Ambiguity and rational expectations equilibria*

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Abstract

This paper demonstrates that when the concept of Rational Expectations Equilibrium (REE) is expanded to allow for agents whose preferences display ambiguity aversion, standard results on REE no longer hold. In particular, REE can be partially revealing over a set of parameters with positive Lebesgue measure. This finding illustrates that models with ambiguity averse investors provide a relatively tractable framework through which partial information revelation may be studied in a general equilibrium setting. Constructive examples provide further insight into the properties of these equilibria.

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1 Introduction

Market prices convey information. In every market equilibrium, positive prices at least inform participants that there is some demand for all goods. In many markets, prices convey much more information, including market participants’ private information about relevant economic variables. The concept of a rational expectations equilibrium (REE) formalized in Radner (1979) was formulated to study the dissemination of privately held information through market prices. One of the startling results of the ensuing literature [Radner (1979) and Allen (1981)] is that when REE exist and the market has no “noise” (see for example Grossman and Stiglitz (1980)), equilibrium market prices almost surely reveal all agents’ private information.

The sharpness of this result proved to limit the application of REE as an equilibrium concept. Generally, one tends to think of the amount of private information held by individuals as both diverse (i.e. varying in quality and the set of markets to which it applies) and diffuse (i.e. held by many different participants in the market). The idea that all of this information is, almost surely, fully conveyed in a finite-dimensional market statistic like the equilibrium price proved too simplistic for many applications.

The aim of this paper is to demonstrate that the full-revelation property of the REE concept need not hold when the class of conceivable preferences for market participants is expanded beyond the standard subjective expected utility model of Savage (1954). Specifically, we demonstrate that when at least one investor has preferences that display ambiguity aversion (see for example Gilboa and Schmeidler (1989), and Schmeidler (1989)) then the concept of REE (suitably generalized to allow for the generalization of preferences) permits robust equilibria that are partially revealing. We show explicitly that there exists a parameter set with positive Lebesgue measure in which partially-revealing REE
exist.

Generally speaking, when applied to markets where investors may display ambiguity aversion, the REE concept permits a richer set of information revelation phenomena than the analogous model populated by only expected utility maximizers. These results also serve to highlight important additional differences between investors with preferences that display ambiguity aversion and investors who have subjective expected utility preferences.

It has been noted, for example by Zambrano (2005), that in Walrasian equilibrium models, subjective expected utility places no restriction on observable market data. As such, in a Walrasian equilibrium one is never able to determine conclusively that a participant has preferences that are anything other than subjective expected utility. While this observational difficulty remains when the stricter concept of REE is employed (since only prices and not the price function is ever observed), this work highlights the fact that the mechanism by which ambiguity averse investors convey privately held information to the market differs from the way that subjective expected utility maximizing agents convey privately held information.

It is interesting to contrast the present examples with the examples of partially-revealing REE given in Grossman and Stiglitz (1980) and the expansive literature that followed. While that literature also gives examples of REE that are partially revealing, these seem to depend on the particular distribution of payoffs (normal) as well as the utility function of investors (constant absolute risk aversion). These models also usually make use of noise traders, or traders whose preferences are not explicitly modeled.\footnote{Diamond and Verrecchia (1981) and Ganguli and Yang (2007) construct partially revealing REE in this framework without the presence of noise traders.}

In the model presented here however, no assumptions beyond those necessary to ensure the existence of equilibrium are placed on the von Neumann-
Morgenstern utility functions of the investors. Additionally, the preferences of all investors are specified ex ante and none of these investors is “irrational” in the sense that each has well-specified preferences and makes decisions that are optimal given these preferences and the constraints faced.²


The remainder of the paper proceeds as follows. Section 2 outlines the model and the definition of an REE while section 3 contains explicit examples of the nature of REE in a market with ambiguity aversion. Section 4 contains results for the more general case and section 5 concludes. The appendix provides analytical results on equilibria with ambiguity averse investors that are necessary but tangential to the discussion in the paper.

2 The model

The market is populated by a finite set $\mathcal{N} = \{1, \ldots, n, \ldots, N\}$ of investors who live for 2 periods labeled 1 and 2. At the end of period 2, one of a finite set $\Omega$²

²See Blume and Easley (forthcoming) and Gilboa, Postlewaite, and Schmeidler (2007) for insightful discussions on rationality.
of possible states of nature is realized and investors in the economy consume.
A typical element of $\Omega$ is denoted $\omega$.

In period 1, some information is revealed to each investor. Investor $n \in \mathcal{N}$ receives a private signal $s^n$ that comes from a finite set $\mathcal{S}^n = \{1, \ldots, S^n\}$. Let the set of all possible collections of private information that might be available to the market be labeled $\Sigma = \times_{n \in \mathcal{N}} \mathcal{S}^n$ with representative element $\sigma$ and let $\mathcal{F}$ be the discrete $\sigma$-algebra over $\Sigma$. The investors’ private signals convey information about how likely each of the elements in $\Omega$, which realize at the end of period 2, are (a more precise formulation is provided in section 2.1).

Each investor has an endowment $e^n \in \mathbb{R}_{+}^{\vert \Omega \vert}$ of a single consumption good and must choose a consumption allocation in $X = \mathbb{R}_{+}^{\vert \Omega \vert}$ for period 2. This allocation is financed by trading contingent claims on consumption over $\Omega$ in the market that opens in the beginning of period 2.

The market opens at the beginning of period 2 and in equilibrium, each investor derives information about the private signals of other investors by observing the prices of the contingent claims that are traded in the market as described in section 2.1. Let $P \subset \mathbb{R}_{+}^{\vert \Omega \vert}$ be the space of possible prices over contingent claims that can be purchased at the beginning of period 2. The conditions imposed on preferences and endowments ensure that $P$ may be normalized so that it is equal to $\Delta^{\vert \Omega \vert} \subset \mathbb{R}_{+}^{\vert \Omega \vert}$ (the $\vert \Omega \vert - 1$-dimensional simplex). This normalization will be assumed throughout the paper but to avoid confusion, the notation $P$ will be used to refer to the price space and $\Delta^{\vert \Omega \vert}$ will refer to the space of probability distributions over $\Omega$.

2.1 Preferences and beliefs

The market is populated by both expected utility and ambiguity averse investors. The set of expected utility maximizing investors is denoted $\mathcal{N}^E$ and
has cardinality $N^E \geq 1$ while $N^A$ is the set of ambiguity averse investors which has cardinality $N^A \geq 1$. Preferences for the ambiguity averse investors will be described first.

To start, define $\mathcal{C}(\Delta^{[\Omega]})$ to be the collection of non-empty, convex, closed subsets of $\Delta^{[\Omega]}$. Let $\gamma^{n^A} : \mathcal{F} \rightarrow \mathcal{C}(\Delta^{[\Omega]})$ be a mapping that for each set of joint signals $f$ assigns a set of probability distributions that is to be interpreted as the information that $f$ conveys to investor $n^A \in N^A$. The set $\gamma^{n^A}(f)$ is the collection of probability distributions that the ambiguity averse investor believes may govern the resolution of uncertainty over $\Omega$ when she knows that the joint signal $\sigma \in f$. Let $\Gamma_0$ be the space of all such mappings.

Furthermore, we will assume that each ambiguity averse investor holds a marginal distribution $\pi^{n^A}$ over the set of joint signals $\Sigma$. This could be extended to allow for ambiguity averse investors whose preferences demonstrate ambiguity over the space of joint signals without any qualitative change in the results. Each $\gamma^{n^A}(\cdot)$ can then be viewed as an update of the set of prior probabilities over $\Sigma \times \Omega$ by an application of Bayes Rule to each prior along the lines suggested by Epstein and Schneider (2003). However, that generalization requires significantly more investment in notation and we do not pursue it here. The space of beliefs for each ambiguity averse investor is then given by $\Gamma = \Delta^{[\Sigma]} \times \Gamma_0$, where $\Delta^{[\Sigma]}$ denotes the space of probability distributions over $\Sigma$.

Investors in $N^E$ have beliefs over the space $\Sigma \times \Omega$. These can be characterized for any investor $n^E$ by functions of the form $\pi^{n^E}_c : \mathcal{F} \rightarrow \Delta^{[\Omega]}$, that for each information set $f$ assign a conditional probability distribution over $\Omega$ and $\pi^{n^E}_m$, which denotes the marginal distribution over the signal space $\Sigma$. The space of such beliefs for each $n^E$ is denoted $\Pi$.

Investors utilize information both from their private signal and from prices.
Abusing notation slightly, we let \( f(s^n) \in \mathcal{F} \) be the set of joint signals \( \sigma \) that have \( \sigma(n) = s^n \), where \( \sigma(n) \) is the \( n \)th component of \( \sigma \). Each investor \( n \) knows by her private signal that \( \sigma \in f(s^n) \).

A price function \( \phi : \Sigma \to P \) defines a price for every joint signal \( \sigma \). In equilibrium, information is gathered from prices by using the equilibrium price function \( \phi \), so if the observed price is \( p \), then \( \phi^{-1}(p) \) is the information revealed by price \( p \) to all investors \( n \in \mathcal{N} \). Combining the information derived from her personal signal and that inferred from prices, investor \( n \) in equilibrium has information \( f(s^n) \cap \phi^{-1}(p) \). The next assumption describes the preferences of the investors in the economy.

**Assumption 1.** Given any information set \( f \in \mathcal{F} \),

1. Investor \( n^A \in \mathcal{N}^A \) has preferences that can be represented by the utility function

   \[
   U^{n^A}(x^{n^A}; f) = \min_{\gamma \in \gamma(f)} E_{\gamma} u^{n^A}(x^{n^A}),
   \]

   with \( \gamma(f) \subset \Delta|\Omega| \cap \mathbb{R}_{++}^{\vert \Omega \vert} \) for all \( x^{n^A} \in X \).

2. Investor \( n^E \in \mathcal{N}^E \) has preferences given by

   \[
   U^{n^E}(x^{n^E}; f) = E_{\pi^{n^E}(f)}[u^{n^E}(x^{n^E})],
   \]

   for all \( x^{n^E} \in X \) with \( \pi^{n^E}(f) \in \Delta|\Omega| \cap \mathbb{R}_{++}^{\vert \Omega \vert} \).

3. For all \( n \in \mathcal{N}^E \cup \mathcal{N}^A \), the von Neumann-Morgenstern (vN-M) utility function \( u^n(\cdot) \) satisfies \( u^n \in C^2, u^n'(\cdot) > 0, u^{nn}(\cdot) < 0, \) and \( \lim_{c \to 0} u^n(c) = \infty \).

It is useful to note that the definition of investor \( n^A \)'s preferences includes as a special case the situation in which \( n^A \) is an expected utility maximizer.
From the characterization of preferences we now move on to the mechanics of the market.

2.2 Equilibrium

For any price vector $p \in P$, the set of feasible consumption bundles (or budget set) of investor $n$ is

$$F(e^n, p) = \{ x \in X : p(e^n - x^n) \geq 0 \}. \quad (2.3)$$

With this notation at hand, we can now define the equilibrium notion of interest.

**Definition 1.** A pair $(x, \phi)$, where $\phi : \Sigma \to P$ is a price function and $x : \Sigma \to R^{N[Ω]}$ an allocation, is a rational expectations equilibrium (REE) if for all $n$ and $σ$, $(x, \phi)$ satisfies

1. $x^n(σ) \in \arg \max U^n(x^n(σ); f(σ(n)) \cap \phi^{-1}(φ(σ)))$ s.t. $x^n \in F(e^n, φ(σ))$
2. $\sum_{n \in N}(e^n(σ) - x^n(σ)) = 0$

**Definition 2.** An REE price function $φ$ is said to be fully-revealing if it is injective. It is said to be partially-revealing if it is not fully-revealing. An REE is called fully-revealing if the corresponding price function is fully-revealing and is called partially-revealing otherwise.

Having fully described the model and defined equilibrium, the next section gives examples by way of introducing the paper’s main results.

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Our notion of partially-revealing REE prices includes the case where the prices are non-revealing, i.e., $φ$ is a constant function.
3 Examples

3.1 Existence and Robustness

Consider a pure exchange economy populated by two investors who will be denoted by \( a \) (for ambiguity averse) and \( e \) (for expected utility). There are two possible payoff-relevant states (labeled 1 and 2) that can be realized in period 2. Investors may receive one of two possible signals that convey information on the environment that will prevail in this market. The signals are labeled \( s_1 \) and \( s_2 \).

The first investor will be denoted investor \( a \). Her preferences, given the information that is available at time 1, can be described by a utility function of the form

\[
U^a = \min_{\pi \in \gamma} E_{\pi}[u^a(x)] = \pi u^a(x(1)) + (1 - \pi) u^a(x(2))
\] (3.1)

where \( \gamma \) is a compact, convex subset of the open unit interval denoting the set of probabilities of state 1 occurring.

The second investor will be denoted investor \( e \). His preferences, given the information available at time 1, can be represented by the utility function

\[
U^e = E[u^e(x)] = \pi u^e(x(1)) + (1 - \pi) u^e(x(2))
\] (3.2)

where \( \pi \in (0, 1) \) denotes the probability of state 1.

Each investor has the endowment \( \bar{e} \in \mathbb{R}_{++} \) in state 1 and state 2. Furthermore, assume that \( u^e \) and \( u^a \) are strictly increasing, strictly concave and that \( \lim_{x \to 0} u^{i'}(x) = \infty \) for \( i \in \{e, a\} \).

For this example the set of possible signal profiles is

\[
\Sigma = \{ \sigma = (s^e, s^a) \in \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\} \}. \] (3.3)
Let $\pi(s^e)$ (respectively $\gamma(s^a)$) represent investor $e$’s (a’s) beliefs about state one being realized given his (her) signal $s^e$ ($s^a$). Let $\pi(s_k)$ represent $e$’s distribution over states given that he knows only that $\sigma \in \{(s_k, s_1), (s_k, s_2)\}$. Suppose that the investors update their beliefs given their individual signals in such a way that the following condition holds

$$\pi(s_1) \in \gamma(s_1, s_1) \cap \gamma(s_1, s_2)$$

$$\pi(s_2) \in \gamma(s_2, s_1) \cap \gamma(s_2, s_2)$$

Then we have the following.

**Claim 1.** A partially revealing REE exists for the given economy.

**Proof.** Consider the allocation $x^e(\sigma) = x^a(\sigma) = \bar{e}$ for all $\sigma \in \Sigma$ and the price function satisfying $\phi((s_1, s_k)) = \pi(s_1)$ and $\phi((s_2, s_k)) = \pi(s_2)$ for $k \in \{1, 2\}$. First it is necessary to check that given the individual signals received by each investor the given allocation and prices are an Arrow-Debreu equilibrium. Then we verify that it is still an equilibrium given the information available in the price function.

First note that the allocation is market clearing. We now show that such an allocation is optimal given market prices. To do so, we investigate the necessary and sufficient conditions for an optimum for each investor. These conditions are demonstrated in the appendix for ambiguity averse investors.

Now, consider the first order conditions for investor $e$. Given the assumption of strict concavity these are necessary and sufficient to ensure that an allocation is optimal for investor $e$ under the given price. Given that investor $e$ has received
signal \( s_k \), his first order conditions evaluated at the proposed equilibrium are

\[
\lambda^e \left( \begin{array}{c} \pi(s_k) \\ 1 - \pi(s_k) \end{array} \right) = \left( \begin{array}{c} \pi(s_k)u^e'(\bar{e}) \\ (1 - \pi(s_k))u^e'(\bar{e}) \end{array} \right) \quad (3.5)
\]

\[
\pi(s_k)\bar{e} + (1 - \pi(s_k))\bar{e} \leq \bar{e}
\]

These conditions are satisfied for the proposed equilibrium values given any signal \( s_k \).

Before stating the related conditions for investor \( a \), a bit of notation is required. Throughout the paper, if \( x \in \mathbb{R}^m \) and \( A \subseteq \mathbb{R}^m \) then the notation \( x \circ A \) is used to denote the set of vectors \( B \subseteq \mathbb{R}^m \) given by \( B = \{ b : b = [xa_1 \cdots xa_m] \text{ for all } a \in A \} \). The first order conditions under the proposed equilibrium for investor \( a \) (see Appendix A.1) given that she has received the signal \( s_l \) and investor \( e \) has received the signal \( s_k \) are

\[
\lambda^a \left( \begin{array}{c} \pi(s_k) \\ 1 - \pi(s_k) \end{array} \right) \in u^a(\bar{e}) \cdot \gamma(s_k, s_l)
\]

\[
\pi(s_k)\bar{e} + (1 - \pi(s_k))\bar{e} \leq \bar{e}
\]

Condition (3.6) is satisfied by letting \( \lambda^a = u^a(\bar{e}) \) and then noting that (3.4) implies that \( (\pi(s_k), (1 - \pi(s_k)) \in \gamma(s_l, s_k) \) for each \( k \) and \( l \). The resource constraint condition also holds.

Thus, if agents do not condition their behavior on market prices then this is an equilibrium. What remains is to verify that no additional information is available from market prices.

To see this, consider investor \( e \). Knowing the market mechanism, he knows that investor \( a \)'s beliefs are such that regardless of the signals that either of them

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4Two additional “complementary slackness” conditions will also hold but given the assumption that \( \pi(\sigma) \in (0, 1) \) for all \( \sigma \in \Sigma \), the multipliers for these conditions will always be zero.
receive, a will fully insure. Furthermore, he knows that she will do so at the 
equilibrium prices that clear markets. Thus, prices differ across the signals that 
investor e receives, but for any signal s^e, prices do not differ across the signal 
that investor a receives and so investor e can garner no further information from 
market prices.

Now consider investor a. She knows her own signal, and can also determine 
investor e’s signal from market prices. However, assumption (3.4) implies that 
even with knowledge of e’s signal, investor a still chooses to fully insure so the 
given allocation is still an equilibrium when the information content of prices is 
considered.

The next result shows that in the space of updated beliefs, the set over which 
such a partially-revealing REE exists has positive Lebesgue measure. This result 
is the direct analog of the proposition in Radner (1979)[page 668]. First, we note 
that an updated belief for investor a given the signal σ can be characterized by 
two points (the lower and upper bound of probabilities of state 1 occurring 
given the joint signal σ). Thus, the set of possible updates that a may have is 
isomorphic to a subset of [0, 1]^8 (two points for each of the four possible joint 
signals). Label this set Θ^a. Likewise, the set of possible updates that e may 
have given the joint signals σ is isomorphic to a point in [0, 1]^4 (one point—
representing the probability of state 1 occurring—for each of the four possible 
joint signals). This set is labeled Θ^e. To state the result, let Θ = Θ^e × Θ^a and 
endow this space with the usual subspace topology inherited from \( \mathbb{R}^{12} \).

**Claim 2.** The set \( \theta \in \Theta \) for which assumption (3.4) holds has positive Lebesgue 
measure.

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5The space of beliefs is characterized by a subset of this space because of the condition 
that given a particular joint signal σ, the upper bound of the probability of state 1 occurring 
must be at least as great as the lower bound of the probability of state 1 occurring.
Proof. Let $\pi \in \Theta^e$ have the property that

$$\pi \notin \text{bd } \Theta^e \tag{3.7}$$

and define $0 < \epsilon_\pi < \min_{x \in \text{bd } \Theta^e} ||x - \pi||$, where $\text{bd } \Theta^e$ denotes the boundary of $\Theta^e$. Consider the set $\bar{\Theta}^e \subseteq \Theta^e$ defined by

$$\bar{\Theta}^e = \{x \in \Theta^e : ||x - \pi|| < \epsilon_\pi\} \quad \tag{3.8}$$

The set $\bar{\Theta}^e$ is open in $\Theta^e$.

Now consider the set $\Theta^a$. For convenience, let us adopt the convention that the coordinates $\{1, 2, 3, 4\}$ will refer to the lower bound of the probabilities of state 1 occurring given the four possible joint signals. Similarly, the coordinates $\{5, 6, 7, 8\}$ will refer to the upper bound of the probability of state 1 occurring given the four possible joint signals.

What remains to be shown is that there is a corresponding open set $\bar{\Theta}^a \in \Theta^a$ that has positive measure (in $\Theta^a$) for which condition 3.4 holds. First we define the following sets.

$$\bar{\Theta}^a_\ast = \{x \in \Theta^a : x(i) < \min_{y \in \bar{\Theta}^e} y(i) \text{ for } i \in \{1, 2, 3, 4\}\}$$

$$\bar{\Theta}^{a\ast} = \{x \in \Theta^a : x(i) > \max_{y \in \bar{\Theta}^e} y(i-4) \text{ for } i \in \{5, 6, 7, 8\}\} \quad \tag{3.9}$$

Now define $\bar{\Theta}^a = \bar{\Theta}^a_\ast \cap \bar{\Theta}^{a\ast}$. By condition [3.7], $\bar{\Theta}^a$ is non-empty and open. It can be seen that for all $y \in \bar{\Theta}^e$, each set of beliefs represented by a point in $\bar{\Theta}^a$ contains $y$ and as such the product $\bar{\Theta}^a \times \bar{\Theta}^e$ is non-empty, and hence (in conjunction with its openness) has positive Lebesgue measure in the space $\Theta$.

Then $\bar{\Theta}^a \times \bar{\Theta}^e$ will be the product of two open sets and hence open under the product topology. Since the product topology and the metric topology are
This example has shown that with ambiguity averse investors, standard REE may be partially revealing. It further shows explicitly the set of economies (parameterized by beliefs) for which this partially-revealing equilibrium exists.

3.2 Comparative Statics

The next example illustrates some of the qualitative properties of these equilibria through comparative statics. Some simplifying assumptions are made so that salient points can be highlighted, but the qualitative properties of the example hold in a broader setting.

Suppose that there are two investors, an expected utility investor labeled $e$ and an ambiguity averse investor labeled $a$. There are two states of the world, labeled $\{1, 2\}$. Aggregate consumption in state 1 is given by $x + \epsilon$ and aggregate consumption in state 2 is $x$. The endowments of the investors are: $\bar{e}^e = (x, x)$ and $\bar{e}^a = (\epsilon, 0)$. Only investor $a$ receives a signal $s \in \{s_1, s_2\}$ that conveys information about the relative likelihood of each state occurring. The vN-M utility function for both investors is $\log(\cdot)$.

Investor $e$’s beliefs are denoted $\pi$. Without knowing $a$’s signal, investor $e$ believes that the probability of state 1 occurring is $\pi = \frac{1}{2}$. On the other hand, $e$’s conditional beliefs are given by $\pi(1|s_1) = \pi_1$ and $\pi(1|s_2) = \pi_2$. These beliefs are interpreted as the “correct” probabilities of state 1 occurring given the signal $s$. We assume that

$$\frac{1}{2} \pi_1 + \frac{1}{2} \pi_2 = \frac{1}{2} = \pi, \text{ with } \pi_1 \leq \pi_2. \quad (3.10)$$

Investor $a$’s signal conveys (to him) information with some ambiguity. When he receives signal $s_k$ he believes that the true probability of state 1 occurring is in the set $[\pi_k - \alpha, \pi_k + \alpha]$. 

14
The parameter $\alpha \geq 0$ can be thought of as describing the amount of ambiguity inherent in investor $a$’s preferences. In the limit as $\alpha$ goes to zero, investor $a$ becomes an expected utility maximizing investor. This comparative statics exercise will analyze the change in equilibrium price that arises from changes in $\alpha$.

We will normalize the price of consumption in state 2 to be 1 and call the price of state 1 consumption $p$.

Given beliefs $\pi$, it can be seen that $e$’s demand for consumption in state 1 is

$$x^e(p) = \frac{\pi(px + x)}{p}.$$  (3.11)

Investor $a$’s demand for consumption in state 1 when the signal $s_k$ has been received is

$$x^a(p) = \begin{cases} 
\frac{(\pi_k + \alpha)p}{p} & p \geq \frac{\pi_k + \alpha}{1-(\pi_k + \alpha)} \\
\frac{\pi_k - \alpha}{1-(\pi_k - \alpha)} & \frac{\pi_k - \alpha}{1-(\pi_k - \alpha)} \leq p \leq \frac{\pi_k + \alpha}{1-(\pi_k + \alpha)} \\
\frac{(\pi_k - \alpha)p}{p} & p \leq \frac{\pi_k - \alpha}{1-(\pi_k - \alpha)}.
\end{cases}$$  (3.12)

From these demands it can be shown that the equilibrium price when $e$ knows only that $s \in \{s_1, s_2\}$ and $a$ fully insures is

$$p_{pr} = \frac{\sqrt{x^2 + \epsilon^2} - \epsilon}{x}.$$  (3.13)

Therefore, if $\alpha$ is large enough that

$$\frac{\sqrt{x^2 + \epsilon^2} - \epsilon}{x} \in \left[ \frac{\pi_1 - \alpha}{1-(\pi_1 - \alpha)}, \frac{\pi_1 + \alpha}{1-(\pi_1 + \alpha)} \right]$$

and

$$\frac{\sqrt{x^2 + \epsilon^2} - \epsilon}{x} \in \left[ \frac{\pi_2 - \alpha}{1-(\pi_2 - \alpha)}, \frac{\pi_2 + \alpha}{1-(\pi_2 + \alpha)} \right]$$  (3.14)

The fact that this price does not depend on $\pi$ ($e$’s beliefs when he doesn’t know $a$’s signal) is an artifact of the assumption that $\pi = \frac{1}{2}$. The qualitative properties of this equilibrium hold when $\pi \neq \frac{1}{2}$, but the expressions become much more cumbersome.
then there will exist a partially-revealing equilibrium with \( \phi(s_1) = \phi(s_2) = p_{pr} \). This REE price is illustrated in figure 1.

However, as \( \alpha \) decreases toward zero, the condition (3.14) will fail to hold. At the point at which either one of the two inclusions in (3.14) do not hold, there is only a fully-revealing REE and it has the form

\[
p_{fr}(s_1) = \frac{\pi_1 x}{(1 - \pi_1) x + (1 - (\pi_1 - \alpha)) \epsilon} \quad \text{and} \quad p_{fr}(s_2) = \frac{\pi_2 x}{(1 - \pi_2) x + (1 - (\pi_2 - \alpha)) \epsilon}.
\]

(3.15)

Of note is that since in general \( p_{fr}(s_1) \neq p_{fr}(s_2) \neq p_{pr} \), as the amount of ambiguity (parametrized by \( \alpha \)) crosses the threshold at which either of the inclusions in (3.14) doesn’t hold (denoted by \( \bar{\alpha} \) in the diagram), there is a discontinuous jump in the equilibrium prices away from the partially-revealing price \( p_{pr} \) to the two fully-revealing prices \( p_{fr}(s_1) \) and \( p_{fr}(s_2) \). This is demonstrated in figure 1.
One may also construct a (self-fulfilling) fully-revealing REE when the PR-REE exists in the following way. Assuming that a's signal is known to both investors, one can find a different price under each signal that equilibrates the market. The function that maps the signals into these prices is an REE price function with the corresponding allocations. The dotted lines in figure 1 show the signal-dependent prices corresponding to this full revelation REE. Consequently, there always exists a fully-revealing REE in this example and this REE's price correspondence is continuous in $\alpha$.

Having provided some intuition on the nature of equilibrium to be demonstrated, the next section gives the main results of this paper.

4 Partially and fully-revealing REE

This section presents the core results of the paper. We first demonstrate the existence of beliefs which permit a partially-revealing REE in proposition 2. The main result on robustness of these partially-revealing REE will then be presented as theorem 1.

We start with the observation that since expected utility maximizers are a special case of ambiguity averse investors, applying Radner (1979) immediately gives the following result.

Observation 1. There exist beliefs $(\gamma^1, \ldots, \gamma^N, \pi^1, \ldots, \pi^N)$ for which an REE exists.

To demonstrate the existence of partially-revealing REE we start by analyzing the behavior of one ambiguity averse investor, who is called investor 1. In particular, we will investigate the beliefs of investor 1. To show that there exist some beliefs for which there is a partially-revealing REE we need a few preliminary results.

For the next few results we restrict attention to class of ambiguity averse
preferences where the set of beliefs satisfy the following assumption (see for example Siniscalchi (2006), who provides an axiomatization for this class of ambiguity-averse preferences).

**Assumption 2.** For any information set \( f \in F \), the conditional beliefs \( \gamma(f) \) of investor 1 are finitely-generated by at most \( L < \infty \) extreme points.

Given assumption 2 any conditional belief \( \gamma(f) \) can be represented by its extreme points. This set of extreme points is a point in \( \Delta^{\Omega L} \) where \( L \) is the number of extreme points. So, given conditional beliefs \( (\gamma(f), \pi(f)) \in \Gamma^N \times \Pi^N \), let \( W(\gamma(f), \pi(f)) \) be the set of Arrow-Debreu equilibria when all investors hold these updated beliefs.

**Proposition 1.** Let \( \{\gamma^2(f), \ldots, \gamma^N(f), \pi^1(f), \ldots, \pi^N(f)\} \) be conditional beliefs for investors 2, \ldots, N when each investor knows that the joint signal \( \sigma \) is in \( f \). There exists a probability distribution \( \tilde{\pi} \in \Delta^{\Omega} \) such that if investor 1’s beliefs satisfy \( \tilde{\pi} \in \gamma^1(f) \) then all equilibria in \( W(\gamma(f), \pi(f)) \) satisfy \( x^1(\omega) = x^1(\omega') \) for all distinct \( \omega, \omega' \in \Omega \).

**Proof.** Consider a hypothetical economy in which investor 1 has Leontief preferences of the form \( U^1(x) = \min_{\omega \in \Omega} x(\omega) \) and all other investors have beliefs given by \( \{\gamma^2(f), \ldots, \gamma^N(f), \pi^1(f), \ldots, \pi^N(f)\} \).

For any strictly positive price vector \( p \in P \), investor 1’s excess demand is

\[
x^1 = pe^1 - e^1
\]

where \( e \) is the \( |\Omega| \)-dimensional vector of 1’s.

An Arrow-Debreu equilibrium exists for this economy. The proof of existence of the Arrow-Debreu equilibrium is essentially identical to that presented in Mas-Colell, Whinston, and Green (1995)[Proposition 17.C.1]. Let \( Z(p) \) be the excess demand correspondence for this economy. It must be shown that

2. $Z(p)$ is homogeneous of degree zero in $p$.

3. $pZ(p) = 0$ for all $p$.

4. There is a $b > 0$ such that $Z_\omega > -b$ for all $\omega \in \Omega$ and all $p$.

5. If $p^k$ is a sequence of prices converging to $p$ where $p(\omega) = 0$ for some $\omega$ then $\max\{Z(1; p^k), \ldots, Z(|\Omega|; p^k)\} \to \infty$ as $p^k \to p$.

The first four items follow from the fact that each of the preferences for investors in this economy are continuous, strictly convex and locally non-satiated. The final property holds because $x^n(p^k) \to \infty$ for all investors but 1 and her demand $x^1(\omega)$ is bounded below by 0, so excess demand is bounded below by $-\max_{\omega \in \Omega} e^1(\omega)$. Thus, there is an equilibrium $(x, p)$ for which $x^1$ satisfies (4.1).

From the equilibrium in this economy one can derive conditions on the beliefs of investor 1 that ensure that her demand is equal to (4.1) at the equilibrium price. From the first order conditions given in (A.38) it can be seen that if for the equilibrium price $p$, beliefs satisfy $p \in \gamma(f)$ then the full-insurance allocation $x^1$ will be optimal for investor 1. Therefore, any beliefs $\gamma$ that satisfy this condition will guarantee that investor 1 fully insures in this economy.

The previous result demonstrates that for any set of updated beliefs for the other investors in the economy, there is always a set of beliefs for investor 1 with the property that if she holds these beliefs then she will hold a riskless portfolio in equilibrium.

The next task is to demonstrate that from any REE one can construct an REE that reveals less information by allowing a single investor’s preferences to demonstrate sufficient ambiguity aversion.

Definition 3. Let $(x, \phi)$ be a rational expectations equilibrium. The partition induced by the equilibrium $(x, \phi)$ is a partition $\{\Sigma_1, \ldots, \Sigma_K\}$ of $\Sigma$ such that
φ(σ) = φ(σ′) for all σ, σ′ ∈ Σ_k for each k ∈ {1, . . . , K}. An equilibrium (x′, φ′) is said to reveal less information than the equilibrium (x, φ) if the σ-algebra generated by the partition induced by (x′, φ′) is strictly coarser than the σ-algebra generated by the partition induced by (x, φ).

Proposition 2. Let (x, φ) be an REE under beliefs \{γ^1, . . . , γ^N_A, π^1, . . . , π^{N_E}\} and let σ and σ′ be two joint signals that differ only in the private signal of investor 1 (who is assumed to be ambiguity averse). There exists a set of beliefs \{γ^1′, . . . , γ^N_A′, π^1′, . . . , π^{N_E′}\} and an REE that reveals strictly less information than (x, φ).

Proof. The proof is constructive. Let σ′, σ″ ∈ Σ be two joint signals that differ only in 1’s private signal. Denote by (¯x, ¯φ) our candidate partially-revealing REE. For all σ /∈ {σ′, σ″}, let (¯x(σ), ¯φ(σ)) = (x(σ), φ(σ)). Denote by

\{γ^2({σ′, σ″}), . . . , γ^N_A({σ′, σ″}), π^1({σ′, σ″}), . . . , π^{N_E}({σ′, σ″})\} \hspace{1cm} (4.2)

the conditional beliefs for investors 2, . . . , N when they know only that σ ∈ \{σ′, σ″\}. In this equilibrium investors 2, . . . , N are unable to precisely distinguish the signal investor 1 has received. Let (¯x(σ′), ¯φ(σ′)) ∈ W(γ^1′(σ′), γ^2({σ′, σ″}), . . . , π^{N_E}({σ′, σ″})) where γ′ satisfies

\bar{φ}(σ′) ∈ γ′(σ′). \hspace{1cm} (4.3)

This implies that \bar{x}^1(ω; σ′) = \bar{x}^1(ω′; σ′) for all ω, ω′ ∈ Ω. By Proposition I such beliefs exist. Define (¯x(σ″), ¯φ(σ″)) = (¯x(σ′), ¯φ(σ′)). The allocation and price functions (¯x, ¯φ) are a partially-revealing REE if γ′ satisfies

\bar{φ}(σ′) = \bar{φ}(σ″) ∈ γ′(σ′) \cap γ′(σ″). \hspace{1cm} (4.4)

Note that prices are the same across joint signals σ′ and σ″. Since these
prices constitute an Arrow-Debreu equilibrium given the beliefs of each trader 2, . . . , \( N \) when they know only that \( \sigma \in \{ \sigma', \sigma'' \} \), their behavior is optimal under the price \( \bar{\phi}(\sigma') = \bar{\phi}(\sigma'') \).

One last qualification must be made. It is possible that the partially-revealing price \( \bar{\phi}(\cdot) \) has the property that \( \bar{\phi}(\sigma') = \bar{\phi}(\sigma''') \) for some \( \sigma''' \neq \sigma'' \). If this occurs, the proof for the generic existence of fully-revealing REE (appendix A.2) can be adapted to show that this occurs with probability zero and thus perturbing beliefs under the signal \( \sigma''' \) will break the equality.

Finally, since by construction \((\bar{x}(\sigma), \bar{\phi}(\sigma))\) is an Arrow-Debreu equilibrium for each joint signal \( \sigma \), it is an REE and because \( \bar{\phi}(\sigma') = \bar{\phi}(\sigma'') \), it is an REE that reveals less information than \((x, \phi)\).

Nothing in the previous theorem requires that ambiguity averse investors have beliefs that are finitely generated. The only constraint is that there exist an REE in the economy. Thus, if the preferences of all investors were generalized to display ambiguity aversion and it were known that a fully-revealing REE existed, then the method of constructing partially-revealing REE from fully-revealing REE used to prove proposition 2 would apply equally well.

The inclusion (4.4) has economic content. For any price vector \( \phi(\sigma) \in P \), if an investor’s beliefs \( \gamma \) satisfy \( \phi(\sigma) \in \gamma(\sigma) \) then the investor believes that it is possible that the price vector represents the true probability distribution over states in \( \Omega \). That is, the investor believes that the market may have assigned relative prices to assets in these states that are exactly the likelihood of these assets paying off. Since investor 1 is ambiguity averse she has a desire to fully-insure as long as prices do not differ from what she thinks might be the true probabilities over states. The condition (4.4) implies that the information that she receives in her own private signal under both \( \sigma' \) and \( \sigma'' \) does not give her any reason to bet against the odds that the market presents her. This happens...
even though it can be the case that the set of distributions that she believes to be possible given $\sigma'$ and the set she believes to be possible under $\sigma''$ differ greatly. The only requirement is that both $\gamma(\sigma)$ and $\gamma(\sigma')$ have the price vector $\phi(\sigma)$ as a common interior element.

Equation (4.4) can be interpreted as a restriction on the relative informativeness of investor 1’s signal. On the other hand, (4.4) says nothing about the absolute informativeness of 1’s portion of the joint signals $\sigma'$ and $\sigma''$. It is entirely consistent with the previous result to assume that if some other investor knew 1’s private signal he would hold beliefs that are very different than he does when he cannot determine the signal. That is, generically in $\Pi$, $\pi_n^E(\sigma') \neq \pi_n^E(\sigma'') \neq \pi_n^E(\{\sigma', \sigma''\})$, meaning that beliefs under these three different states of information are likely to vary.

It can be seen that the existence of these partially-revealing REE does not fall out of the realm of the results established by Radner (1979). In particular, investor 1 could be an expected utility maximizing investor who happens to believe that $\sigma'$ and $\sigma''$ convey the same information to her. Then her conditional beliefs over $\Omega$ will be the same under these two signals. However, it is apparent that in a world in which all investors maximize expected utility this phenomenon is not robust. That is, perturbing 1’s beliefs slightly must then induce a fully-revealing REE. In a world in which investors are ambiguity averse however, perturbation of beliefs does not necessarily remove the possibility of a partially-revealing REE.

In the following, we use $\text{int}[\gamma(\sigma)]$ to refer to the interior of $\gamma(\sigma)$ as a subset of $\Delta^{\Omega}$.

**Lemma 4.1.** Suppose that for some $p \in P$, $p \in \text{int}[\gamma(\sigma) \cap \gamma(\sigma')]$ for some $\gamma \in \Gamma^{N_A}$ and some $\sigma, \sigma' \in \Sigma$, with $\sigma \neq \sigma'$. Then there exists an open neighborhood $B_\gamma \subset \Gamma^{N_A}$ of beliefs with positive Lebesgue measure in $\Gamma^{N_A}$ and an open neigh-
borhood $B_p \subset P$ with positive Lebesgue measure for which $B_p \in \text{int}[\gamma'(\sigma) \cap \gamma'(\sigma')]$ for all $\gamma' \in B_\gamma$.

**Proof.** This is a direct corollary of Lemma A.11 in the appendix. ■

We now show for a special case that partial revelation is a robust property when there is an ambiguity-averse investor present. We denote by $\Gamma$ the space of ambiguity averse beliefs when $N^A = 1$.

**Proposition 3.** If there is a single ambiguity averse investor and at least one expected utility maximizing investor, then there exists a set $A \in \Gamma \times \Pi^{N^E}$ with positive Lebesgue measure such that for each $(\gamma, \pi) \in A$ there exists a partially-revealing REE.

**Proof.** Let $(\gamma, \pi) \in \Gamma \times \Pi^{N-1}$ permit a partially-revealing REE. By proposition 2 such a $(\gamma, \pi)$ exists. The proof will show that there exists an $\epsilon > 0$ such that for all $(\gamma', \pi')$ that satisfy $||(\gamma', \pi') - (\gamma, \pi)|| < \epsilon$, there is a partially-revealing REE.

From the proof of proposition 2 it can be seen that a sufficient condition, under our assumptions, for a partially-revealing REE to exist is condition (4.4). It will be shown that such a condition holds locally around $(\gamma, \pi)$ by showing that both sides of the inclusion are “continuous” in $(\gamma, \pi)$ in a specific sense. Let $\sigma'$ and $\sigma''$ be two signals for which condition (4.4) hold.

First, it is demonstrated that the equilibrium price function $\phi(\sigma') = \phi(\sigma'')$ is locally continuous around $(\gamma, \pi)$. To see this, notice that for $\sigma \in \{\sigma', \sigma''\}$, $Z(p, \pi(\sigma), \gamma(\sigma))$ is locally differentiable in $p$. Furthermore, by lemma A.11 there exists a neighborhood $B_\gamma$ of $\gamma$ in which $D_\gamma Z(p, \gamma, \pi)$ exists and is the zero matrix. Assume for now that $Z(p, \gamma(\sigma), \pi(\sigma))$ is regular (i.e. $\det D_p Z \neq 0$) at the equilibrium price $p$. Then there exists a neighborhood $B_\pi$ of $\pi$ and $B_\gamma$ of $\gamma$ and a differentiable function $\hat{\phi}(\pi, \gamma)$ such that $Z(\hat{\phi}(\pi, \gamma), \gamma, \pi) = 0$ for all $\gamma', \pi' \in B_\gamma \times B_\pi$. Thus, if $p$ is a regular point then the equilibrium price
function \( \phi \) is locally continuous in \((\gamma, \pi)\) and thus for any \( \epsilon \) there exist \( \delta_\gamma \) and \( \delta_\pi \) such that if \( \pi' \in B(\pi, \delta_\pi) \) and \( \gamma \in B(\gamma, \delta_\gamma) \) then \( ||\phi(\gamma', \pi') - \phi(\gamma, \pi)|| < \epsilon \).

Now, it must be verified that there exists a partially-revealing REE for which the equilibrium price \( \phi(\sigma) = \phi(\sigma') \) is a regular point. To see this, consider an economy populated by one investor with Leontief preferences and \( N^E \geq 1 \) expected utility maximizing investors. Excess demand \( Z(p, \pi) \) for this economy is differentiable for all prices \( p \in \text{int } P \) and is given by

\[
Z(p, \pi) = \sum_{n \in N^E} (x^n(p) - e^n) + pe^1
\]  

(4.5)

where prices are assumed to be normalized to sum to one and \( 1 \) is a vector of ones.

Furthermore, the Jacobian of excess demand is

\[
DZ(p, \pi) = [D_p Z(\cdot), D_\pi Z(\cdot)].
\]  

(4.6)

The term \( D_\pi Z(\cdot) \) has rank \( N^E(|\Omega| - 1) \), which implies that \( DZ(p, \pi) \) always has at least rank \( (|\Omega| - 1) \). Then, by the Transversality Theorem, for almost all \( \pi \), \( \text{det} D_p Z(p, \pi) \neq 0 \) when \( Z(p, \pi) = 0 \). By proposition 2 for all possible beliefs \( \pi \) there exists a partially-revealing REE, so for almost all of these \( Z(p, \pi) \) is determinate which implies that for almost all of these \( \pi \) there exists a function \( \hat{\phi}(\gamma, \pi) \) that is continuous in \((\gamma, \pi)\).

Now, by Lemma 4.1 we may choose neighborhoods \( B_\phi \) around \( \phi(\sigma) \) and \( B_\gamma \) around \( \gamma \) for which \( B_\phi \in \text{int } \gamma' \cap \gamma'(\sigma'') \) for all \( \gamma' \in B_\gamma \). Since \( \phi(\gamma, \pi) \) is continuous in \( \gamma, \pi \), there exists a neighborhood \( B((\gamma, \pi), \epsilon_{\gamma,\pi}) \) such that \( \phi(\gamma', \pi') \in B_\phi \) for all \((\gamma', \pi') \in B((\gamma, \pi), \epsilon_{\gamma,\pi})\).

Combining the continuity of \( \phi \) with Lemma 4.1 choose \( \epsilon \) so that \( B((\gamma, \pi), \epsilon) \cap B_\gamma \subset B_\gamma \) and \( B((\gamma, \pi), \epsilon) \cap B((\gamma, \pi), \epsilon_{\gamma,\pi}) \subset B((\gamma, \pi), \epsilon_{\gamma,\pi}) \). This implies that
for all \((\gamma', \pi') \in B((\gamma, \pi), \epsilon), \phi(\gamma', \pi') \in \gamma'(\sigma') \cap \gamma'('(\sigma''))\) this set contains an open \(\epsilon\)-ball in \(\Gamma \times \Pi^{N_E}\) and thus has positive measure. \(\blacksquare\)

Before proceeding to show the generality of the preceding special case, we note the following result which will be used later.

**Lemma 4.2.** Let \(x(p, \gamma)\) represent demand for an ambiguity averse investor with beliefs \(\gamma\). Assume that \(\gamma\) is finitely-generated by the points \(\{\bar{\pi}, \pi_1, \ldots, \pi_{L-1}\}\) and that for a particular price \(\bar{p}\)

\[
\bar{\pi} \circ u'(x(p, \gamma)) - \lambda p = 0
\]

\(p(e - x(p, \gamma)) = 0\)

\(\bar{\pi} = \arg\min_{\pi \in \gamma} E_{\pi} u(x)\) \hspace{1cm} (4.7)

(that is, \(\bar{\pi}\) is the unique minimizing element of the linear functional \(E_{\pi} u(x(p, \gamma))\)).

Then \(x(p, \gamma)\) is locally differentiable in the vertices \(\{\bar{\pi}, \pi_1, \ldots, \pi_{L-1}\}\) of the beliefs \(\gamma\).

**Proof.** First we demonstrate that \(\bar{\pi}\) is the unique local minimizer (in \(\gamma\)) of \(E_{\pi} u(x)\) if and only if \(E_{\pi} u(x) < E_{\pi_l} u(x)\) for all \(l\). Necessity of the condition follows directly from the definition.

For sufficiency, by assumption we have that \(E_{\pi} u(x(p, \gamma)) < E_{\pi} u(x(p, \gamma))\) for all \(\pi \in \gamma\) with \(\pi \neq \bar{\pi}\). Since every \(\pi \in \gamma\) can be characterized as a linear combi-
nation $\alpha \bar{\pi} + \sum_{l=1}^{L-1} \alpha_l \pi_l$ with $\sum_{l=0}^{L-1} \alpha_l = 1$, we have that for $\alpha \neq (1, 0, \ldots, 0)$

$$E_\pi u(x) = \sum_{\omega \in \Omega} \pi(\omega)u(x(\omega))$$

$$= \sum_{\omega \in \Omega} \left[ \alpha_0 \bar{\pi}(\omega) + \sum_{l=1}^{L-1} \alpha_l \pi_l(\omega) \right] u(x(\omega))$$

$$= \alpha_0 \sum_{\omega \in \Omega} \bar{\pi}(\omega)u(x(\omega)) + \sum_{\omega \in \Omega} \sum_{l=1}^{L-1} \alpha_l \pi_l(\omega)u(x(\omega)) \quad (4.8)$$

$$\geq \alpha_0 \sum_{\omega \in \Omega} \bar{\pi}(\omega)u(x(\omega)) + \sum_{\omega \in \Omega} \sum_{l=1}^{L-1} \alpha_l \pi_l(\omega)u(x(\omega))$$

$$= \sum_{\omega \in \Omega} \bar{\pi}(\omega)u(x(\omega)).$$

Thus, if $\bar{\pi}$ minimizes $E_\pi u(x)$ over $\{\bar{\pi}, \pi_1, \ldots, \pi_{L-1}\}$ then $\bar{\pi}$ minimizes $E_\pi u(x)$ over $\gamma$.

Now, let $x(p, \pi)$ represent the demand of an expected utility maximizing investor who holds beliefs $\pi$ and has the same endowment and von Neumann-Morgenstern utility function as the ambiguity averse investor who has demand $x(p, \gamma)$. It will be shown that if $\gamma(\pi, \pi_1, \ldots, \pi_{L-1})$ are the beliefs of the ambiguity averse investor generated by the set of points $\{\pi, \pi_1, \ldots, \pi_{L-1}\}$, then in a neighborhood of $\pi$ we have that $x(p, \pi) = x(p, \gamma(\pi, \pi_1, \ldots, \pi_{L-1}))$. That is, $x(p, \gamma(\pi, \pi_1, \ldots, \pi_{L-1}))$ is locally differentiable in $\pi$.

Applying the implicit function theorem to this expected utility investor’s first order conditions shows that $x(p, \pi)$ is differentiable (and hence continuous) in $\pi$. Furthermore $x(p, \bar{\pi}) = x(p, \gamma(\bar{\pi}, \pi_1, \ldots, \pi_{L-1}))$ by assumption. Since $E_\pi u(x)$ is continuous in $\pi$ and $x$ (by the continuity of $u$), we have that for every $\pi'$ in some open neighborhood $B_{\bar{\pi}}$ of $\bar{\pi}$

$$E_{\pi'} u(x(p, \pi')) < E_{\pi} u(x(p, \pi')),$$ for all $l \in \{1, \ldots, L-1\}. \quad (4.9)$$
By the definition of $x(p, \pi)$, for all $\pi' \in B_{\bar{\pi}}$

$$\pi' \circ u(x(p, \pi')) - \lambda p = 0$$

$$p(e - x(p, \pi')) = 0. \quad (4.10)$$

Investigating the first order conditions for the ambiguity averse investor given
in Corollary 2 of section A.1 of the appendix we see that for all $\pi' \in B_{\bar{\pi}}$, the
investor’s first order conditions are satisfied by the allocation $x(p, \pi')$.7 Thus
$x(\pi, \gamma(\pi, \pi_1, \ldots, \pi_{N-1})) = x(p, \pi)$ in $B_{\bar{\pi}}$ and is hence differentiable in $\pi$.

To see that it is locally differentiable in the other vertices of the set $\gamma$,
simply note that locally changing these vertices does not change the minimizing
distribution in the set $\gamma$ and hence the derivative with respect to these vertices
is zero. ■

And finally, we present the main result of the paper.

**Theorem 1.** If assumption 3 holds, there exists a set $A$ of positive measure in
$\Gamma^{N_A} \times \Pi^{N_E}$ such that for each $(\gamma, \pi) \in A$, a partially-revealing REE exists.

**Proof.** Suppose $(x, \phi)$ is a partially-revealing REE, under the beliefs $(\gamma^1, \ldots, \gamma^{N_A}, \pi^1, \ldots, \pi^{N_E})$, that does not distinguish the distinct signals $\sigma$ and $\sigma'$. Suppose further that $\sigma$ and $\sigma$ differ only in the signal received by investor 1. By proposition 3 such REE exist. Then, for all ambiguity averse investors other than investor 1, one may assign beliefs such that $(x, \phi)$ is still an equilibrium, but for which $x$ is locally differentiable in $\pi$. To see how this is done, consider the first order conditions for ambiguity averse investors given in corollary 2 in the appendix.

Let $\bar{\pi}$ be the unique set of beliefs satisfying

$$\lambda p = \pi \circ u'(x) \quad (4.11)$$

---

7In terms of the Corollary, from equation (4.9) we see that $\gamma_0(x(p, \pi')) = \pi'$ in the neighborhood $B_{\bar{\pi}}$.
Then we define a closed, convex set of beliefs $\gamma(\sigma)$ which has the property that

$$E_{\bar{\pi}} u(x) < E_{\pi} u(x) \text{ for all } \pi \in \gamma(\sigma), \pi \neq \bar{\pi}. \quad (4.12)$$

To see that this can be done, we notice that the set of beliefs satisfying equation (4.12) is the intersection of an $|\Omega|$-dimensional half space and the $|\Omega| - 1$ dimensional simplex in $\mathbb{R}^{|\Omega|}$. Then select a finite set of points $(\pi_1, \pi_2, \ldots, \pi_{L-1})$ such that for each $l \in \{1, \ldots, L - 1\}$ equation (4.12) holds.

By the proof of Lemma 4.2 we have that $x^n(p, \gamma)$ is locally differentiable in the vertices that define $\gamma$ for all $n \geq 2$. Thus, the result from Proposition 3 applies and demonstrates the existence of an open ball in $\Gamma_{N}^{A} \times \Pi_{N}^{E}$ for which there is a partially-revealing REE. ■

Since the set of beliefs $(\gamma, \pi)$ for which such a partially-revealing REE exists has positive Lebesgue measure, it is not an artifact of carefully chosen model parameters. In this sense, the inclusion of ambiguity-averse investors in a traditional REE framework provides for fundamentally different equilibria than those robustly found when all investors have subjective expected utility preferences.

It is worth pointing out that assumption 2 is needed for the application of the Transversality Theorem. Although we restrict the attention to beliefs that satisfy this assumption, the existence of a partially-revealing REE can be established without it. It may also be possible to establish that the inclusion condition (4.4) is robust without it. To our knowledge, there is no version of the Transversality Theorem that can be applied when the space of parameters (the space of conditional beliefs in our case) is infinite dimensional and the function of interest is not Frechet-differentiable in the parameter.

As mentioned earlier, Ausubel (1990) provides an interesting class of economies without ‘noise’, where partial revelation is robust. However, in that model, the closest possibility to our knowledge is theorem 3.7 in Shannon (2006) which requires Frechet equi-differentiability.
unlike the present framework, the space of private information has higher dimension than the space of prices. In particular, the state space, which is the same as the space of private information in Ausubel (1990), is described by two random variables one of which is continuously distributed while the other takes on elements in a finite set. The investors in the model conform to the Savage expected utility framework, but have state-dependent vonNeumann-Morgenstern utilities, and are either informed about one or both of the random variables or completely uninformed.

In contrast, here, the space signals and that of payoff-relevant is assumed to be finite. Indeed, we operate within the framework of Radner (1979), where only full revelation is generic with Savage expected utility investors, except for allowing ambiguity averse investors. We also allow for possibly larger information (non) revelation by allowing for the private signals to be arbitrary across investors, other than the restriction that they take on values in a finite set.

A similarity between the partial revelation equilibria of Ausubel (1990) and the ones constructed here is that the REE prices are two-to-one mappings. There is partial revelation in that two signals (or states of information) are not revealed by the equilibrium prices. While proposition 2 here provides an existence result, it is clear how one could increase the cardinality of the set of signals that are not revealed for the present model. The construction would work along the lines of the inclusion condition (4.4) and the increase in non-revelation could be due to the same ambiguity averse investor or due to two (or more) ambiguity averse investor. In Ausubel (1990), there may exist other partial revelation REE that reveal less than the ones constructed explicitly, but it is not clear how to obtain them.

Ausubel (1990) does establish that the REE with price functions that are

9However, the condition would become harder to satisfy.
10One possibility of course is to expand the space of private information.
two-to-one mappings are the only REE within a subclass of the economies he considers. We do not establish uniqueness of the partial revelation REE that we construct. Indeed, one would expect full revelation REE to exist in our adaptation of the Radner (1979) framework and as we show in appendix A.2, these will exist generically for an interesting subclass of the economies we consider. Also, we conjecture that partial revelation equilibria with ambiguity averse investors may take on other forms than the one we establish using full-insurance in the present setting.

Before concluding, we point to three papers that are of related interest. Ozsoylev and Werner (2007) introduce an ambiguity averse uninformed trader into an extension of the basic set-up of Grossman and Stiglitz (1980) that allows for ambiguity. They find that the unique partially revealing REE is piecewise linear unlike the unique linear partial revelation REE price in Grossman and Stiglitz (1980). However, the partial revelation in the basic model of Ozsoylev and Werner (2007) occurs due to ‘noise traders’ as in Grossman and Stiglitz (1980). When they introduce an ambiguity averse informed trader, they achieve partial revelation without ‘noise’ in a manner analogous to the more general results here.

One of the earliest papers that introduced ambiguity-averse traders into the REE framework is Tallon (1998). He showed by example that an ambiguity-averse uninformed trader would buy ‘redundant’ private information even when the REE price is fully-revealing. This suggests that ambiguity aversion might also help understand the Grossman and Stiglitz (1980) paradox - no investor will have an incentive to acquire private information if all information is revealed through the equilibrium price system - in a manner that is different from partial revelation.

\footnote{We conjecture that full revelation will be generic in the entire class of economies we consider, but have not been able to establish this so far.}
Epstein and Schneider (forthcoming) assess the impact of ambiguous information on portfolio choice and asset prices in a dynamic representative investor setting. They find several interesting features such as stronger portfolio reactions to bad news than good, ambiguity premia that depend on idiosyncratic risk and skewness in returns, and persistent negative effects on prices from negative shocks to information quality without changes in fundamentals. These features and our results on the informational role of prices in the presence of ambiguity averse investors suggest that the effects of ambiguous information in asset markets is a promising area of future research.

5 Conclusion

This paper has shown that when the REE concept is extended to include investors whose preferences are not of the expected utility form that the information content of prices can vary drastically. It is clear that the implications of the presence of ambiguity averse agents in financial, insurance or other information-centric markets are yet to be fully understood. Models that explicitly allow for the presence of asymmetric information and ambiguity aversion in such markets may yield results that differ from those of more “traditional” models. In such markets it is not always (or even almost always) true that all privately held information is revealed through market prices, a result that stands in sharp contrast to previous work.

While we focus on the multiple-priors model developed by Gilboa and Schmeidler (1989), the methods we use to construct partially revealing REE can be extended to other models of decision-making. Some possibilities include convex preferences that are represented by the Choquet expected utility model of Schmeidler (1989) or by the $\alpha$-maxmin model of Ghirardato, Maccheroni, and Marinacci (2004).
A Appendix

A.1 Decision making under ambiguity

The goal of this appendix is to collect several results which are used throughout the paper. Some of the results may be found in Aubin (1979) and in cases where the proof is not expositionally important and can be found elsewhere we have cited an appropriate reference.

For what follows, let $X = \mathbb{R}^W_+$ and let $f : X \to \mathbb{R}$ be a continuous and concave function. The set $\partial f(x_0)$ for $x_0 \in X$ is defined to be

$$\partial f(x_0) = \{ p \in \mathbb{R}^W : f(x) - f(x_0) \leq p \cdot (x - x_0) \text{ for all } x \in X \} \quad (A.1)$$

is called the superdifferential of $f$ at $x_0$.

It can be seen that if $x_0$ maximizes the function $f$ over the set $X$ and $x_0 \in \text{int} X$ then

$$0 \in \partial f(x_0). \quad (A.2)$$

If $f$ is a strictly concave function and has a maximum in $\text{int} X$ then the condition (A.2) is both necessary and sufficient for $x_0$ to be a maximum.

The rest of the results are applied specifically to the utility functions of interest, i.e. they apply for functions of the form $U(x) = \min_{\pi \in \Gamma} E_\pi u(x) = \min_{\pi \in \Gamma} \sum_{\omega \in \Omega} \pi(\omega)u(x(\omega))$.

Lemma A.1. If $u(\cdot)$ is strictly increasing and strictly concave then $U(x)$ is increasing in each of its arguments and strictly concave. Furthermore, if $\gamma \cap \text{bd} \Delta^{[\Omega]} = \emptyset$, then $U(x)$ is strictly increasing in each of its arguments.

Proof. Let $x, x'$ and $\omega'$ be such that for all $\omega \neq \omega'$, $x(\omega) = x'(\omega)$ but $x(\omega') < x'(\omega')$. Suppose that $U(x') < U(x)$ and let $\pi^* \in \arg \min_{\pi \in \Gamma} \sum_{\omega \in \Omega} \pi(\omega)u(x'(\omega))$. 

32
Then
\[
\min_{\pi \in \gamma} \sum_{\omega \in \Omega} \pi(\omega)u(x'(\omega)) < \min_{\pi \in \gamma} \sum_{\omega \in \Omega} \pi(\omega)u(x(\omega)) \\
\leq \sum_{\omega \in \Omega} \pi^*(\omega)u(x(\omega)).
\]

This last inequality implies that
\[
0 < \sum_{\omega \in \Omega} \pi^*(\omega)(u(x(\omega)) - u(x'(\omega))) \tag{A.3}
\]
which with the definition of \(\pi^*\) gives
\[
0 < \pi^*(\omega')(u(x(\omega')) - u(x'(\omega'))) \tag{A.4}
\]
which is a contradiction since \(u(\cdot)\) is strictly increasing. Notice that if \(\gamma \cap \text{bd } \Delta^{[\Omega]} = \emptyset\) and we start with the weak inequality \(U(x') \leq U(x)\) then all of the strict inequalities become weak and the final inequality can be satisfied only for \(\pi^*(\omega) = 0\) which is again a contradiction.

To show strict concavity, notice that for each \(\pi \in \gamma\), the function \(\sum_{\omega \in \Omega} \pi(\omega)u(x(\omega))\) is strictly concave. Define \(\pi^*(x) \in \arg \min_{\pi \in \gamma} \sum_{\omega \in \Omega} \pi(\omega)u(x(\omega)).\)
Let \( x, x' \in X \). Then for \( \alpha \in (0, 1) \) we have that

\[
\alpha U(x) + (1 - \alpha) U(x') = \alpha \min_{\pi \in \gamma} \sum_{\omega \in \Omega} \pi(\omega) u(x(\omega)) + (1 - \alpha) \min_{\pi \in \gamma} \sum_{\omega \in \Omega} \pi(\omega) u(x'(\omega)) \quad (A.5)
\]

\[
\leq \min_{\pi \in \gamma} \left\{ \alpha \sum_{\omega \in \Omega} \pi(\omega) u(x(\omega)) + (1 - \alpha) \sum_{\omega \in \Omega} \pi(\omega) u(x'(\omega)) \right\} \quad (A.6)
\]

\[
= \min_{\pi \in \gamma} \left\{ \sum_{\omega \in \Omega} \pi(\omega) [\alpha u(x(\omega)) + (1 - \alpha) u(x'(\omega))] \right\} \quad (A.7)
\]

\[
< \min_{\pi \in \gamma} \left\{ \sum_{\omega \in \Omega} \pi(\omega) (\alpha x(\omega) + (1 - \alpha) x'(\omega)) \right\} \quad (A.8)
\]

\[
= U(\alpha x + (1 - \alpha) x'). \quad (A.9)
\]

Equation \( (A.6) \) holds by definition. Equations \( (A.7) \) and \( (A.8) \) follow from the definition of the minimum and an algebraic manipulation. \( (A.9) \) follows from the strict concavity of \( u(\cdot) \) and \( (A.10) \) again holds by definition. \( \blacksquare \)

For the remainder of the appendix we will assume that the conditions present in assumption 1 hold. The next lemma is a direct result of these assumptions.

**Lemma A.2.** Assume that for investor \( n \), Assumption 1 in the body of the paper holds. Then for any price vector \( p \in \text{int} X \), if

\[
x \in \arg\max_{x \in B(e^n, p)} U^n(x) \quad (A.11)
\]

then \( x \in \text{int} X \).

**Proof.** Assumption 1 ensures that the Inada condition at 0 holds for every possible state \( \omega \). \( \blacksquare \)

**Lemma A.3.** For the strictly concave function \( U \) and a compact, convex set \( Y \subset X \) that is defined by the system of linear inequalities \( g(x) \geq 0 \) there exists
a Lagrange multiplier $\lambda \in \mathbb{R}_+$ such that

$$\arg\max_{x \in X} L(x, \lambda) = U(x) + \lambda g(x) = \arg\max_{x \in Y} U(x). \quad (A.12)$$

In particular, for each investor $n$ there exists a Lagrange multiplier $\lambda^n$ such that

$$\max_{x \in B(e^n, p)} U^n(x) = \max_{x \in X} U^n(x) + \lambda^n p(e^n - x) = L^n(x, \lambda^n) \quad (A.13)$$

Furthermore, if $u^n(\cdot)$ is strictly concave then $L(x, \lambda)$ is also strictly concave in $x$.

**Proof.** $U^n$ is continuous and concave and meets the necessary constraint qualification in (Aubin 1979)[5.3.1, Theorem 1]

Let $L(x, \lambda) = U(x) + \lambda p(e - x)$. By the strict concavity of $U(x)$ and the linearity of the constraint it can be seen that $L(x, \lambda)$ is strictly concave in $x$.

As such a necessary condition for a solution to max $L(x, \lambda)$ for a given $\lambda$ is that

$$0 \in \partial L(x, \lambda) \quad (A.14)$$

To derive the first order conditions for an ambiguity averse investor the following lemma is needed.

**Lemma A.4.** Let $f : X \to \mathbb{R}$ be superdifferentiable and $g : X \to \mathbb{R}$ be affine (and thus Gateaux differentiable). If $L = f + g$ then

$$\partial L(x) = \partial f(x) + \partial g(x) = \partial f(x) + g'(x). \quad (A.15)$$

**Proof.** It is straightforward to show that $\partial f(x) + \partial g(x) \subseteq \partial L(x)$. What remains to be shown is that $\partial L(x) \subseteq \partial f(x) + \partial g(x)$.

Suppose that there exists $p \in \partial L(x)$ such that $p \notin \partial f(x) + \partial g(x)$. By
Since $g$ is Gateaux differentiable, $\partial g(x) = \{g'(x)\}$ (i.e. it is a single vector in $\mathbb{R}^W$ labeled $g'(x)$.) Without loss of generality one may define $p = g'(x) + p'$. Since $p' + g'(x) \not\in \partial f(x) + \partial g(x) = \partial f(x) + g'(x)$ this implies that $p' \not\in \partial f(x)$. Therefore, there exists a $y^* \in X$ such that

$$f(y^*) - f(x) > p'(y^* - x). \quad (A.17)$$

Since $g(x)$ is affine,

$$g(y^*) - g(x) = g'(x)(y^* - x). \quad (A.18)$$

Summing (A.17) and (A.18) yields

$$f(y^*) + g(y^*) - f(x) - g(x) > (p' + g'(x))(y^* - x) \quad (A.19)$$

which contradicts (A.16). 

Of note in the previous lemma is the fact that it is not true in general that $\partial L(x) = \partial f(x) + \partial g(x)$ when $g$ is not affine.

Applying the lemma [A.4] implies the following necessary condition for ambiguity averse investors.

$$0 \in \partial U(x_0) - \lambda p. \quad (A.20)$$

Assumption 1 ensures that all optimal allocations are interior. This fact, combined with the previous lemma implies the following result.
Proposition 4. Let $x_0$ be a solution to the problem

$$\max_{x \in B(e,p)} U(x)$$ (A.21)

If $x_0 \in \text{int } X$ then

$$\lambda p \in \partial U(x_0)$$ (A.22)

$$p(e - x) = 0$$

The condition (A.22) are necessary for utility maximization. If $U(\cdot)$ is strictly concave then these will also be sufficient.

The next corollary provides the intuition behind the robustness of the partially-revealing REE discussed in the paper. Essentially, it gives a sufficient condition for two investors to behave identically in equilibrium.

Corollary 1. Suppose that for two investors $m$ and $n$, $e^m = e^n$ and that for the allocation $x_0$, $\partial U^m(x_0) \subseteq \partial U^n(x_0)$. Then if $x_0$ is a solution to

$$\max_{x \in B(e^m,p)} U^m(x)$$ (A.23)

then it is also a solution to

$$\max_{x \in B(e^n,p)} U^n(x)$$ (A.24)

Proof. If (A.22) holds for investor $m$ then it must hold for investor $n$ as well since the budget constraints are the same and the Euler condition (A.20) is a weaker condition for investor $n$. ■

To obtain an idea of the geometry of the set $\partial U(c)$ it is helpful to characterize it in terms of quantities that are somewhat easier to calculate.

**Definition 4.** For the continuous function $U(x)$ define the **derivative from the**
right of $U$ at $x_0$ in the direction $y$ to be

\[
DU(x_0)(y) = \lim_{\alpha \to 0^+} \frac{U(x_0 + \alpha y) - U(x_0)}{\alpha}
\]  

(A.25)

Of note in the preceding definition is that $y$ need not be in $X$, although for sufficiently small $\alpha$ it must be true that $x_0 + \alpha y \in X$ since $U$ is only defined over $X$.

**Lemma A.5.**

\[
DU(x_0)(y) = \min_{p \in \partial U(x_0)} py
\]  

(A.26)

*Proof.* Aubin (1979)[Section 4.3.2, proposition 4]

Therefore we have that for $x_0$

\[
\partial U(x_0) = \{ p \in \mathbb{R}_+^W : py \geq D(x_0)(y) \text{ for all } y \in \mathbb{R}^W \}
\]  

(A.27)

The function $DU(x)(\cdot)$ is the support function of the $\partial U(x)$. Since compact and convex sets may be completely characterized as the intersection of all of the half spaces in which they are contained and the support function characterizes such half spaces, we turn to the function $DU(x)(\cdot)$ to better understand $\partial U(x)$.

**Lemma A.6.** Let $\gamma_0(x) = \{ \pi \in \gamma : U(x) = E_{\pi} u(x) \}$. Then

\[
DU(x_0)(y) = \min_{\pi \in \gamma_0(x)} DE_{\pi} u(x)(y)
\]  

(A.28)

*Proof.* Aubin (1979)[Section 4.3.3, proposition 6]
Standard calculus shows that
\[
DE\pi u(x)(y) = \lim_{\alpha \to 0^+} \frac{E\pi u(x + \alpha y) - E\pi u(x)}{\alpha} = \frac{d}{d\alpha} E\pi u(x + \alpha y)|_{\alpha=0^+} = E\pi u'(x) y
\]  
(A.29)

Applying this to the definition of \( \partial U(x_0) \) gives
\[
\partial U(x_0) = \{ p \in \mathbb{R}^W_+ : py \geq \min_{\pi \in \gamma_0(x)} E\pi u'(x)(y) \text{ for all } y \in \mathbb{R}^W \}. \quad (A.30)
\]

From this, the superdifferential for some allocations can be calculated directly.

**Lemma A.7.** Consider an investor with beliefs given by \( \gamma \in \mathcal{C}(\Delta|\Omega|) \). Let \( x \) be an allocation such that \( x(\omega) = x(\omega') \) for all \( \omega, \omega' \in \Omega \).

\[
\partial U(x) = u'(x) \circ \gamma. \quad (A.31)
\]

**Proof.** Noting that \( u'(x(\omega)) = u'(x(\omega')) \) for all \( \omega, \omega' \) and applying equation \( (A.30) \) gives
\[
\partial U(x) = \{ p \in \mathbb{R}^W_+ : py \geq u'(x) \min_{\pi \in \gamma_0(x)} E\pi y \text{ for all } y \in \mathbb{R}^W \}. \quad (A.32)
\]

As defined in lemma \( A.6 \), \( \gamma_0 = \gamma \) for the allocation \( x \) since all probability distributions in \( \gamma \) give the same expected value for the constant random variable \( u(x) \). By definition, \( E\pi y = \pi y \) so one can rewrite equation \( (A.32) \) as
\[
\partial U(x) = \{ p \in \mathbb{R}^W_+ : \frac{py}{u'(x)} \geq \min_{\pi \in \gamma} \pi y \text{ for all } y \in \mathbb{R}^W \}. \quad (A.33)
\]

But the right hand side of the inequality in the definition is the support function

39
for the set $\gamma$. As such, is equivalent to $pu'(x) \in \gamma$. ■

Alternatively, if for a particular allocation $x$ the set $\gamma_0(x)$ is a singleton, the equation \text{(A.30)} reduces to the single vector

$$\partial U(x) = \{[\pi(\omega)u'(x(\omega))]_{\omega \in \Omega}\}. \quad \text{(A.34)}$$

The general case follows.

**Lemma A.8.** The superdifferential of $U$ at the point $x$ is given by

$$\partial U(x) = u'(x) \circ \gamma_0(x) \quad \text{(A.35)}$$

**Proof.** The following manipulation of the definition of $\partial U(x)$ gives the result.

$$\partial U(x) = \{p \in \mathbb{R}^W_+: py \geq \min_{\pi \in \gamma_0(x)} E_\pi u'(x)y \text{ for all } y \in \mathbb{R}^W\}$$

$$= \{p \in \mathbb{R}^W_+: py \geq \min_{\pi \in \gamma_0(x)} [\pi \cdot u'(x)]y \text{ for all } y \in \mathbb{R}^W\} \quad \text{(A.36)}$$

$$= \{p \in \mathbb{R}^W_+: py \geq \min_{q \in u'(x)-\gamma_0(x)} qy \text{ for all } y \in \mathbb{R}^W\}$$

■

**Corollary 2.** Let $n$ be an investor whose preferences satisfy Assumption 1. An allocation $x_0 \in X$ is a solution to the problem

$$\max U^n(x) \text{ s.t. } x \in B(e^n, p) \quad \text{(A.37)}$$

if and only if

$$\lambda p \in \gamma_0(x_0) \circ u'(x_0) \quad \text{(A.38)}$$

$$p(e - x_0) = 0$$

**Corollary 3.** Let $n$ and $m$ be two different investors having beliefs $\gamma^m$ and $\gamma^n$, but identical vN-M utility functions $u(\cdot)$ and equal endowments ($e^m = e^n$). If
\[ \gamma^m_0(x) \subseteq \gamma^n_0(x) \text{ then if } x \in X \text{ solves } \]
\[ \max U^m(x) \text{ s.t. } x \in B(e^m, p) \]  
\[ \text{(A.39)} \]
then \( x \) also solves
\[ \max U^n(x) \text{ s.t. } x \in B(e^n, p) \]  
\[ \text{(A.40)} \]

**Proof.** Inspecting equations (A.22) shows that any solution to these equations for beliefs \( \gamma^m \) is also a solution for beliefs \( \gamma^n \). \[ \blacksquare \]

**Proposition 5.** Let \((x, p) \in X^N \times P\) be an Arrow-Debreu equilibrium for the economy characterized by \( g = (\gamma^1, \ldots, \gamma^N) \in \Gamma^N \). If \( \hat{g} = (\hat{\gamma}^1, \ldots, \hat{\gamma}^N) \) satisfies \( \gamma^n_0(x^n) \subseteq \hat{\gamma}^n_0(x^n) \) then \((x, p)\) is an equilibrium for the economy \( \hat{g} \) as well.

**Proof.** Applying corollary \[ \blacksquare \] for each investor shows that the allocation \( x \) continues to be utility maximizing for all investors. Markets must also clear since \((x, p)\) is an equilibrium for the economy \( g \). \[ \blacksquare \]

**Lemma A.9.** Assume that \( e^n \gg 0 \) for all \( n \), and that each investor satisfies assumption \[ \blacksquare \]. Let \( g = \{\gamma^1, \ldots, \gamma^n\} \in G^n \). Let \( \mathcal{E}(g) \) be an economy where beliefs for each investor \( n \) are given by \( g(n) \). Then a Walrasian equilibrium exists for each economy \( \mathcal{E}(g) \) for all \( g \in G^n \).

**Proof.** It suffices to check that the conditions presented in (Debreu 1959)[Section 5.7] are satisfied for every economy \( \mathcal{E}(g) \). It is straightforward to show that

1. \( X \) is closed, convex and has a lower bound for each investors’ preference ordering

2. No investor is satiated in \( X \)

3. For each \( x' \in X \) the sets \( \{x \in X \mid U^n(x) \geq U^n(x')\} \) and \( \{x \in X \mid U^n(x) \leq U^n(x')\} \) are closed for all investors \( n \).
4. If $U^n(x') \geq U^n(x)$ then $U^n(tx' + (1-t)x) \geq U^n(x)$ for $t \in (0, 1)$.

5. There exists $x \in X$ such that $x \ll e^n$ for all $n$ (by assumption).

For an endowment economy these conditions are sufficient to guarantee the existence of equilibrium. ■

**Lemma A.10.** For any economy $E(g)$ let $E : G \rightarrow P$ represent the equilibrium price correspondence. Then $E(g)$ is upper-hemicontinuous in $g$.

**Proof.** Let $g^k$ be a sequence of economies converging to $g$ and let $p^k$ be the equilibrium price vector for economy $g^k$. Then $Z(p^k, g^k) = 0$ for all $k$. By the continuity of $Z(\cdot, g^k)$ if $p^k$ converges then its limit is also a market clearing price. ■

From general equilibrium results we now move on to more specific results on the space of beliefs for both ambiguity averse and expected utility investors. In order to analyze the size of the set of economies for which an REE exists it is necessary to investigate the properties of the space of economies that are considered in this paper. The fact that partially-revealing REE occur for a set of beliefs that has positive measure requires that this set of beliefs be explicitly described.

First a characterization of the beliefs of the ambiguity averse investor is given and it is shown that these beliefs can be embedded in a finite-dimensional Euclidean space. This embedding allows us to use Lebesgue measure on the space of beliefs. After characterizing the beliefs of the ambiguity averse investor the space of beliefs for expected utility maximizing investors will be characterized.

As noted in Assumption 2 the conditional beliefs of investor 1 can always be represented as the convex hull of a finite set of extreme points. Therefore, if the beliefs of investor 1 are given by $\{\gamma(\sigma)\}_{\sigma \in \Sigma}$, each of these conditional beliefs can be characterized by a set of $L$ probability distributions in $\Delta^{(\Omega)}$. So for any
joint signal $\sigma$, beliefs for the investor can be identified with an element of the space $\Delta^{[O][L]}$. Since the cardinality of the set of joint signals is $|\Sigma|$, the beliefs for investor 1 can be characterized by an element of the space $\Delta^{[O][L][\Sigma]}$. It is clear that any point in $\Delta^{[O][L][\Sigma]}$ represents a set of beliefs that meet our assumptions for investor 1. However, since the order of the distributions $(\pi_1, \ldots, \pi_L)$ that define $\gamma(\sigma)$ doesn’t matter when generating the convex hull of these points, there are always multiple elements of $\Delta^{[O][L][\Sigma]}$ that represent the same beliefs over $\Omega$.

The fact that the map from points in $\Delta^{[O][L][\Sigma]}$ into beliefs on $\Sigma$ is not injective will not matter for the applications in this paper. To ease the exposition, for this appendix we will abuse notation slightly and let $\Gamma = \Delta^{[O][L][\Sigma]}$ keeping in mind that $\Gamma$ as defined in the body is the function space of beliefs related to the set $\Delta^{[O][L][\Sigma]}$.

This formulation of beliefs is used because it allows for the discussion of convergence of beliefs in the traditional sense. A sequence $\gamma^k \subseteq \Gamma$ is said to converge to some element $\gamma \in \Gamma$ if it converges in the standard (Euclidean, metric) sense.

To obtain the robustness results of Lemma 4.1 which is in turned used to prove Theorem 3, we need the following lemma.

**Lemma A.11.** Let $\gamma$ be any set of beliefs in $\Gamma$. Let $\pi \in \text{int}[\gamma(\sigma) \cap \gamma(\sigma')]$ for some $\sigma, \sigma' \in \Sigma$. Then there exist open neighborhoods $B(\gamma, \epsilon_\gamma) \subset \Gamma$ and $B(\pi, \epsilon_\pi) \subset \Delta^{[O]}$ such that for all $\gamma' \in B(\gamma, \epsilon_\gamma)$, $B(\pi, \epsilon_\pi) \subset \text{int}[\gamma'(\sigma) \cap \gamma'(\sigma')]$.

**Proof.** Define

$$d(\gamma(\sigma), B(\pi, \epsilon_\pi^1)) = \inf_{\pi' \in \text{bd} \gamma(\sigma), \pi \in B(\pi, \epsilon_\pi)} ||\pi' - \pi||. \quad (A.41)$$

Without loss of generality we may select $\epsilon_\pi^1$ so that $d(\gamma(\sigma), B(\pi, \epsilon_\pi^1)) > 0$. This implies that if $\overline{B}(\pi, \epsilon_\pi^1)$ is the closure of $B(\pi, \epsilon_\pi^1)$ then $\overline{B}(\pi, \epsilon_\pi^1) \subset \text{int} \gamma(\sigma)$. 43
It can be seen that 
\[ d(\gamma(\sigma), B(\pi, \epsilon_\pi^1)) \geq d(\gamma(\sigma), \bar{B}(\pi, \epsilon_\pi^1)). \]

The function \( d(\gamma(\sigma), B) \) is continuous in \( \gamma \) if \( B \) is a compact set.\(^{12}\) To see this, let \( \{\pi_1, \ldots, \pi_L\} \) be the extreme points of the set \( \gamma(\sigma) \) and \( Q \) be the finite set of facets (\( L - 1 \) dimensional faces) of the convex polytope \( \gamma(\sigma) \), equation (A.41) can be rewritten as

\[
d(\gamma(\sigma), B) = \min_{\pi' \in \partial \gamma(\sigma), \pi \in B} \|\pi' - \pi\|
\]

\[
= \min_{q \in Q, \pi \in B} \|q - \pi\|
\]

\[
= \min_{q \in Q} \min_{\pi' \in q, \pi \in B} \|\pi' - \pi\|
\]

\[
= \min_{q \in Q} \min_{\alpha \in \Delta^L, \pi \in B} \left\| \sum_l \alpha_l \pi_l - \pi \right\|
\]

The facets of \( \gamma(\sigma) \) are found by taking convex combinations of sets of \( (L - 1) \) vertices of \( \gamma(\sigma) \). For each facet \( q \), the function \( \min_{\pi' \in q, \pi \in B} \|\pi' - \pi\| = \min_{\alpha \in \Delta^L, \pi \in B} \left\| \sum_l \alpha_l \pi_l - \pi \right\| \) is absolutely continuous over \( q \) in its vertices (since \( q \) is a compact set). Therefore the distance \( \min_{q \in Q} \|q - \pi\| \) is the minimum of a finite set of absolutely continuous functions and is therefore absolutely continuous in the set of vertices.

Since \( d(\gamma(\sigma), B) \) is continuous in \( \gamma \), there exists a neighborhood \( B(\gamma, \epsilon_\gamma^1) \) such that for all \( \gamma' \in B(\gamma, \epsilon_\gamma^1) \), \( d(\gamma'(\sigma), \bar{B}(\pi, \epsilon_\pi^1)) > 0 \). From this it can be shown that if \( \gamma' \in B(\gamma, \epsilon_\gamma^1) \), then \( B(\pi, \epsilon_\pi^1) \subset \text{int} \gamma'(\sigma) \).

This process is then repeated for \( \gamma(\sigma') \) to show that there exists an \( \epsilon_\pi^2 \) and \( \epsilon_\gamma^2 \) such that \( B(\pi, \epsilon_\pi^2) \subset \text{int} \gamma'(\sigma') \) for all \( \gamma' \in B(\gamma, \epsilon_\gamma^2) \).

Then, choosing \( \epsilon_\pi \) and \( \epsilon_\gamma \), so that \( B(\pi, \epsilon_\pi) \subset B(\pi, \epsilon_\pi^1) \cap B(\pi, \epsilon_\pi^2) \) and \( B(\gamma, \epsilon_\gamma) \subset B(\gamma, \epsilon_\gamma^1) \cap B(\gamma, \epsilon_\gamma^2) \) it is seen that for any \( \gamma' \in B(\gamma, \epsilon_\gamma) \), \( B(\pi, \epsilon_\pi) \subset \gamma'(\sigma) \cap \gamma'(\sigma') \).

\(^{12}\)Recall that \( \gamma \) is an element of the finite-dimensional space \( \Gamma \) meaning that \( \gamma(\sigma) \) and \( \gamma(\sigma') \) belong to the finite dimensional space \( \Delta^{|\Omega|L} \).
Characterizing the space of beliefs for investors $2, \ldots, N$ is more intuitive. Beliefs for these investors are given by $\pi_m, \pi_e \in \Delta^{[2]} \times \Delta^{[0][2]}$. The following result is used implicitly in Theorem 3.

**Lemma A.12.** Let $\Pi$ be the space of beliefs for investor $n \geq N^E$. For any $\epsilon > 0$ there exists a $\delta > 0$ such that if $||\pi'' - \pi|| \leq \delta$ with $\pi' \in \Pi$ then $||\pi'({\sigma, \sigma'}) - \pi({\sigma, \sigma'})|| < \epsilon$.

**Proof.** By Bayes rule, given any beliefs $\pi$,

$$\pi({\sigma, \sigma'}) = \frac{\pi_m(\sigma)\pi_c(\sigma') + \pi_m(\sigma')\pi_c(\sigma)}{\pi_m(\sigma) + \pi_m(\sigma')}$$

(A.42)

In light of Assumption [1] it is seen that $\pi({\sigma, \sigma'})$ is continuous in $\pi_c$ and $\pi_m$ which proves the lemma. ■

### A.2 The generic existence of fully-revealing REE

The goal of this section is to lay out the proof of the generic existence of fully-revealing rational expectations equilibrium. It is largely self-contained.

**Definition 5.** Let $X$ be a subset of $\mathbb{R}^m$. A function $f : X \to \mathbb{R}^n$ is Lipschitz continuous of rank $K$ if for all $x, y \in X$,

$$|f(y) - f(x)| \leq K|y - x|.$$  

(A.43)

The next result follows from the properties of the Lebesgue measure on $\mathbb{R}^m$.

**Lemma A.13.** A set $A \subset \mathbb{R}^m$ has Lebesgue measure zero iff for each $\epsilon > 0$ there exists a countable set of cubes $\{S_i^\epsilon\}_{i=1}^\infty$ such that for each $\epsilon$

1. $A \subseteq \bigcup_{i=1}^\infty S_i^\epsilon$
The following lemma is adapted from Boothby (2003)[Section 6, lemma 1.12, p227].

**Lemma A.14.** Let $A \subset \mathbb{R}^n$ and suppose that $A$ has Lebesgue measure 0 in $\mathbb{R}^m$. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz continuous function then $f(A)$ has Lebesgue measure 0 in $\mathbb{R}^n$.

**Proof.** Let $\mu$ represent Lebesgue measure in $\mathbb{R}^n$.

By assumption, since $\mu(A) = 0$, for any $\epsilon > 0$ there exists a set of cubes $\{S_\epsilon^i\}_{i=1}^\infty$ that satisfies

\begin{align*}
1. & \quad A \subseteq \cup_{i=1}^\infty S_\epsilon^i \\
2. & \quad \sum_{i=1}^\infty \mu(S_\epsilon^i) < \epsilon.
\end{align*}

Let $\delta_\epsilon^i$ be side length of the cube $S_\epsilon^i$. The volume of each cube $S_\epsilon^i$ is given by

$$V(S_\epsilon^i) = (\delta_\epsilon^i)^m$$ \hfill (A.44)

By the Lipschitz continuity of $f$, there exists an $K > 0$ such that for each $x, y \in A$, $|f(x) - f(y)| < K|x - y|$. Hence, denoting the center of the cube $S_\epsilon^i$ by $a_\epsilon^i$, it follows that $f(S_\epsilon^i)$ must be contained in a cube $R_\epsilon^i$ with center $f(a_\epsilon^i)$ and side length less than or equal to $K\delta_\epsilon^i$.

As such, the volume of $f(S_\epsilon^i)$ satisfies

$$V(f(S_\epsilon^i)) \leq V(R_\epsilon^i) = (K\delta_\epsilon^i)^m = K^mV(S_\epsilon^i).$$ \hfill (A.45)

Finally, for any $\epsilon > 0$, one may select a set of cubes $\{R_\epsilon^i\}_{i=1}^\infty$ that satisfies the conditions of lemma A.13 by using the above procedure and selecting a cover of $A$ that has total volume less than or equal to $\epsilon/(K^m)$. \qed
The next two definitions and the next lemma follow Clarke (1983). By Rademacher’s Theorem, a function $F : \mathbb{R}^m \to \mathbb{R}^k$ that is Lipschitz continuous on an open subset of $\mathbb{R}^m$ is differentiable almost anywhere on that subset. Let $\Lambda_F$ be the set of points in the domain for the function $F$ at which $F$ is not differentiable.

**Definition 6.** The generalized Jacobian of $F$ at $x$, denoted $\partial F(x)$ is given by

$$
\partial F(x) = \text{co}\{\lim JF(x_i) : x_i \to x, x_i \notin \Lambda_F\}.
$$

(A.46)

where $JF(x_i)$ is the Jacobian of $F$ at the point of differentiability $x_i$.

The generalized Jacobian is a set of matrices (being a singleton if $F$ is differentiable at $x$) defined for all $x$ in the domain of a Lipschitz function $F$.

**Definition 7.** Let $G : \mathbb{R}^k \to \mathbb{R}^k$. The generalized Jacobian $\partial G(x_0)$ is said to be of maximal rank if every matrix $M \in \partial G(x_0)$ is non-singular.

In order to state the next lemma and proposition, some notation must be clarified. Let $F : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k$. Define $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_m)$ and let $(\hat{x}, \hat{y})$ solve $F(x, y) = 0$.

Let $\partial_x F(x, y)$ be the set of all $k \times k$ matrices $M$ such that for some $k \times m$ matrix $M'$, the $k \times (k + m)$ matrix $[M, M']$ belongs to $\partial F(x, y)$.

**Lemma A.15** (Clarke, p256, Corollary). Suppose that $\partial_x F(\hat{x}, \hat{y})$ is of maximal rank. Then there exists a neighborhood $Y$ of $\hat{y}$ and a Lipschitz function $\zeta : Y \to \mathbb{R}^k$ such that $\zeta(\hat{y}) = \hat{x}$ and such that, for all $y \in Y$, $F(\zeta(y), y) = 0$.

Recall from Radner (1979) that a full communication equilibrium (FCE) is a set of Walrasian equilibria (one for each joint signal $\sigma$) where in the equilibrium corresponding to each joint signal $\sigma$, investors hold the beliefs that they would hold if they knew all private information (i.e. they know that the joint signal is $\sigma$). An FCE is said to be confounding if $\phi(\sigma) = \phi(\sigma')$ for some distinct
An FCE is a fully-revealing REE if it is not confounding. It will be shown that the set of \((\gamma, \pi) \in \Gamma^N \times \Pi^N\) for which an FCE is confounding is of measure zero. We assume that \(N^E \geq 1\).

First, let \(\hat{\Pi} = \Delta^{[\Omega]}N^E\) be the space of EU investors’ conditional beliefs over \(\Omega\) given any set of information. Likewise, let \(\hat{\Gamma} = C(\Delta^{[\Omega]}N^A)\) be the space of conditional beliefs for ambiguity averse investors. We assume that the beliefs of each ambiguity averse investor are the core of a convex capacity. Let \(Z^E : P \times \hat{\Pi} \rightarrow \mathbb{R}^{[\Omega]}_+\) represent the excess demand function of all expected utility maximizing investors. That is, \(Z^E(p, \pi)\) represents excess demand when EU investors hold beliefs \(\pi \in \hat{\Pi}\) and the price is \(p\).

Let \(Z^A : P \times \hat{\Gamma} \rightarrow \mathbb{R}^{[\Omega]}_+\) represent aggregate excess demand for the ambiguity averse investors. From theorem 3 of Rigotti and Shannon (2006), the excess demand is Lipschitz continuous. As above, \(Z^A(p, \gamma)\) gives excess demand when prices are \(p\) and the ambiguity averse investors hold beliefs \(\gamma \in \hat{\Gamma}\).

The price vector \(p\) is an equilibrium price vector given beliefs \((\pi, \gamma) \in \hat{\Pi} \times \hat{\Gamma}\) if and only if \(Z(p, \pi, \gamma) = Z^E(p, \pi) + Z^A(p, \gamma) = 0\). To examine whether an FCE is confounding, we attempt to determine the size of the set of beliefs that will generate identical prices.

Consider the following system of equations.

\[
F(p_1, p_2, \pi_1, \pi_2, \gamma_1, \gamma_2) = \begin{pmatrix}
Z(p_1, \pi_1, \gamma_1) \\
Z(p_2, \pi_2, \gamma_2) \\
p_1 - p_2
\end{pmatrix} = 0. \quad (A.47)
\]

For an FCE to be confounding it must be true that for some distinct \(\sigma', \sigma'' \in \mathcal{F}\), there exists a solution to the system \(F(p_1, p_2, \pi(\sigma'), \pi(\sigma''), \gamma(\sigma'), \gamma(\sigma'')) = 0\).

Let \(B\) be the set of all \((p_1, p_2, \pi_1, \pi_2, \gamma_1, \gamma_2) \in P^2 \times \hat{\Pi}^2 \times \hat{\Gamma}^2\) that solve the system \((A.47)\). Let \(T(B)\) be the projection of this set into the space \(\hat{\Pi}^2 \times \hat{\Gamma}^2\). We
show that the set $T(B)$ has measure zero in $\hat{\Pi}^2 \times \hat{\Gamma}^2$. The following proposition provides the key result for this proof.

**Proposition 6.** Let

$$F(p_1, p_2, \pi_1, \pi_2, \gamma_1, \gamma_2) = \begin{pmatrix} Z(p_1, \pi_1, \gamma_1) \\ Z(p_2, \pi_2, \gamma_2) \\ p_1 - p_2 \end{pmatrix} = 0. \quad (A.48)$$

For any $(\gamma_1, \gamma_2) \in \hat{\Gamma}^2$, let $\partial F(p_1, p_2, \pi_1, \pi_2)$ be the generalized Jacobian corresponding to the arguments $(p_1, p_2, \pi_1, \pi_2)$. Then, every $M \in \partial F(p_1, p_2, \pi_1, \pi_2)$ is of rank $3(|\Omega| - 1)$.

**Proof.** To see this, note that for a fixed $\gamma \in \hat{\Gamma}^2$ the system is differentiable in $\pi_1$ and $\pi_2$ but because of the possible non-differentiability of $Z^A(\cdot, \gamma)$, it need not be strictly differentiable in $(p_1, p_2)$. However, it is true that every possible $M \in \partial F(p_1, p_2, \pi_1, \pi_2)$, will have the following form:

$$\begin{array}{cccc} p_1 & p_2 & \pi_1 & \pi_2 \\
1: & A & 0 & C & 0 \\
2: & 0 & B & 0 & D \\
3: & I & -I & 0 & 0 \end{array} \quad (A.49)$$

where $I$ is the $(|\Omega| - 1) \times (|\Omega| - 1)$ identity matrix. The $(|\Omega| - 1) \times (|\Omega| - 1)$ matrices $A$ and $B$ will vary across elements of $\partial F(\cdot)$, but the identity matrices and the matrices $C$ and $D$ will not. Now consider the $(|\Omega| - 1) \times N^E(|\Omega| - 1)$ matrix $C$. The form of matrix $D$ is similar. Since only $Z^E(\cdot)$ depends on $\pi_1$ and only investor $n$'s demand is affected by $n$'s beliefs, $C$, will have the form

$$C = \left( C^1, C^2, \ldots, C^{N^E} \right). \quad (A.50)$$

49
Applying the implicit function theorem to the investor’s first order conditions reveals that each $C_n$ is given by

$$C_n = \begin{pmatrix} -\frac{u'(x(1))}{u''(x(1))} & 0 & \cdots & 0 \\ 0 & -\frac{u'(x(2))}{u''(x(2))} & 0 & \cdots \\ \vdots \\ 0 & \cdots & 0 & -\frac{u'(x(|\Omega|-1))}{u''(x(|\Omega|-1))} \end{pmatrix}.$$  \hspace{1cm} (A.51)

By inspection the matrix $C_n$ spans a space of dimension $(|\Omega| - 1)$. Thus the columns corresponding to $p_1$, $C_1$, and $D_1$ will span a space of dimension $3(|\Omega| - 1)$, regardless of the entries in the matrix $A$. Thus, all matrices in $\partial F(p_1, p_2, \pi_1, \pi_2)$ span a space of dimension $3(|\Omega| - 1)$. \hspace{1cm} ■

**Proposition 7.** Consider the following system of equations.

$$F(p_1, p_2, \pi_1, \pi_2, \gamma_1, \gamma_2) = \begin{pmatrix} Z(p_1, \pi_1, \gamma_1) \\ Z(p_2, \pi_2, \gamma_2) \\ p_1 - p_2 \end{pmatrix} = 0. \hspace{1cm} (A.52)$$

Let $\mu_{|\Pi|}$ be Lebesgue measure on $\Pi^2$ and let $T(B(\gamma_1, \gamma_2))$ be the set of $\pi_1, \pi_2 \in \Pi^2$ for which this system has a solution for a fixed $(\gamma_1, \gamma_2) \in \Gamma^2$. Then for a fixed $\gamma_1, \gamma_2$, $\mu_{|\Pi|}(T(B(\gamma_1, \gamma_2))) = 0$.

**Proof.** By Proposition $6$ and Lemma $A.15$ the set of solutions $B(\gamma_1, \gamma_2)$ is a $2(|\Omega| - 1) + 2N_E(|\Omega| - 1) - 3(|\Omega| - 1) = (2N_E - 1)(|\Omega| - 1)$ dimensional manifold in $P^2 \times \Pi^2$. \hspace{1cm} \footnote{That is, every solution point is locally homeomorphic to a subset of $R(2N_E - 1)(|\Omega| - 1)$. See Sternberg (1983) for this definition.}

Let $\tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2$ be the space corresponding to the exogenous variables in Proposition $6$. Let $\{S_i\}_{i=1}^\infty$ be a countable covering of the space $\tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2$. Proposition $6$ and Lemma $A.15$ then tell us that there exist Lipschitz functions $\{\zeta_i\}_{i=1}^\infty$ such that $\cup_{i=1}^\infty \{\zeta_i(S_i)\}$ covers $T(B(\gamma_1, \gamma_2))$. \hspace{1cm} \footnote{To see how this is done, start with an arbitrary countable covering $\{A_i\}_{i=1}^\infty$ of $\tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2$.}
By Lemma A.13 the set $\zeta_i(S_i)$ has measure zero in $\Pi^2$ since the set $\tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2$ is homeomorphic to Euclidean space of dimension $(2N^E - 1)(|\Omega| - 1)$ while the dimension of $\Pi^2$ is $2N^E(|\Omega| - 1)$.

Then since $\{\zeta_i(S_i)\}_{i=1}^\infty$ covers $T(B(\gamma_1, \gamma_2))$,

$$\mu_{\Pi^2}(T(B(\gamma_1, \gamma_2))) \leq \mu_{\Pi^2}(\bigcup_{i=1}^\infty \zeta_i(S_i)) = \sum_{i=1}^\infty \mu_{\Pi^2}(\zeta_i(S_i)). \quad (A.53)$$

By Lemma A.13 $\mu_{\Pi^2}(\zeta_i(S_i)) = 0$ for all $i$, which proves the result. ■

Before stating the theorem, one clarification must be made about the space of beliefs. If each ambiguity averse investors’ beliefs can be generated by the core of a convex capacity, then each conditional belief can be generated by a set of the form:

$$\gamma(\sigma) = \{\pi \in \Delta^{|\Omega|-1} : \pi(\omega) \geq a(\omega)\}. \quad (A.54)$$

As such, beliefs for each joint signal can be described by a point $a = \{a(1), 2, \ldots, |\Omega|\}$ under the restriction that the resulting set must be convex and non-empty. The set of such points is a subset of Euclidean space, and so it makes sense to impose Lebesgue measure on this set. This clarification leads to the following theorem.

**Theorem 2.** Let $Z(p, \pi, \gamma)$ represent aggregate excess demand. If $Z(p, \pi, \gamma)$ is Lipschitz continuous in $p$ then the set of beliefs in $\Pi \times \Gamma$ for which there is not a

For each $A_i$, let $B_i$ be the set of points in $B$ such that the projection of each point into $\tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2$ is in $S_i$. For each of these points $b$, Lemma A.15 says that there is an open set $V_b \subseteq P^2 \times \Pi^2 - (P \times \Pi_1 \times \Pi_2)$, an open set $W_b \in P \times \Pi_1 \times \Pi_2$ and a Lipschitz function $\zeta_b$ that maps points in $(P \times \Pi_1 \times \Pi_2) \cap W_b$ into points in $(B - (P \times \Pi_1 \times \Pi_2)) \cap V_b$. By definition $\{W_b\}_{b \in B_i}$ is an open cover of $A_i$ and $\{V_b\}_{b \in B_i}$ is an open cover of $B_i$. Since $A_i$ and $B_i$ are subsets of Euclidean space, each of these open covers has a countable subcover (see Munkres (2000)[Theorems 30.2 and 30.3]) which we label $\{V_{b_k}\}_{k=1}^\infty$ and $\{V_{b_k}\}_{k=1}^\infty$ respectively. Let $B_i$ be the collection of $b \in B_i$ such that either $V_b \in \{V_{b_k}\}_{k=1}^\infty$ or $W_b \in \{W_{b_k}\}_{k=1}^\infty$. Let us then replace each element $A_i$ with the countable covering $W_i = \{W_{b_k}\}_{b \in B_i}$. The countable covering over $\tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2$ formed in this way is countable since it is a countable collection of countable sets. The collection $\{Q_i\}_{b \in B_i}$ is a countable collection of Lipschitz functions. Define another countable collection of Lipschitz function from $\tilde{P} \times \tilde{\Pi}_1 \times \tilde{\Pi}_2$ into $\Pi^2$, $\{\zeta_i\}_{i=1}^\infty = \{\{\zeta_{b_k}\}_{b \in B_i}\}_{i=1}^\infty$ to be $\zeta_{b_k}(p, c, d) = (p, c, d, T(\zeta_{b_k}(p, c, d)))$ where $(p, c, d) \in W_b$ and $T(\zeta_{b_k}(p, c, d))$ is the component projection of $\zeta_{b_k}(p, c, d)$ into $\Pi^2$. The collection $\{\zeta_i\}_{i=1}^\infty$ along with the sets $\{S_i\}_{i=1}^\infty$ have the stated properties.
fully-revealing rational expectations equilibrium has measure zero in $\Gamma^N \times \Pi^N$.

**Proof.** Let $\mu_{\hat{\Gamma}_2, \hat{\Pi}_2}$ be the product measure over $\hat{\Pi}_2 \times \hat{\Gamma}_2$. Let $\mu_{\hat{\Gamma}_2}$ be the Lebesgue measure defined over $\hat{\Gamma}_2$ and let $\mu_{\hat{\Pi}_2}$ be the Lebesgue measure over $\hat{\Pi}_2$. We define

$$T(B(\gamma_1, \gamma_2)) = \{ \pi_1, \pi_2 \in \hat{\Pi}_2 : (\pi_1, \pi_2, \gamma_1, \gamma_2) \in T(B) \}. \quad (A.55)$$

We now employ the properties of the product measure and note that (see theorem 8.6 in Rudin (1987))

$$\mu_{\hat{\Pi}_2, \hat{\Gamma}_2}(T(B)) = \int_{\hat{\Gamma}_2} \mu_{\hat{\Pi}_2}(T(B(\gamma_1, \gamma_2))) \, d\mu_{\hat{\Gamma}_2} \quad (A.56)$$

From this it can be seen that if $\mu_{\hat{\Pi}_2}(T(B(\gamma_1, \gamma_2))) = 0$ for $\mu_{\hat{Gamma}_2}$-almost all $\gamma_1, \gamma_2$, then $\mu_{\hat{\Pi}_2, \hat{\Gamma}_2}(T(B)) = 0$. In fact, from Proposition 7 we see that $\mu_{\hat{\Pi}_2}(T(B(\gamma_1, \gamma_2))) = 0$ for all $(\gamma_1, \gamma_2) \in \hat{\Gamma}_2$.

Thus, in the space $\hat{\Pi}_2 \times \hat{\Gamma}_2$, the set $T(B)$ of confounding beliefs is of $\mu_{\hat{\Pi}_2, \hat{\Gamma}_2}$ measure zero.

As in Radner (1979), one may then extend this result to show that for any finite set of joint signals, the set of beliefs that lead an FCE to be confounding has measure zero. ■

**References**


54


