z'_{b_1}$. By *peak-only,* $F(R_{B\setminus\{b_1\}}, R'_{b_1}, R_{S'}, T) = z'$ and $F(R_{B\setminus\{b_1\}}, R'_{b_1}, R_S, T) = z$. Then, we have $z_{b_1}\ P'_{b_1}\ z'_{b_1}$ and $z'_{b_n}\ P_{b_n}\ z_{b_n}$, a contradiction to F satisfying *population monotonicity.*

Second, suppose $\Omega(R_B, R_S, T) < \sum_{S'} p(R_{s'}) + T$. Then, similar argument proves that in each case there is a violation of *population monotonicity.* Thus, by *Pareto optimality*, $\Omega(R_B, R_S, T) \in \{\sum_B p(R_b), \sum_S p(R_s) + T\}$. Similar argument proves the other case in which $h(R_B, R_S, T) = B$, just replace S with B. Thus, $\Omega = \Omega_{long,short}$.

To prove(i), suppose $h(R_B, R_S, T) = B$ and $\Omega(R_B, R_S, T) = \sum_B p(R_b)$. Let $(B' \cup S') \subseteq (B \cup S)$. Suppose $h(R_{B'}, R_{S'}, T) = S'$ and there is $s'_t \in S'$ such that $z_{s'_t} \neq p(R_{s'_t})$. Suppose for a contradiction, $\Omega(R_{B'}, R_{S'}, T) = \sum_{B'} p(R_{b'})$. Without loss of generality, let $S' = \{s'_1, \cdots, s'_r\}$ and $s'_1 \leq \cdots \leq s'_r$. Let $z' = F(R_{B'}, R_{S'}, T)$. Since $z_{s'_t} \neq p(R_{s'_t})$, $z_{s'_r} < p(s'_r)$. Then we have the following cases:

Case 1: Let $z_{s'_1} = p(s'_1)$, $z_{s'_r} < p(s'_r)$, $z'_{s'_1} > p(s'_1)$, and $z'_{s'_r} = p(s'_r)$. Then, $z_{s'_1}\ P_{s'_1}\ z'_{s'_1}$ and

$z'_{s'_r}\ P_{s'_r}\ z_{s'_r}$, a contradiction to F satisfying *population monotonicity.*

Case 2: Let $z_{s'_1} = p(s'_1)$, $z_{s'_r} < p(s'_r)$, and $z'_{s'_1} = z'_{s'_r} > p(s'_r)$. Then, consider $R'_{s'_r}$ such that $p(R'_{s'_r}) = p(s'_r)$ and $z'_{s'_r}\ P'_{s'_r}\ z_{s'_r}$. By *peak-only,* $F(R_B, R_{S\setminus\{s'_r\}}, R'_{s'_r}, T) = z'$ and $F(R_B, R_{S\setminus\{s'_r\}}, R'_{s'_r}, T) = z$. Then, we have $z_{s'_1}\ P_{s'_1}\ z'_{s'_1}$ and $z'_{s'_r}\ P'_{s'_r}\ z_{s'_r}$, a contradiction to F satisfying *population monotonicity.*

Case 3: Let $z_{s'_1} = z_{s'_r} < p(s'_1)$, and $z'_{s'_1} > p(s'_1)$, and $z'_{s'_r} = p(s'_r)$. Then, consider $R'_{s'_1}$ such that $p(R'_{s'_1}) = p(s'_1)$ and $z_{s'_1}\ P'_{s'_1}\ z'_{s'_1}$. By *peak-only,* $F(R_B, R_{S\setminus\{s'_1\}}, R'_{s'_1}, T) = z'$ and $F(R_B, R_{S\setminus\{s'_1\}}, R'_{s'_1}, T) = z$. Then, we have $z_{s'_1}\ P'_{s'_1}\ z'_{s'_1}$ and $z'_{s'_r}\ P_{s'_r}\ z_{s'_r}$, a contradiction to F satisfying *population monotonicity.*

Case 4: Let $z_{s'_1} = z_{s'_r} < p(s'_1)$, and $z'_{s'_1} = z'_{s'_r} > p(s'_r)$. Then, consider $R'_{s'_1}$ and $R'_{s'_r}$ such that $p(R'_{s'_1}) = p(s'_1)$, $p(R'_{s'_r}) = p(s'_r)$, $z_{s'_1}\ P'_{s'_1}\ z'_{s'_1}$, and $z'_{s'_r}\ P'_{s'_r}\ z_{s'_r}$. By *peak-only,* $F(R_B, R_{S\setminus\{s'_1,s'_r\}}, R'_{s'_1}, R'_{s'_r}, T) = z'$ and $F(R_B, R_{S\setminus\{s'_1,s'_r\}}, R'_{s'_1}, R'_{s'_r}, T) = z$. Then, we have $z_{s'_1}\ P'_{s'_1}\ z'_{s'_1}$ and $z'_{s'_r}\ P'_{s'_r}\ z_{s'_r}$, a contradiction to F satisfying *population monotonicity.*

A similar argument proves the other cases. Thus, (i) and (ii) also hold.

■

Proof. (Theorem 8) Let F be a trade rule that satisfies *Pareto optimality, no-envy,* and *peak-only.* Lemma 3 implies that f also satisfies those properties. Then, by Lemma 8, $f = U$. Now, we will show that there is $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^N$ such that independent of $\Omega(R_B, R_S, T)$, $U \circ \Omega$ does not satisfy *strong population monotonicity.* Let $B = \{b_1, b_2, b_3\}$ and $S = \{s_1, s_2, s_3\}$. Let $p(R_{b_1}) = 1$, $p(R_{b_2}) = 2$, $p(R_{b_3}) = 6$, and $p(R_{s_1}) = 1$, $p(R_{s_2}) = p(R_{s_2}) = 5$. Let $B' = \{b_1, b_3\}$ and $S' = \{s_1, s_2\}$. Let $z \equiv F(R_B, R_S, T)$ and $z' \equiv F(R_{B'}, R_{S'}, T)$. Note that $6 \le \Omega(R_{B'}, R_{S'}, T) \le 7$. Then, if $\Omega(R_{B'}, R_{S'}, T) = 6$, then $z'_{b_1} = 1$, $z'_{b_3} = 5$, $z'_{s_1} = 1$, and $z'_{s_2} = 5$. If $6 < \Omega(R_{B'}, R_{S'}, T) < 7$, then $z'_{b_1} = 1$, $z'_{b_3} < 6$, $z'_{s_1} > 1$, and $z'_{s_2} = 5$. If $\Omega(R_{B'}, R_{S'}, T) = 7$, then $z'_{b_1} = 1$, $z'_{b_3} = 6$, $z'_{s_1} = 2$, and $z'_{s_2} = 5$. Thus, we have the following claims:

Claim 1. $(\Omega(R_B, R_S, T) \neq 9)$ Suppose $\Omega(R_B, R_S, T) = 9$. Then, $z_{b_1} = 1$, $z_{b_2} = 2$, $z_{b_3} = 6$, $z_{s_1} = 1$, and $z_{s_2} = z_{s_3} = 4$. If $\Omega(R_{B'}, R_{S'}, T) = 6$, then $z_{b_3}\ P_{b_3}\ z'_{b_3}$ and $z'_{s_2}\ P_{s_2}\ z_{s_2}$, a contradiction to *strong population monotonicity.* If $6 < \Omega(R_{B'}, R_{S'}, T) < 7$, then $z_{s_1}\ P_{s_1}\ z'_{s_1}$ and $z'_{s_2}\ P_{s_2}\ z_{s_2}$, a contradiction to *strong population monotonicity.* If $\Omega(R_{B'}, R_{S'}, T) = 7$, then $z_{s_1}\ P_{s_1}\ z'_{s_1}$ and $z'_{s_2}\ P_{s_2}\ z_{s_2}$, a contradiction to *strong population monotonicity.* Thus,

$\Omega(R_B, R_S, T) \neq 9$.

Claim 2. ($\Omega(R_B, R_S, T) \notin (9, 11]$) Suppose $\Omega(R_B, R_S, T) \in (9, 11)$. Then, $z_{b_1} > 1$, $z_{b_3} = 6$, and $z_{s_1} = 1$. If $\Omega(R_{B'}, R_{S'}, T) = 6$, then $z'_{b_1}\ P_{b_1}\ z_{b_1}$ and $z_{b_3}\ P_{b_3}\ z'_{b_3}$, a contradiction to *strong population monotonicity.* If $6 < \Omega(R_{B'}, R_{S'}, T) < 7$, then $z'_{b_1}\ P_{b_1}\ z_{b_1}$ and $z_{b_3}\ P_{b_3}\ z'_{b_3}$, a contradiction to *strong population monotonicity.* If $\Omega(R_{B'}, R_{S'}, T) = 7$, then $z'_{b_1}\ P_{b_1}\ z_{b_1}$ and $z_{s_1}\ P_{s_1}\ z'_{s_1}$, a contradiction to *strong population monotonicity.* Thus, $\Omega(R_B, R_S, T) \notin (9, 11]$.

Thus, $\Omega(R_B, R_S, T) \notin [9, 11]$. However, since F is *Pareto optimal*, Lemma 1 implies, $\Omega(R_B, R_S, T) \in [9, 11]$, a contradiction. ■

Proof. (Theorem 9) Let F be a trade rule that satisfies *Pareto optimality, no-envy*, and *strategy proofness.* Lemma 3 implies that f also satisfies those properties. Then, by Lemma 8, $f = U$. Note that, for each Ω, $U \circ \Omega$ does not satisfy *strong population monotonicity* for $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^N$ in the proof of Theorem 8.

■