Large games: structural robustness

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ABSTRACT

In simultaneous-move Bayesian games with many semi-anonymous players all the equilibria are structurally robust. The equilibria survive under structural alterations of the rules of the game and its information structure, even when the game is embedded in bigger games. Structural robustness implies ex-post Nash conditions and a stronger condition of information proofness. It also implies fast learning, self-purification and strong rational expectations in market games. Structurally robust equilibria may be used to model games with highly unspecified structures, such as games played on the web.

0. Background on large games

Earlier literature on large (many players) cooperative games is surveyed in Aumann and Shapley (1974). For large strategic games, see Schmeidler (1973) and the follow-up literature on the purification of Nash equilibria. There is also substantial literature on large games with special structures, for example large auctions as reported in Rustichini, Satterthwaite, and Williams (1994).

Unlike the above, this survey concentrates on the structural robustness of (general) Bayesian games with many semi-anonymous players, as developed in Kalai (2004, 2005). (For additional notions of robustness in game theory, see Bergemann and Morris, 2005).

1. Main message and examples

In simultaneous-move Bayesian games with many semi-anonymous players, all Nash equilibria are structurally robust. The equilibria survive under structural alterations that relax the simultaneous-play assumptions, and permit information transmission, revisions of choices, communication, commitments, delegation, and more.

Large economic and political systems and distributive systems such as the Web are examples of environments that give rise to such games. Immunity to alterations means that Nash equilibrium predictions are valid even in games whose structure is largely unknown to modellers or to players.

The next example illustrates immunity of equilibrium to revisions, or being ex-post Nash, see Cremer and McLean (1985), Green and Laffont (1987) and Wilson (1987) for early examples.

Example 1. Ex post *stability illustrated in match pennies*

Simultaneously, each of *k* males and k females chooses one of two options, *H* or *T*. The payoff of every male is the proportion of females his choice matches and the payoff of every female is the proportion of males her choice *mis*matches. (When $k = 1$ this is the familiar match-pennies game.) Consider the mixed-strategy equilibrium where every player chooses *H* or *T* with equal probabilities.

Structural robustness implies that the equilibrium must be *ex post* Nash: it should survive in alterations that allow players to revise their choices after observing their opponents' choices. Clearly this is not the case when k is small. But as k becomes large, the equilibrium becomes arbitrarily close to being *ex post* Nash. More precisely, the Prob[some player can improve his payoff by more than ε *ex-post*] decreases to zero at an exponential rate as k becomes large.

Example 2. *Invariance to sequential play illustrated in a computer choice game*

Simultaneously, each of n players chooses one of two computers, *I* or *M*. But before choosing, with 0.50 – 0.50 i.i.d. probabilities, every player is privately informed that she is an *I*-type or an *M*-type. The payoff of every player is 0.1 if she chooses the computer of her type (zero otherwise) plus 0.9 times the proportion of opponents whose choices she matches. (Identical payoffs and prior probabilities are assumed only to ease the presentation. The robustness property holds without these assumptions.) Consider the favorite-computer equilibrium (FC) where every player chooses the computer of her type.

Structural robustness implies that the equilibrium must *be invariant to sequential play*: it should survive in alterations in which the (publicly observed) computer choices are made sequentially. Clearly this is not the case for small *n*, where any equilibrium must involve herding. But as n becomes large, the structural robustness theorem below implies that FC becomes an equilibrium in all sequential alterations. More precisely, the Prob[some player, by deviating to her non favorite computer, can achieve an ε-improvement at her turn] decreases to zero at an exponential rate.

The general definition of structural robustness, presented next, accommodates the above examples and much more.

2. Structural robustness

A mixed-strategy (Nash) equilibrium $\sigma = (\sigma_1, \ldots, \sigma_n)$ of a one-simultaneous-move *n*-person strategic game *G* is structurally robust if it *remains an equilibrium* in *every* structural *alteration* of *G*. Such an alteration is described by an extensive game, A, and for σ to remain an equilibrium in A means that *every adaptation* of σ to A, σ^{A}, must be an equilibrium in A.

Consider any *n*-person one-simultaneous-move Bayesian game *G*, like the Computer Choice game above.

Definition 1. *A (structural) alteration of G* is any finite extensive game A with the following properties:

1. A *includes the (original) G-players*: The players of A constitute a superset of the G-players (the players of *G*).

2. *Unaltered type structure*: At the first stage of A, the *G*-players are assigned a profile of types by the same prior probability distribution as in *G*. Every player is informed of his own type.

3. *Playing* A *means playing G*: with every final node of A, z, there is an associated unique profile of G pure-strategies, $a(z) = (a_1(z), \ldots, a_n(z))$.

4. *Unaltered payoffs*: the payoffs of the *G*-players at every final node z are the same as their payoffs in *G* (at the profile of realized types and final pure-strategies *a*(*z*)).

5. *Preservation of original strategies*: every pure-strategy a_i of a *G*-player *i* has at

least one $\mathcal{A}$ adaptation. That is, an $\mathcal{A}$-strategy $a_i^{\mathcal{A}}$ that guarantees (w.p. 1) ending at a final node *z* with $a_i(z) = a_i$ (no matter what strategies are used by the opponents).

In the computer choice example, every play of an alteration $\mathcal{A}$ must produce a profile of computer allocations for the *G*-players. Their preferences in $\mathcal{A}$ are determined by their preferences over profiles of computer allocations in *G*. Moreover, every *G*-player *i* has at least one $\mathcal{A}$-strategy $I_i^{\mathcal{A}}$ (which guarantees ending at a final node where she is allocated *I*), and at least one $\mathcal{A}$-strategy $M_i^{\mathcal{A}}$ (which guarantees ending at a final node where she is allocated *M*).

Definition 2. *An* $\mathcal{A}$ *(mixed) strategy-profile,* $\sigma^{\mathcal{A}}$*, is an adaptation of* a G (mixed) strategy-profile σ, if for every G-player i, every $\sigma_i^{\mathcal{A}}$ is an $\mathcal{A}$-adaptation of σ_i. That is, for every G pure-strategy a_i, $\sigma_i(a_i) = \sigma_i^{\mathcal{A}}(a_i^{\mathcal{A}})$ for some $\mathcal{A}$-adaptation $a_i^{\mathcal{A}}$ of a_i.

In the computer choice example, for a *G*-strategy where player *i* randomizes 0.20 to 0.80 between *I* and *M*, an $\mathcal{A}$ adaptation must randomize 0.20–0.80 between a strategy of the type $I_i^{\mathcal{A}}$ and a strategy of the type $M_i^{\mathcal{A}}$.

Definition 3. An equilibrium σ of *G is structurally robust* if in every alteration of *G*, $\mathcal{A}$, and in every adaptation of σ, $\sigma^{\mathcal{A}}$, the strategy of every *G*-player i, $\sigma_i^{\mathcal{A}}$, is best response to $\sigma_{-i}^{\mathcal{A}}$.

Remark 1. The structural robustness theorem, discussed later, presents an asymptotic result: the equilibria are structurally robust up to two positive numbers (ε,ρ), which can be made arbitrarily small as n becomes large. The notion of approximate robustness is the following.

An equilibrium is (ε,ρ)-structurally robust if in every alteration and every adaptation as above, Prob[visiting an information set where a *G*-player can improve his payoff by more than ε] $\leq \rho$. (ε-improvement is computed conditional on being at the information set. To gain such improvement the player may coordinate his deviation: he may make changes at the information set under consideration together with changes at forthcoming ones.)

For the sake of brevity, the next section discusses full structural robustness. But all the observations presented above also hold for the properly defined approximate counterparts. For example, the fact that structural robustness implies *ex post* Nash also implies that approximate structural robustness implies approximate *ex post* Nash. The implications of approximate (as opposed to full) structural robustness are important, due to the asymptotic nature of the structural robustness theorem.

3. Implications of structural robustness

Structural robustness of an equilibrium σ in a game G is a strong property, because the set of G-alterations that σ must survive is rich. The simple examples below are meant to suggest the richness of its implications, with the first two examples showing how it implies the notions already discussed (see Dubey and Kaneko, 1984, for related issues).

Remark 2. *Ex post Nash and being information-proof*

G with revisions, GR, is the following n-person extensive game. The n players are assigned types as in G (using the prior type distribution of G and informing every player of his own type). In a first round of simultaneous play, every player chooses one of his G pure strategies; the types realized and pure strategies chosen are all made public knowledge. Then, in a second round of simultaneous play, the players again choose pure strategies of G (to revise their first round choices). The payoffs are as in G, computed at the profile of realized types with the profile of pure strategies chosen in the *second* round.

Clearly GR satisfies the definition of an alteration (with no additional players), and every equilibrium σ of G has the following GR adaptation, σ^{NoRev}: in the first round the players choose their pure strategies according to σ, just as they do in G; in the second round nobody revises his first round choice.

Structural robustness of σ implies that σ^{NoRev} must be an equilibrium of GR, that is, σ is *ex post* Nash.

Moreover, the above reasoning continues to hold even if the information revealed between the two rounds is partial and different for different players. The fact that σ^{NoRev} is an equilibrium in all such alterations shows that σ is *information-proof*: no revelation of information (even if strategically coordinated by G-players and outsiders) could give any player an incentive to revise. Thus, structural robustness is substantially stronger than all the variants of the *ex post* Nash condition. (In the non-approximate notions, being *ex post* Nash is equivalent to being information proof. But in the approximate notions information proofness is substantially stronger.)

Remark 3. *Invariance to order of play*

G played sequentially, GS , is the following *n*-person extensive game. The *n* players are assigned types as in *G*. The play progresses sequentially, according to a fixed publicly known order. Every player, at his turn, knows all earlier choices.

Clearly, GS is an alteration of *G*, and every equilibrium σ of *G* has the following GS adaptation: At his turn, every player *i* chooses a pure-strategy with the same probability distribution σ_i as he does in the simultaneous-move game *G*. Structural robustness of σ implies that this adaptation of σ must be an equilibrium in every such GS .

Moreover, the above reasoning continues to hold even if the order of play is determined dynamically, and even if it is strategically controlled by *G*-players and outsiders. Thus, a structurally robust equilibrium is invariant to the order of play in a strong sense.

Remark 4. *Invariance to revelation and delegation*

G with delegation, GD , is the following (n + 1)-players game. The original *n* *G*-players are assigned types as in *G*. In a first round of simultaneous play, every *G*-player chooses between (1) self-play and (2) delegate-the-play and report a type to an outsider, player n + 1. In a second round of simultaneous play all the self-players choose their own *G* pure strategies, and the outsider chooses a profile of *G* pure strategies for all the delegators. The payoffs of the *G* players are as in *G*; the outsider may be assigned any payoffs.

Clearly, GD is an alteration of *G*, and every equilibrium σ of *G* has adaptations that involve no delegation.

In the computer choice game, for example, consider an outsider with incentives to coordinate: his payoff equals one when he chooses the same computer for all delegators, zero otherwise. This alteration has a new (more efficient) equilibrium, not available in *G*: everybody delegates and the outsider chooses the most-reported type.

Nevertheless, as structural robustness implies, FC remains an equilibrium in GD (nobody delegates in the first round and they choose their favorite computers in the second). Moreover, FC remains an equilibrium under any scheme that involves reporting and voluntary delegation of choices.

Remark 5. *Partially specified games*

Structurally robust equilibria survive under significantly more complex alterations than the ones above. For example, one could have multiple opportunities to revise, to delegate, to affect the order of play, to communicate, and more. Because of these strong invariance properties, such equilibria may be used in games which are only partially specified as illustrated by the following example.

Example 3. *A game played on the Web*

Suppose that instead of being played in one simultaneous move, the Computer Choice game has the following instruction: 'Go to Web site xyz before the end of the week, and click in your computer choice.' This instruction involves substantial structural uncertainty: In what order would the players choose? Who can observe whom? Who can talk to whom? Can players sign binding agreements? Can players revise their choices? Can players delegate their choices? And so forth.

Because it is unaffected by the answers to such questions, a structurally robust equilibrium σ of the one-simultaneous-move game can be played on the Web in a variety of ways without losing the equilibrium property. For example, players may make their choices according to their σ_i probabilities prior to the beginning of the click-in period, then go to the Web and click in their realized choices at individually selected times.

Remark 6. *Competitive prices in Shapley–Shubik market games*

For a simple illustration, consider the following n-trader market game (see Shapley and Shubik, 1977, and later references in Dubey and Geanakoplos, 2003, and McLean, Peck and Postlewaite, 2005). According to individual independent prior probability distributions, each trader is born to be one of four types: a banana owner or an apple owner and a banana lover or an apple lover. Each trader knows his own type, and his payoff depends on his own type and the fruit he ends up with, as well as on the distribution of types and fruit ownership of his opponents (externalities allowed). In one simultaneous move, every player has to choose between (1) keeping his fruit and (2) trading it for the other kind.

The banana/apple price is determined proportionately (with one apple and banana added in to avoid division by zero). For example, if 199 bananas and 99 apples were traded, the price of bananas to apples would be (199 + 1)/(99 + 1) = 2, that is, every apple trader gets two bananas and every banana trader gets 0.5 apples.

With a small number of traders, the price is unlikely to be competitive. If players are allowed to re-trade after the realized price becomes known, they would, and a new price would emerge.

However, when n is large, approximate structural robustness implies being approximately information-proof. So even when the realized price becomes known, no player has significant incentive to re-trade, that is, the price is approximately competitive (Prob[some player can ε-improve his expected payoff by re-trading at the observed price] $\leq \rho$).

This is stronger than classical results relating Nash equilibrium to Walras equilibrium (for example, Dubey, Mas Colell and Shubik, 1980). First, being conducted under incomplete information, the above relates Bayesian equilibria to rational expectations equilibria (rather than Walras). Also the competitive property described here is substantially stronger, due to the immunity of the equilibria to alterations represented by extensive games. If allowance is made for spot markets, coordinating institutions, trade on the Web, and so on, the Nash-equilibrium prices of the simple simultaneous-move game are sustained through the intermediary steps that may come up under such possibilities.

Remark 7. *Embedding a game in bigger worlds*

Alterations allow the inclusion of outside players who are not from G. Moreover, the restrictions imposed on the strategies and payoffs of the outsiders are quite limited. This means that alterations may describe bigger worlds in which G is embedded. Structural robustness of an equilibrium means that the small-world (G) equilibrium remains an equilibrium even when the game is embedded in such bigger worlds.

Remark 8. *Self-purification*

Schmeidler (1973) shows that in a normal-form game with a continuum of anonymous players, every strategy can be purified, that is, for every mixed-strategy equilibrium one can construct a pure-strategy equilibrium (Ali Khan and Sun, 2002 survey some of the large follow-up literature).

The *ex post* Nash property above constitutes a stronger (but asymptotic) result. Since the resulting play of a mixed strategy equilibrium yields pure-strategy profiles that are Nash equilibria (of the perfect information game), one does not need to construct pure-strategy equilibria: simply playing a mixed-strategy equilibrium yields pure-strategy profiles that are equilibria.

The approximate statement is: for every (ε,ρ) for sufficiently large n, Prob[ending at a pure strategy profile that is not an ε Nash equilibrium of the realized perfect information game] $\leq \rho$. Since both ε and ρ can be made arbitrarily small, this is asymptotic purification. Note that the model of Schmeidler, with a continuum of players, requires non-standard techniques to describe a continuum of independent random variables (the mixed strategies of the players). The asymptotic result stated here, dealing always with finitely many players, does not require any non-standard techniques.

Remark 9. *'As if' learning*

Kalai and Lehrer (1993) show that in playing an equilibrium of a Bayesian repeated game, after a sufficiently long time the players best-respond as if they know their opponents' realized types and, hence, their mixed strategies.

But being information-proof, at a structurally robust equilibrium (even of a one shot game) players' best respond (immediately) as if they know their opponents' realized types, their mixed strategies and even the pure-strategies they end up with.

4. Sufficient conditions for structural robustness

Theorem 1. *Structural Robustness* (rough statement): the equilibria of large one-simultaneous-move Bayesian games are (approximately) structurally robust if

1. the players' types are drawn independently, and

2. payoff functions are anonymous and continuous.

Payoff anonymity means that in addition to his own type and pure-strategy, every player's payoff may depend only on aggregate data of the opponents' types and pure-strategies. For example, in the computer choice game a player's payoff may depend on her own type and choice, and on the *proportions* of opponents in the four groups: *I*-types who chose *I*, *I*-types who chose *M*, *M*-types who chose *I*, and *M*-types who chose *M*.

The players in the games above are only semi-anonymous, because there are no additional symmetry or anonymity restrictions other than the restriction above. In particular, players may have different individual payoff functions and different prior probabilities (publicly known).

The continuity condition relates games of different sizes and rules out games of the type below.

Example 4. *Match the expert*

Each of n players has to choose one of two computers, *I* or *M*. Player 1 is equally likely to be one of two types: 'an expert who is informed that *I* is better' (*I*-better) or 'an expert who is informed that *M* is better' (*M*-better). Players $2,\ldots,n$ are of one possible 'non-expert' type. Every player's payoff is one if he chooses the better computer, zero otherwise. (Stated anonymously: choosing computer *X* pays one, if the proportion of the *X-better* type is positive, zero otherwise.)

Consider the equilibrium where player 1 chooses the computer he was told was better and every other player chooses *I* or *M* with equal probabilities. This equilibrium fails to be *ex post* Nash (and hence, fails structural robustness), especially as n becomes large, because after the play approximately one-half of the players would want to revise their choices to match the observed choice of player 1. (With a small n there may be 'accidental *ex post* Nash', but it becomes extremely unlikely as n becomes large.)

This failure is due to discontinuity of the payoff functions. The proportions of *I*-better types and *M*-better types in this game must be either $(1/n, 0)$ or $(0, 1/n)$, because only one of the n players is to be one of these types. Yet, whatever n is, every player's payoff is drastically affected (from 0 to 1 or from 1 to 0) when we switch from $(1/n, 0)$ to $(0, 1/n)$ (keeping everything else the same).

As n becomes large, this change in the type proportions becomes arbitrarily small, yet it continues to have a drastic effect on players' payoffs. This violates a condition of uniform equicontinuity imposed simultaneously on all the payoff functions in the games with $n = 1,2,\ldots$ players.

Bibliography

Ali Khan, M. and Sun, Y. 2002. Non-cooperative games with many players. In *Handbook of Game Theory with Economic Applications*, vol. 3, ed. R. J. Aumann and S. Hart. Amsterdam: North-Holland.

Aumann, R. J. and Shapley, L. S. 1974. *Values of Nonatomic Games*. Princeton, NJ: Princeton University Press.

Bergemann, D. and Morris, S. 2005. Robust mechanism design. *Econometrica* 73, 1771–813.

Cremer, J. and McLean, R. P. 1985. Optimal selling strategies under uncertainty for a discriminating monopolist when demands are interdependent. *Econometrica* 53, 345–61.

Dubey, P., Mas-Colell, A. and Shubik, M. 1980. Efficiency properties of strategic market games: an axiomatic approach. *Journal of Economic Theory* 22, 339–62.

Dubey, P. and Kaneko, M. 1984. Information patterns and Nash equilibria in extensive games: 1. *Mathematical Social Sciences* 8, 111–39.

Dubey, P. and Geanakoplos, J. 2003. From Nash to Walras via Shapley–Shubik. *Journal of Mathematical Economics* 39, 391–400.

Green, J. R. and Laffont, J. J. 1987. Posterior implementability in a two-person decision problem. *Econometrica* 55, 69–94.

Kalai, E. 2004. Large robust games. *Econometrica* 72, 1631–66.

Kalai, E. 2005. Partially-specified large games. *Lecture Notes in Computer Science*, 3828, 3–13.

Kalai, E. and Lehrer, E. 1993. Rational learning leads to Nash equilibrium. *Econometrica* 61, 1019–45.

McLean, R., Peck, J. and Postlewaite, A. 2005. On price-taking behavior in asymmetric information economies. In *Essays in Dynamic General Equilibrium: Festschrift for*

David Cass, ed. A. Citanna, J. Donaldson, H. Polemarchakis, P. Siconolfi and S. Spear. Berlin: Springer. Repr. in *Studies in Economic Theory* 20, 129–42.

Rustichini, A., Satterthwaite, M.A. and Williams S.R. 1994. Convergence to Efficiency in a Simple Market with Incomplete Information. *Econometrica* 62, pp 1041-64.

Schmeidler, D. 1973. Equilibrium points of nonatomic games. *Journal of Statistical Physics* 17, 295–300.

Shapley, L. S. and Shubik, M. 1977. Trade using one commodity as a means of payment. *Journal of Political Economy* 85, 937–68.

Wilson, R. 1987. Game-theoretic analyses of trading processes. In *Advances in Economic Theory: Fifth World Congress*, ed. T. Bewley. Cambridge: Cambridge University Press.