Stable Solutions on Majority Relation

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November 6, 2009

A – finite set of alternatives, $|A|=m>2$.

N – finite set of agents, $|N|=n>1$

R_i, $R_i \subseteq A \times A$ ($i \in N$) – agent's preferences over alternatives from A.

$\forall x, y \in A$ either xR_iy or yR_ix (completeness).

$R_i=P_i \cup I_i$:

1) $P_i \cap I_i = \varnothing$

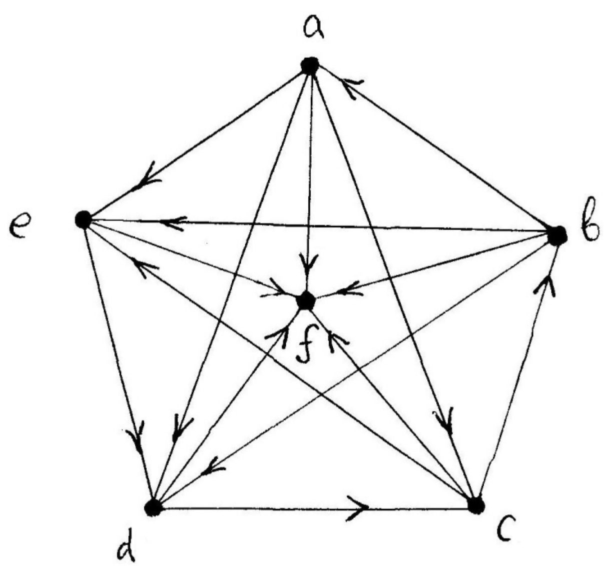
2) $\forall x, y \in A$ $(x, y) \in P_i \Rightarrow (y, x) \notin P_i$ (strong preference)

3) $\forall x, y \in A$ $xI_iy \Rightarrow yI_ix$ (indifference).

Majority relation μ: $x\mu y \Leftrightarrow \text{card}\{i\in N\ xP_iy\}>\text{card}\{i\in N\ yR_ix\}$

If neither $(x, y)\in\mu$, nor $(y, x)\in\mu$ holds, then (x, y) is a *tie* - $x\tau y$.

μ - *tournament* $\Leftrightarrow \forall x, y\in A$ either $x\mu y$ or $y\mu x \Leftrightarrow \tau=\varnothing$.



A *step* $x \rightarrow y$ – when $x \mu y$.

A *path* $x \rightarrow y_1 \rightarrow y_2 \rightarrow \ldots \rightarrow y_{k-2} \rightarrow y_{k-1} \rightarrow y$: $x \mu y_1$, $y_1 \mu y_2, \ldots, y_{k-1} \mu y$.

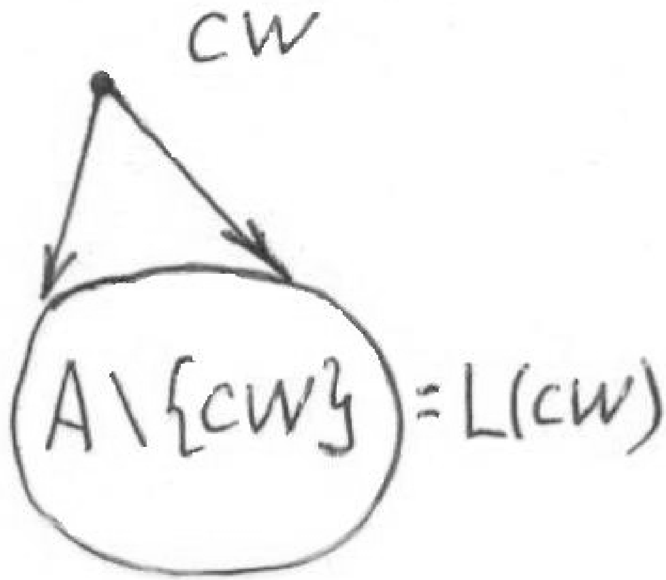
Length of a path - a number of steps in the sequence.

y is *reachable in k steps* from x $\Leftrightarrow$ $\exists$ a path of length k from x to y.

Lower contour set L(x) - $\{y \in A: x\mu y\}$. *Upper contour set* D(x) - $\{y \in A: y\mu x\}$.

$\forall x \Rightarrow L(x) \cup D(x) \cup \{x\} = A$

Condorcet winner CW - a: $\forall x \in A,\ x \neq a \Rightarrow a\mu x$



A set of all undominated alternatives is called a *(majority) core* Cr,

$x \in Cr \Leftrightarrow \forall y \in A\ y \neq x \Rightarrow (x\mu y \text{ or } x\tau y)$.

Dominant set (Condorcet set) D

$D \subseteq A$ - $\forall x \in D \Leftrightarrow \{\forall y \in A \backslash D \Rightarrow x \mu y\}$.

Minimal dominant set MD

$\forall$ B: $B \subset MD \Leftrightarrow$ B is not a dominant set.

Minimal dominant set of degree 2 $MD_{(2)}$

$MD_{(2)}$ is a minimal dominant set in $A \backslash MD_{(1)}$.

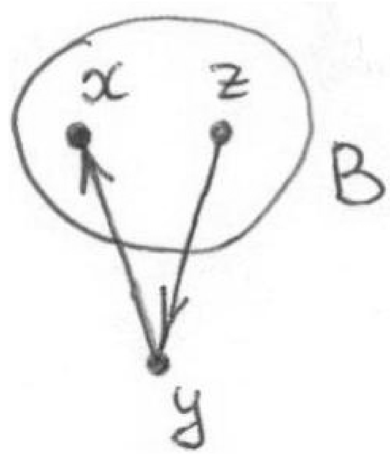
Minimal dominant set of degree i $MD_{(i)}$

$MD_{(i)}$ is a minimal dominant set in $A \backslash (\cup MD_{(j)})$, $1 \leq j \leq i-1$

Weakly stable set (Aleskerov, Kurbanov, 1999)

$\forall x \in B \Leftrightarrow (\exists y \in A \backslash B: y\mu x \Rightarrow$

$\exists z \in B: z\mu y)$.



Minimal weakly stable set MWS

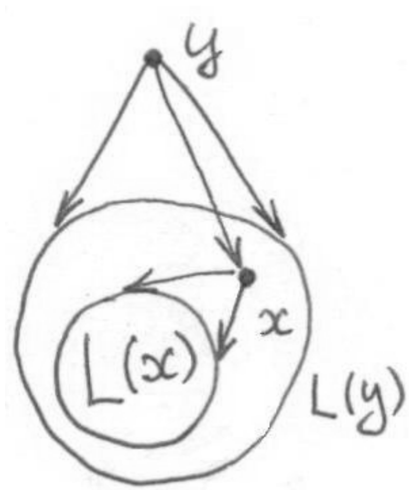
$\forall$ B: B$\subset$MWS $\Leftrightarrow$ B is not weakly stable. If MWS is not unique, then

the social choice is defined as $\cup MWS_i$.

y *covers* x

$y \mu x$ & $L(x) \subseteq L(y)$

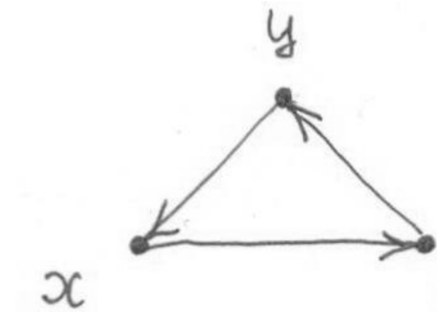
(Miller, 1980)



x is uncovered

$\forall y$: $y \mu x \Rightarrow \exists z$:

$x \mu z$ & $z \mu y$



Uncovered set (Miller, 1980) UC - $\forall x \in$ UC $\Leftrightarrow$ x is uncovered.

UC is unique and is always non-empty (Miller, 1980).

Lemma 1. If D_1 and D_2 are dominant sets then either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.

Lemma 2. MD always exists and is unique.

Lemma 3. D - dominant set $\Rightarrow D = MD + MD_{(2)} + \ldots + MD_{(j-1)} + MD_{(j)} + \ldots + MD_{(i)}$

A set of all dominant sets for any μ is a sequence of s sets

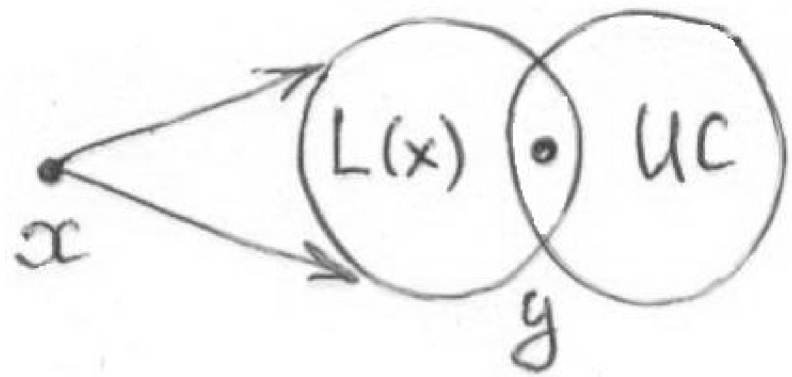
$MD \subset D_{(2)} \subset \ldots \subset D_{(i-1)} \subset D_{(i)} \subset \ldots \subset D_{(s)} = A$.

Lemma 4. $\cup MWS_i \subseteq MD$, $UC \subseteq MD$.

Lemma 5. B is weakly stable iff $\forall y \notin B \Rightarrow B \cap D(y) \neq \varnothing$.

Theorem 1. $x \in \cup MWS_i$ iff

1) either x is uncovered or 2) some y: $y \in L(x)$ is uncovered.



Corollary. The uncovered set is a subset of the union of minimal

weakly stable sets, $UC \subseteq \cup MWS_i \subseteq MD \subseteq A$.

There are tournaments for which inclusion is strict,

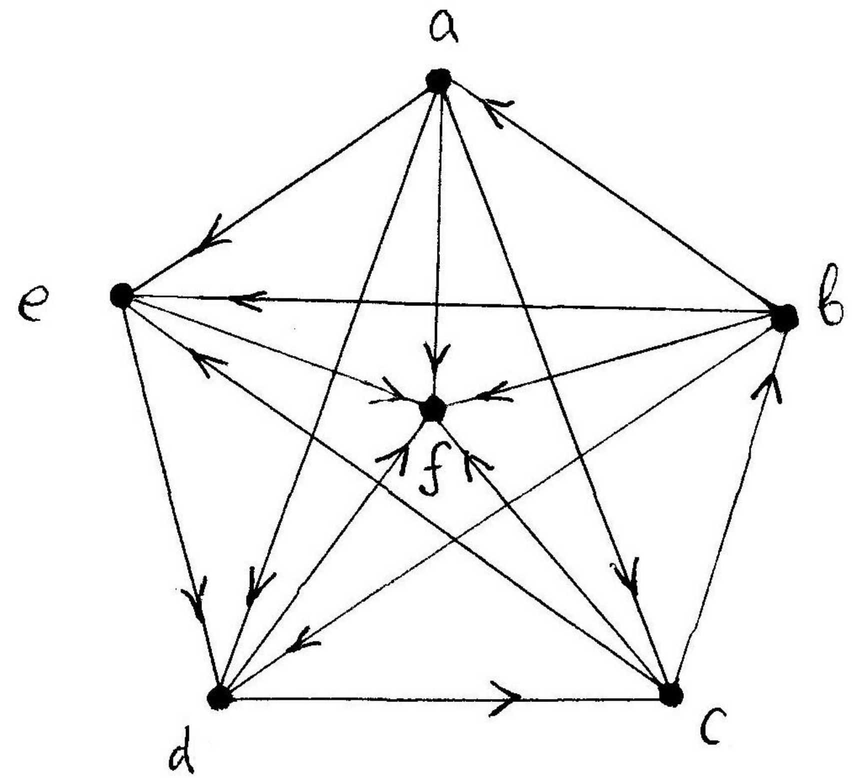
UC⊂MWS⊂MD⊂A.

UC={a, b, c},

MWS={a, b, c, d},

MD={a, b, c, d, e},

A={a, b, c, d, e, f}.



k-stable points and k-stable sets

and their relationship

a) k-stable points

x - *generally stable* if $\forall y \in A$, $y \neq x$ y is reachable from x, otherwise x is *unstable*.

$l(x, y)$ - *minimal length function*

$l_{max}(x) = max_{y \in A} l(x, y)$.

If $l_{max}(x) = k < \infty$ then $\forall y \in A$, $y \neq x$ is reachable from x in $i \leq k$ steps,

but $\exists y \in A$, $y \neq x$ not reachable from x in $i \geq k$ steps.

$l_{max}(x)$ - *degree of stability* of x.

x is *k-stable* $\Leftrightarrow$ the degree of stability of x is k, $k < \infty$.

$SP_{(k)}$ - a class of k-stable points

$x \in SP_{(k)} \Leftrightarrow l_{max}(x)=k.$

$SP_{(1)}=\{CW\}$

If $SP_{(1)}=\varnothing$, then $SP_{(2)}=UC$.

A is finite $\Rightarrow \exists m=\max_{x \in MD} l_{max}(x)$.

Consequently,

1) $\forall SP_{(k)}=\varnothing$, $k>m$;

2) $SP_{(m)} \neq \varnothing$;

3) $MD=SP_{(1)}+SP_{(2)}+SP_{(3)}+\ldots+SP_{(m)}$

Theorem 2. (Nonemptiness of point-classes) If CW does not exist, then $\forall SP_{(k)} \neq \varnothing$, $2 \leq k \leq m = \max_{x \in MD} l_{max}(x)$.

$P_{(k)}$: by definition $x \in P_{(k)} \Leftrightarrow \forall y \in A$, y is reachable from x in *no more* than k steps.

$P_{(k)} = SP_{(1)} + SP_{(2)} + \ldots + SP_{(k)}$.

From the definition of $P_{(k)}$ follows

1) $P_{(1)} = \{CW\} = MD$;

2) If $P_{(1)} = \varnothing$, then $P_{(2)} = UC$;

3) $P_{(1)} \subset P_{(2)} \subset P_{(3)} \subset \ldots \subset P_{(m-1)} \subset P_{(m)} = MD$, $m = \max_{x \in MD} l_{max}(x)$,

b) k-stable sets

$X \subseteq A$ - *generally stable set* if $\forall y \in A \setminus X\ \exists x \in X$: it is possible to reach y from x, otherwise X is *unstable*. $y \in A \setminus X$ is *reachable in k steps* from X if $\exists x \in X$: it is possible to reach y from x in k steps.

$\forall X$: $X \cap MD \neq \varnothing$ - generally stable,

$\forall X$: $X \cap MD = \varnothing$ - unstable.

In terms of $l(x, y)$ X - generally stable $\Leftrightarrow \forall y \in A \setminus X\ \exists x \in X: l(x, y) < \infty$.

minimal length function $l(X, y) = \min_{x \in X} l(x, y)$, $\forall y \in A \setminus X$. $l(X, y) = \infty$ when y is not reachable from X.

$l_{max}(X)=max_{y \in A} l(X, y)$. $l_{max}(X)=k<\infty \Rightarrow \forall y \in A \backslash x \ \exists x \in X$: $l(x, y) \leq k$ & $\exists y \in A \backslash X$ $\forall x \in X$ $l(x, y) \geq k$. $l_{max}(X)$ - *degree of stability* of X.

X is *k-stable set* $\Leftrightarrow$ the degree of stability of X is k, $k<\infty$.

X is *minimal k-stable set* iff $\forall$ B: $B \subset X \Leftrightarrow$ B is not k-stable.

$SS_{(k)}$ - a class of minimal k-stable sets.

$x \in SS_{(k)} \Leftrightarrow \exists$ B, $B \subseteq A$: $x \in B$ and B is minimal k-stable, but $\forall C$, $C \subseteq A$ $x \in C \Rightarrow$ C is not a minimal stable set of degree $i<k$.

$S_{(k)}$ - a union of minimal stable sets of degrees of stability $i \leq k$

$S_{(k)}=SS_{(1)}+SS_{(2)}+\ldots+SS_{(k)}$, $S_{(1)} \subseteq S_{(2)} \subseteq S_{(3)} \subseteq \ldots \subseteq S_{(p-1)} \subseteq S_{(p)}=MD$, $p \leq m$, $m=max_{x \in MD} l_{max}(x)$

$S_{(1)}=SS_{(1)}=\cup MWS_i$.

Theorem 3. $P_{(2)}\subseteq S_{(1)}\subseteq P_{(3)}$, $P_{(k)}\subseteq S_{(k)}\subseteq P_{(k+2)}$, $k>1$

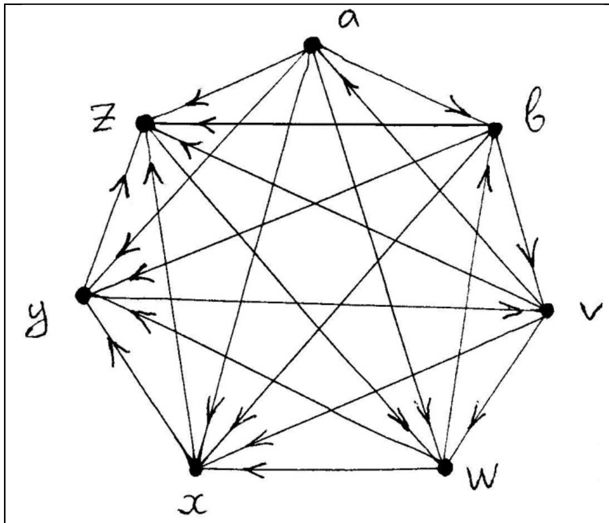
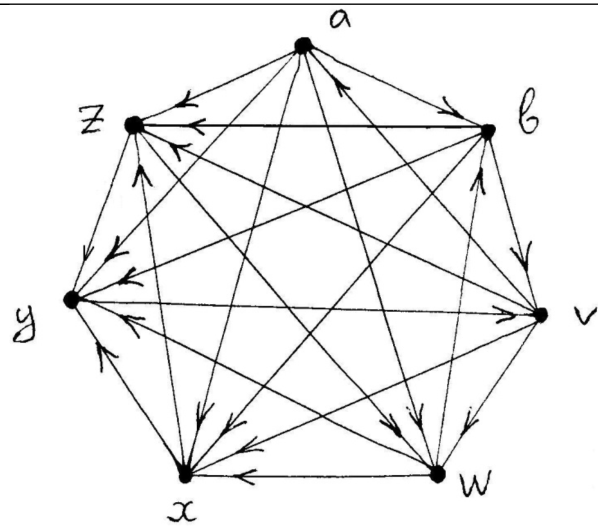
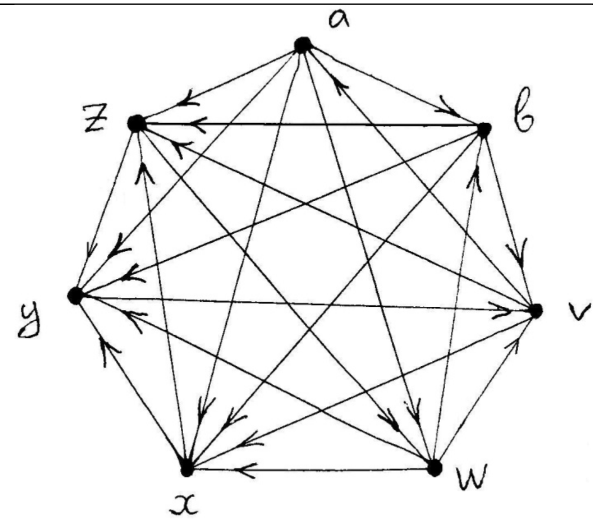
Corollary: If $x\in SP_{(k)}$ then

either $x\in SS_{(k-2)}$;

or $x\in SS_{(k-1)}$;

or $x\in SS_{(k)}$.

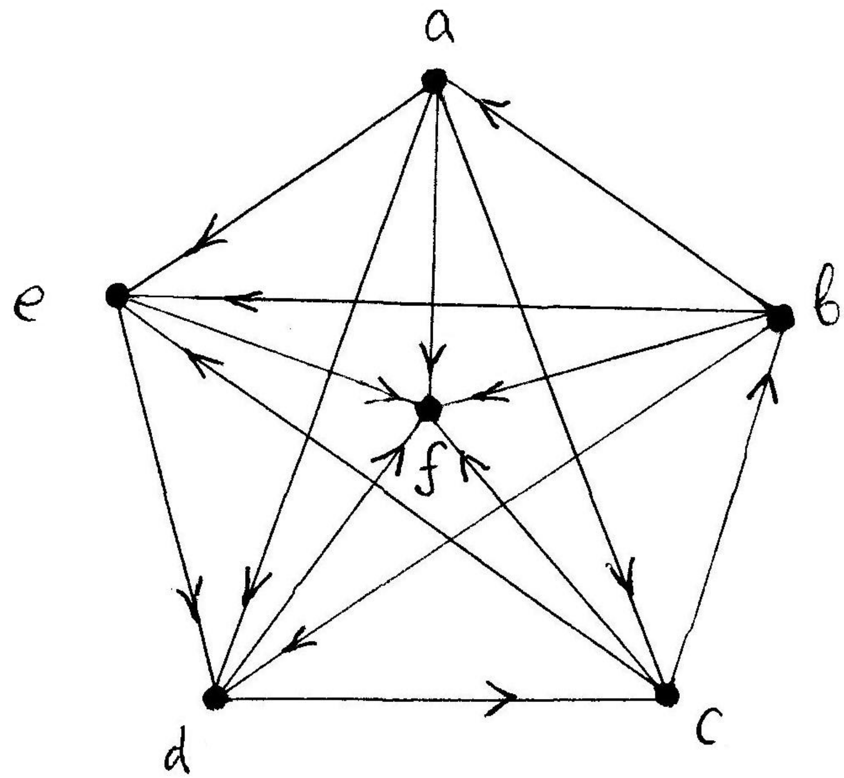
A={a, b, v, w, x, y, z};

 **Figure 1**	 **Figure 2**	 **Figure 3**

Distribution of alternatives by point-classes $\{SP_{(k)}\}$ and set-classes $\{SS_{(k)}\}$

Figure 1			**Figure 2**				**Figure 3**		
	$\mathbf{SP_{(2)}}$	$\mathbf{SP_{(3)}}$		$\mathbf{SP}_{(2)}$	$\mathbf{SP}_{(3)}$	$\mathbf{SP}_{(4)}$		$\mathbf{SP}_{(2)}$	$\mathbf{SP}_{(3)}$
$\mathbf{SS_{(1)}}$	a, b, v	y, w	$\mathbf{SS}_{(1)}$	a, b, v	y, w		$\mathbf{SS}_{(1)}$	a, b, v, w	y, z
$\mathbf{SS_{(2)}}$		z	$\mathbf{SS}_{(2)}$			z	$\mathbf{SS}_{(2)}$		
$\mathbf{SS_{(3)}}$		x	$\mathbf{SS}_{(3)}$		x		$\mathbf{SS}_{(3)}$		x

	$SP_{(2)}$	$SP_{(3)}$	$SP_{(4)}$
$SS_{(1)}$	a, b, c	d	
$SS_{(2)}$			
$SS_{(3)}$			
$SS_{(4)}$			e



Theorem. Let **cw**, **cr**, $\mathbf{uc}^N$ (N=I÷V), **ucp**, **ut** and **md**, respectively, denote characteristic vectors of the following solutions: the Condorcet winner {CW}, the core Cr, the uncovered set UC^N (N=I÷V), the uncaptured set UCp, the union of minimal undominated sets (strong top-cycles) MU, the untrapped set UT, the minimal dominant set (weak top-cycle) MD. Let $\mathbf{sp}_{(k)}$, $\mathbf{ss}_{(k)}$, $\mathbf{p}_{(k)}$ and $\mathbf{s}_{(k)}$ denote characteristic vectors for classes of k-stable alternatives $SP_{(k)}$, classes of k-stable sets $SP_{(k)}$, and their sums $P_{(k)}=SP_{(1)}+SP_{(2)}+\ldots+SP_{(k)}$, $S_{(k)}=SS_{(1)}+SS_{(2)}+\ldots+SS_{(k)}$, respectively. Let **a** denote a characteristic vector of a universal set A. ε denotes the relation of identity, which is represented by the matrix $\mathbf{E}=[\delta_{ij}]$. d=d(ρ) is a ρ-diameter of A. Let **M**, **T**, **U** denote Boolean matrices representing relations μ, τ and $\upsilon=\mu\cup\tau\cup\varepsilon$ on A. Finally, let $\mathbf{M}_{(k)}=\sum_{i=1}^{k}\mathbf{M}^i+\mathbf{E}$ and $\mathbf{U}_{(k)}=\sum_{i=1}^{k}\mathbf{U}^i$.

Then

1) $\mathbf{cw}=\overline{\overline{(\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}$,

$\mathbf{cr}=\overline{\overline{(\mathbf{M}+\mathbf{T}+\mathbf{E})}\cdot\mathbf{a}}=\overline{\overline{\mathbf{U}}\cdot\mathbf{a}}=\overline{\mathbf{M}^{tr}\cdot\mathbf{a}}$,

$\mathbf{uc}^{I}=\overline{\overline{(\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E})}\cdot\mathbf{a}}=\overline{\overline{(\mathbf{M}\cdot\mathbf{M}+\mathbf{U})}\cdot\mathbf{a}}$,

$\mathbf{ucp}=\overline{\overline{(\mathbf{M}\cdot\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{M}\cdot\mathbf{M}+\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E})}\cdot\mathbf{a}}=$

$=\overline{\overline{(\mathbf{M}\cdot\mathbf{U}\cdot\mathbf{M}+\mathbf{U}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{U}+\mathbf{U})}\cdot\mathbf{a}}$,

$\mathbf{mu}=\overline{\overline{(\mathbf{M}_{(d)}+\overline{\mathbf{M}}^{tr}_{(d)})}\cdot\mathbf{a}}$, d=d(μ): $(\mathbf{M}_{(d)}\neq\mathbf{M}_{(d-1)})$ & $(\mathbf{M}_{(d)}=\mathbf{M}_{(d+1)})$,

$\mathbf{ut}=\overline{\overline{(\mathbf{M}_{(d)}+\mathbf{T})}\cdot\mathbf{a}}$, d=d(μ): $(\mathbf{M}_{(d)}\neq\mathbf{M}_{(d-1)})$ & $(\mathbf{M}_{(d)}=\mathbf{M}_{(d+1)})$,

$\mathbf{md}=\overline{\overline{\mathbf{U}}_{(d)}\cdot\mathbf{a}}$, d=d(υ): $(\mathbf{U}_{(d)}\neq\mathbf{U}_{(d-1)})$ & $(\mathbf{U}_{(d)}=\mathbf{U}_{(d+1)})$.

2) If μ is a tournament, then $\mathbf{T}=\mathbf{O}$, $\mathbf{U}=\mathbf{M}+\mathbf{E}$, $\mathbf{M}_{(k)}=\mathbf{U}_{(k)}=\mathbf{U}^k$ and

$$\mathbf{p}_{(k)}=\overline{\overline{\mathbf{M}}_{(k)}\cdot\mathbf{a}}=\overline{\overline{\mathbf{U}^k}\cdot\mathbf{a}},$$

$$\mathbf{sp}_{(k)}=\overline{\overline{\mathbf{p}_{(k)}}+\mathbf{p}_{(k-1)}}=\overline{\overline{\mathbf{M}}_{(k)}\cdot\mathbf{a}+\overline{\overline{\mathbf{M}}_{(k-1)}\cdot\mathbf{a}}}=\overline{\overline{\mathbf{U}^k}\cdot\mathbf{a}+\overline{\overline{\mathbf{U}^{k-1}}\cdot\mathbf{a}}},$$

$$\mathbf{cw}=\mathbf{p}_{(1)}=\mathbf{sp}_{(1)}=\overline{\overline{(\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}=\overline{\overline{\mathbf{U}}\cdot\mathbf{a}},$$

$$\mathbf{uc}=\mathbf{p}_{(2)}=\overline{\overline{(\mathbf{M}^2+\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}=\overline{\overline{\mathbf{U}^2}\cdot\mathbf{a}},$$

$$\mathbf{ucp}=\mathbf{p}_{(3)}=\overline{\overline{(\mathbf{M}^3+\mathbf{M}^2+\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}=\overline{\overline{\mathbf{U}^3}\cdot\mathbf{a}},$$

$$\mathbf{mu}=\mathbf{ut}=\mathbf{md}=\mathbf{p}_{(m)}=\overline{\overline{(\mathbf{M}_{(m)}+\mathbf{E})}\cdot\mathbf{a}}=\overline{\overline{\mathbf{U}^m}\cdot\mathbf{a}},\ m:\ \mathbf{p}_{(m-1)}\neq\mathbf{p}_{(m)}\ \&\ \mathbf{p}_{(m)}=\mathbf{p}_{(m+1)},$$

$$\mathbf{s}_{(1)}=\mathbf{ss}_{(1)}=\mathbf{mws}=(\mathbf{M}+\mathbf{E})\cdot\mathbf{p}_{(2)}=\mathbf{U}\cdot\overline{\overline{\mathbf{U}^2}\cdot\mathbf{a}},$$

Thank you!