Optimal Combinatorial Mechanism Design

[Job Market Paper][1]

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Abstract: We study an adverse selection model in which a principal will allocate a set of nonidentical objects among privately informed agents. Combinatorial auctions provide an important class of examples. Agents have private information that is parametrized by a one dimensional type. The principal collects type reports from the agents, computes their valuations for different sets, and then decides on an allocation and payments so as to maximize revenue. Working with one dimensional private information allows us to isolate the model from the well known problems of multidimensional types and focus on the problems associated with multidimensional outcomes. In particular we reformulate the optimal mechanism design problem using the methodology of Myerson (1981) and identify *regularity* conditions under which the reformulated problem can be solved without the constraints. We find that the solution to a regular problem involves solving a combinatorial optimization problem. We provide conditions under which the mechanism design problem is regular and obtain results for two large classes of valuation functions. The first class consists of functions which satisfy an increasing differences property. All supermodular valuations belong to this class. The second class consists of functions which scalarize sets independently from types and have a single crossing property. This class involves many interesting examples, as well as some valuations that are not in the first class.

Keywords: Mechanism design, Adverse selection, Combinatorial auctions, Supermodularity, Single crossing

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1 Introduction

In his seminal paper on optimal mechanism design, Myerson (1981) considers an environment in which a principal interacts with several privately informed agents in order to allocate a single object and, in return, collect payments, with the goal of maximizing revenue. In this paper, we consider a multi-object extension of Myerson's model, i.e., we study a combinatorial mechanism design problem in which the principal has a finite set of nonidentical objects to be allocated.

In Myerson's single-object model, as well as in subsequent work, the optimal mechanism design problem is that of constructing a mechanism, i.e., a Bayesian game, which will maximize the principal's expected revenue while inducing participation (individual rationality) and honest reporting (incentive compatibility) on the part of the agents. The revelation principle implies that the mechanism can be chosen from among those which collect valuation reports from the agents and then determine an allocation and payments. Myerson characterizes the incentive constraints in the mechanism design problem via "monotonicity" and "envelope" conditions. For each agent i and each of his types t_i, he defines $Q_i(t_i)$, the expected probability of winning the object and $U_i(t_i)$, the expected payoff from reporting truthfully. Loosely speaking, monotonicity requires Q_i to be nondecreasing and the envelope condition requires $Q_i(t_i)$ to be the derivative of U_i at t_i. Myerson then reformulates the principal's problem as one of maximizing the expected sum of "virtual" utilities of the agents subject to his monotonicity constraint, where an agent's virtual valuation is his actual valuation less the inverse of the hazard rate of the distribution of his valuation. Next, he asks when the constraints in the reformulated problem will not be binding. Myerson calls the condition under which this happens "regularity" and he shows that nondecreasing hazard rates are sufficient for regularity. Regular problems can easily be solved by choosing an allocation function that maximizes, for each reported type vector, the sum of virtual valuations. Such an allocation rule satisfies the monotonicity condition which constrains the reformulated problem.

A challenging way to model a combinatorial mechanism design problem is to let the private information possessed by the agents be a vector of valuations for each subset of the grand set. See Krishna and Perry (1998) and Jehiel et al. (1999), among others, for treatments of multidimensional problems of this kind, which reveal the technical difficulties associated with this approach. The main difficulty with type multidimensionality concerns the characterization of incentive compatibility.

Combinatorial auctions provide an important class of combinatorial mechanism design problems which has received a substantial amount of attention from economists as well as computer scientists. Cramton et al. (2006) is an interdisciplinary collection of papers on combinatorial auctions. It brings together work on practical combinatorial auctions that may be used in practice as well as work on computational aspects of the problem including efficient bidding languages and algorithms to solve winner determination problems in combinatorial auctions. Lehmann et al. (2006) study algorithms for Pareto efficient allocations under complete information if agents have submodular utilities. Palfrey (1983) shows that selling a number of nonidentical objects as a bundle in a second price auction is revenue superior to selling each item in separate single-object second price auctions. Jehiel and Moldovanu (2001) show that efficient combinatorial auctions are not revenue maximizing even when all bidders are symmetric. Holzman and Monderer (2004) and Holzman et al. (2004) study the ex post equilibria and communicational aspects of the Vickrey-Clark-Groves mechanisms with private values. In fact, the prevailing treatments of combinatorial auctions almost always assume multidimensional types and therefore the characterization of the optimal combinatorial auction has been elusive.

In this paper, we study the optimal combinatorial mechanism design problem in an attempt to extend Myerson's analysis in two directions, to a multi-object, combinatorial setting and to a larger class of valuation functions. To this end, we work with several objects but one dimensional type spaces. Our approach allows us to focus on the multidimensionality associated with allocating sets to the agents, without having to worry about the problems of multidimensional types. We show that the kind of multidimensionality problem we analyze is analytically tractable. We first characterize incentive compatibility using monotonicity and envelope conditions in a way that mimics Myerson's analysis. Next we find a reformulation of the principal's combinatorial mechanism design problem, the solution to which, when coupled with the right payments, will solve the original problem. Next, we identify sufficient conditions for regularity for two large classes of valuations. The first class contains valuations that satisfy a nondecreasing differences property and the second class consists of valuations that evaluate sets after scalarizing them.

Besides Myerson (1981), the papers that are most closely related to our paper are Branco (1996) and Levin (1997) both of which circumvent multidimensional type problems by assuming one dimensional types.

The description of the optimal behavior of the principal is generally difficult even in regular combinatorial problems as it involves an optimal partitioning problem for each reported type vector: sets should be allocated such that the sum of virtual valuations achieve a maximum. In fact, in many problems exhaustive iteration over all possible allocations may be the only way to solve this problem. However in certain special cases a solution to the partitioning problem can easily be obtained using the greedy algorithm of allocating each subsequent object to the agent with the highest virtual valuation for it, as exemplified by Branco's (1996) analysis of the optimal mechanism design problem with multiple identical units. Following Myerson, Branco reformulates this problem using virtual valuations and a monotonicity constraint and studies regular problems for which the monotonicity constraint is not binding. Branco is also able to describe the principal's behavior at an optimal mechanism and this is facilitated by the special structure of his setup. In Section 7 we show how to map Branco's model into ours and point out similarities and differences.

Levin (1997) considers the optimal mechanism design problem with two complementary objects, but takes a more direct approach than Myerson. Instead of reformulating the problem and solving it under regularity conditions, Levin solves the original problem without the incentive constraints and shows that the incentive constraints are satisfied. We point out in Section 5 that Levin's approach works because his assumptions on the data of his model are enough to ensure regularity the way we define it in Section 4.

The paper proceeds as follows. In the next section we introduce the model. In Section 3 we define the optimal combinatorial mechanism design problem faced by the principal. In Section 4 we reformulate this problem and define regularity. In Section 5 we study sufficient conditions for regularity for a class of valuations that satisfy a nondecreasing differences property (Condition A). In Section 6 we focus on another class of valuation functions that involve scalarization of sets (Condition B) and analyze regularity for this class. We point out via examples the usefulness of both classes of valuation functions in Section 7. Section 8 discusses our model and findings and Section 9 contains concluding remarks. Some proofs are relegated to the Appendix.

2 Basic Notation and Assumptions

We consider an optimal mechanism design problem, in which the set of social alternatives consists of all allocations of a set Ω of nonidentical objects between a set of agents and a principal. A typical example is a combinatorial auction where a seller will allocate a finite number of nonidentical indivisible objects among bidders so as to maximize expected revenue. Let $N = \{1, ..., n\}$ be the set of agents. The space of outcomes is $\mathcal{C} \times \Re^n$ where

$$\mathcal{C} := \{\mathcal{S} = (S_1, ..., S_n) : \cup_i S_i \subseteq \Omega \text{ and } S_i \cap S_j = \emptyset \text{ if } i \neq j\}$$

is the set of all collections of n pairwise disjoint subsets of Ω. The set S_i in the list $\mathcal{S}$ identifies the objects allocated to agent i. Note that the collection $\mathcal{S}$ need not cover Ω, i.e., some members of Ω may remain unallocated to any agent. The requirement that the sets S_i and S_j be disjoint for different agents i and j ensures that no single object is allocated to multiple agents. Note that if $n = 1$, then $\mathcal{C} = 2^{\Omega}$.

One way of formalizing private information and valuations of agents in this setup is to work with multidimensional types. In this approach a type for agent i is a vector $(t_i^S)_{S \subseteq \Omega}$ whose entries indicate the valuation of the agent if he is allocated different subsets of Ω. As noted in the introduction, the kind of multidimensional mechanism design problem that this approach leads to has been recognized as a very difficult problem and is an area of active current research.

In this paper we follow a different path in which multidimensionality arises not because of types, but because of outcomes. We assume that agents have independent private information in the form of one dimensional types. The type space of agent i is an interval $T_i = [a_i, b_i]$ of the real line and f_i is the strictly positive density defining the distribution F_i of i's type. We let $T \equiv T_1 \times \cdots \times T_n$ and $T_{-i} \equiv T_1 \times \cdots T_{i-i} \times T_{i+1} \times \cdots \times T_n$. For each $t \in T$, we let f and f_{-i}

be the joint densities for $\{f_j : j = 1, ..., n\}$ and $\{f_j : j \neq i\}$ respectively, with associated distributions F and F_{-i}.

As a matter of interpretation, we can take i's privately known type to be his income, or any other unidimensional parameter that influences his valuation for subsets of Ω.

The payoffs of agents depend on the set of objects they receive, the size of their payments, and their private information t_i. In particular an agent does not care about the objects he does not receive or the information he does not have. Given an outcome $(S_1, ..., S_n, x_1, ..., x_n) \in \mathcal{C} \times \Re^n$, and a type t_i, we assume that agent i's payoff is $v_i(S_i, t_i) - x_i$ where $v_i : 2^{\Omega} \times T_i \to \Re$ is his valuation function. We assume that the principal attaches no value to the objects and hence his utility is simply the sum of payments $\sum_i x_i$. At the cost of additional notation, all our results extend to a setting in which the principal has positive valuations for various sets of objects.

We maintain the following assumption regarding the valuations $v_i : 2^{\Omega} \times T_i \to \Re$ throughout the paper.

Assumption For each (i, S), $v_i(S, \cdot)$ is differentiable, monotone increasing and absolutely continuous.[3]

This is a technical condition which allows us to characterize incentive compatibility in terms of a well known *envelope* condition and an appropriate *monotonicity* condition. In the next section we present the optimal combinatorial mechanism design problem faced by the principal.

3 The Optimal Mechanism Design Problem

The principal is interested in finding a way to allocate the objects and collect payments that will maximize his expected revenue. Using the *Revelation Principle*, he can restrict attention to direct mechanisms of the form (q, x) where $q : \mathcal{C} \times T \to \Re$ and $x : T \to \Re^n$ determine probabilities of different allocations and sizes of payments, respectively. Since $q(\mathcal{S}, t)$ is the probability that allocation $\mathcal{S}$ is chosen when the profile of announced types is t, it follows that a mechanism (q, x) satisfies the following condition:

Condition F For each $(\mathcal{S}, t)$, $q(\mathcal{S}, t) \geq 0$ and for each t, $\sum_{\mathcal{S} \in \mathcal{C}} q(\mathcal{S}, t) = 1$.

Note that the second set of conditions in F stipulates equalities because the definition of $\mathcal{C}$ allows some objects to remain with the principal.

As usual, a direct mechanism induces a Bayesian game. In this game, agents learn and then report their types to the principal and the principal chooses allocations (or randomizes over them) and payments. The payoffs are as described in the previous section.

In designing a mechanism the principal has to take into account incentive problems caused by asymmetric information. In particular he must make sure

[3] For each (i, S, t_i) we let $\partial_2 v_i(S, t_i)$ denote the value of the derivative of $v_i(S, \cdot)$ at t_i.

that agents are willing to participate in the game induced by the chosen mechanism, and that they have the right incentives to report their types truthfully. To address these concerns we define, for any given mechanism (q, x), the following expectations for each i, t_i and t'_i:

$$\begin{aligned} U_i(t_i) &= \int_{T_{-i}} \left[\sum_{\mathcal{S}} v_i(S_i, t_i) q(\mathcal{S}; t_i, t_{-i}) - x_i(t_i, t_{-i}) \right] dF_{-i}(t_{-i}) \\ \mathcal{U}_i(t'_i; t_i) &= \int_{T_{-i}} \left[\sum_{\mathcal{S}} v_i(S_i, t_i) q(\mathcal{S}; t'_i, t_{-i}) - x_i(t'_i, t_{-i}) \right] dF_{-i}(t_{-i}) \end{aligned}$$

These expressions are the expected utilities to i of type t_i from reporting truthfully and from reporting t'_i, respectively. Note that for each $\mathcal{S}$, v_i depends only on the ith component S_i of $\mathcal{S}$.

Definition 1 *A mechanism (q, x) is incentive compatible if*

$$\text{for each } i, t_i \text{ and } t'_i,\ U_i(t_i) \geq \mathcal{U}_i(t'_i; t_i) \tag{IC}$$

and individually rational if

$$\text{for each } i \text{ and } t_i, U_i(t_i) \geq 0. \tag{IR}$$

If a mechanism is incentive compatible (IC), then truthful reporting is a Bayes Nash equilibrium in the induced Bayesian game. If a mechanism is individually rational (IR), then the aforementioned equilibrium of the induced game gives each agent at least his reservation utility, which we normalize to 0. Note that we implicitly assume reservation utilities to be independent of agents' types.

Now we can write down the problem faced by the principal. The principal wants to choose a mechanism that will maximize his expected utility with the constraints that the mechanism be feasible, incentive compatible and individually rational. Thus the optimal combinatorial mechanism design problem becomes:

$$\max_{(q,x)} \int_T \sum_i x_i(t) dF(t) \text{ subject to Condition } F, IC \text{ and } IR. \tag{P}$$

We now move on to a reformulation of P.

4 Reformulation of the Mechanism Design Problem

4.1 Reformulating the Constraints

The formulation of the optimal mechanism design problem in the previous section does not suggest any straightforward solution procedure. Hence, as is typical in the literature, we move on to reformulate the problem in such a way that under a regularity condition, which we later define, a solution may more easily be obtained.

First we reformulate the constraint set. To do this, we must identify conditions on mechanisms that are necessary and sufficient for incentive compatibility and individual rationality. It is standard in mechanism design analysis to characterize incentive compatibility by means of an envelope and a monotonicity condition. If the valuations are affine in types, a subgradient condition can also be used to characterize incentive compatibility as in Krishna (2002) and Figuera and Skreta (2006). We define and discuss this subgradient condition in Section 8 and prove that it is, in general, not sufficient for incentive compatibility. Hence we need a stronger set of conditions in our setup.

We first state the following well known condition on mechanisms (q, x) which is instrumental in the characterization of incentive compatibility, in the reformulation of the objective function of P and also in a subsequent revenue equivalence result.

Envelope Condition For each i, t_i and t_i'

$$U_i(t_i) = U_i(t_i') + \int_{t_i'}^{t_i} \left[\int_{T_{-i}} \sum_{\mathcal{S}} \partial_2 v_i(S_i, y) q(\mathcal{S}; y, t_{-i}) dF_{-i}(t_{-i}) \right] dy. \quad (E)$$

As Proposition 1 below records, E is a necessary condition for incentive compatibility. This result is driven by absolute continuity and differentiability of the maps $v_i(S, \cdot) : T_i \to \Re$.

Condition E has been used in the characterization of incentive compatibility extensively in varied setups. Restricting attention to only continuously differentiable mechanisms, one can see that incentive compatibility implies condition E by applying the envelope formula on

$$U_i(t_i) = \max_{t_i'} \mathcal{U}(t_i', t_i).$$

However this entails an obvious loss of generality in the analysis since the principal may wish to resort to mechanisms that are not continuously differentiable. Alternatively, if for each i, $v_i(S, \cdot)$ is a convex function, a direct Riemann sum argument shows that incentive compatible mechanisms satisfy the envelope condition. This is essentially the method employed by Myerson (1981) in his analysis of an optimal mechanism design problem with a single object, albeit with linearity in types. Krishna and Maenner (2002) show that if valuations are convex in types, then incentive compatible mechanisms obey the envelope condition using the theory of convex potentials. A very wide class of valuations for which the said implication is valid has been identified by Milgrom and Segal (2002) and it consists of valuations for which the maps $v_i(S, \cdot)$ are absolutely continuous and differentiable for each (i, S), as required by our assumption in Section 2.

Monotonicity of the outcome function q also plays a crucial role in mechanism design both in the reformulation of the constraint set and in the definition and solution of regular problems. In our setup the following notion of monotonicity is useful, which is an adaptation of a monotonicity condition used by Branco (1996) in the analysis of auctions with multiple identical units.

Monotonicity For each i, t_i and t'_i

$$\int_{T_{-i}} \sum_{\mathcal{S}} (v_i(S_i, t_i) - v_i(S_i, t'_i)) q(\mathcal{S}; t'_i, t_{-i}) dF_{-i}(t_{-i})$$
$$\leq \int_{t'_i}^{t_i} \left[\int_{T_{-i}} \sum_{\mathcal{S}} \partial_2 v_i(S_i, y) q(\mathcal{S}; y, t_{-i}) dF_{-i}(t_{-i}) \right] dy. \quad (M)$$

Note that M is a condition on q only and in the single object case with valuations linear in t_i it coincides with the monotonicity condition used by Myerson (1981). In Section 8 we also define and discuss a different generalization of Myerson's monotonicity to our multi-object setting and compare it with M.

Next we record the use of conditions M and E in the characterization of incentive compatible mechanisms.

Proposition 1 *A mechanism (q, x) is incentive compatible if and only if it satisfies E and M.*

Proof. Suppose (q, x) is incentive compatible. Condition E follows from Corollary 1 in Milgrom and Segal (2002). To see that M is also satisfied note that for all i, t_i and t'_i

$$\begin{aligned}
&\int_{T_{-i}} \sum_{\mathcal{S}} q(\mathcal{S}, t'_i, t_{-i})(v_i(S_i, t_i) - v_i(S_i, t'_i) dF_{-i}(t_{-i}) \\
&\leq U_i(t_i) - U_i(t'_i) \\
&= \int_{t'_i}^{t_i} \left[\int_{T_{-i}} \sum_{\mathcal{S}} \partial_2 v_i(S_i, y) q(\mathcal{S}; y, t_{-i}) dF_{-i}(t_{-i}) \right] dy.
\end{aligned}$$

and this proves the only if direction. Now suppose that (q, x) satisfies E and M. For all i, t_i and t'_i

$$\begin{aligned}
U_i(t_i) - U_i(t'_i) &= \int_{t'_i}^{t_i} \left[\int_{T_{-i}} \sum_{\mathcal{S}} \partial_2 v_i(S_i, y) q(\mathcal{S}; y, t_{-i}) dF_{-i}(t_{-i}) \right] dy. \\
&\geq \int_{T_{-i}} \sum_{\mathcal{S}} q(\mathcal{S}, t'_i, t_{-i})(v_i(S_i, t_i) - v_i(S_i, t'_i)) dF_{-i}(t_{-i})
\end{aligned}$$

from which incentive compatibility follows. ■

It is common and useful to interpret the *if* direction of the preceding result as follows: for any map q that satisfies condition M, we can find a map x for payments via condition E that will make the pair (q, x) incentive compatible. In this sense M may be thought of as an implementability condition for an allocation function q.

In order to characterize the constraint set, we have yet to study individual rationality. In a model without externalities, nonparticipation payoffs can be taken to be independent of what would happen in case an agent chooses not to participate, and therefore they can be normalized to zero. Although it may be difficult to come up with conditions on a mechanism that characterize individual rationality alone, this task is much easier in the presence of incentive compatibility, as all we have to concern ourselves with is the expected utility of the "lowest" type of each agent.

Proposition 2 *An incentive compatible mechanism is individually rational if and only if for each* i, $U_i(a_i) \geq 0$.

Proof. Condition E and the monotonicity of $v_i(S, \cdot)$ implies that if the mechanism is incentive compatible then $U_i(\cdot)$ is weakly increasing, and the result follows. ■

Note that the monotonicity of the functions $v_i(S, \cdot)$ is used only in the characterization of individual rationality. In particular Proposition 1 did not require this property of valuation functions.

As a corollary of Propositions 1 and 2 we get the following characterization of feasibility in Problem P.

Corollary 1 *A mechanism* (q, x) *is feasible in* P *if and only if the following conditions hold:*

1. (q, x) *satisfies* F, E *and* M, *and*
2. *for each* i, $U_i(a_i) \geq 0$.

4.2 Reformulating the Objective

Having reformulated the constraints we turn now to the problem of reformulating the objective function in P. But first we need to define virtual valuations of the agents, which, as is typical in the literature, will play a central role in our analysis.

Definition 2 *The virtual valuation of agent* i *is the map* $u_i : 2^{\Omega} \times T_i \to \Re$ *defined for each* (S, t_i) *by*

$$u_i(S, t_i) = v_i(S, t_i) - \partial_2 v_i(S, t_i) \frac{1 - F_i(t_i)}{f_i(t_i)}. \tag{1}$$

The next result indicates that if the principal is going to choose an incentive compatible mechanism, then his expected payoff can be stated in terms of virtual valuations and the payments of the lowest type of each agent.

Proposition 3 *If (q, x) is incentive compatible, then the principal's expected utility can be written as*

$$\int_T \sum_i x_i(t)dF(t) = \int_T \left(\sum_i \sum_{\mathcal{S}} u_i(S_i, t_i) q(\mathcal{S}; t) \right) dF(t) - \sum_i U_i(a_i). \qquad (2)$$

Proof. See the Appendix. ■

The proof of this lemma hinges on condition E and not on incentive compatibility. We could alternatively state it for mechanisms that satisfy E. Note that the payments will appear only in the term $\sum_i U_i(a_i)$ in 2. This is going to be extremely helpful in our reformulation: to maximize the expected revenue, the principal will choose payments so as to equate the expected payoff of the lowest type of each agent to zero.

As an immediate consequence of Proposition 3 we get the following revenue equivalence result.

Corollary 2 *(Revenue Equivalence Theorem) If (q, x) and $(\widehat{q}, \widehat{x})$ are incentive compatible mechanisms such that $q(\cdot;\cdot) = \widehat{q}(\cdot;\cdot)$ and if the lowest type of each agent gets the same expected utility in both mechanisms, then*

$$\int_T \sum_i x_i(t)dF(t) = \int_T \sum_i \widehat{x}_i(t)dF(t).$$

Proof. Direct application of Proposition 3. ■

4.3 The Reformulated Mechanism Design Problem

Recalling that M is a condition on q alone, consider the following problem R:

$$\max_q \int_T \left[\sum_i \sum_{\mathcal{S}} q(\mathcal{S}, t) u_i(S_i, t_i) \right] dF(t) \text{ subject to } F \text{ and } M \qquad (R)$$

As the next result shows problems P and R are closely related.

Proposition 4 *If q solves R and*

$$x_i(t) = \sum_{\mathcal{S}} v_i(S_i, t_i) q(\mathcal{S}, t) - \int_{a_i}^{t_i} \sum_{\mathcal{S}} \partial_2 v_i(S_i, y) q(\mathcal{S}, y, t_{-i}) dy \qquad (3)$$

for each i and $t = (t_i, t_{-i})$, then the mechanism (q, x) solves P.

Proof. See the Appendix ■

In view of this result, to find the optimal combinatorial mechanism, the principal will only need to solve problem R and determine payments as in 3.

Note that the payments are such that the lowest type of each agent will not benefit from possessing private information.

As the argument in the Appendix shows, the proposition could be strengthened to work in the reverse direction as well: if (q, x) solves P then q solves R and for each (i, t_i) the following expected payments condition holds:

$$\int_{T_{-i}} x_i(t_i, t_{-i}) dF_{-i}(t_{-i}) = \int_{T_{-i}} \left[\sum_{\mathcal{S}} v_i(S_i, t_i) q(\mathcal{S}, t_i, t_{-i}) \right] dF_{-i}(t_{-i}) - \int_{T_{-i}} \left[\int_{a_i}^{t_i} \sum_{\mathcal{S}} \partial_2 v_i(S_i, y) q(\mathcal{S}, y, t_{-i}) dy \right] dF_{-i}(t_{-i})$$

4.4 Regular Problems

Problem R would obviously be easier to solve without the constraint M. To maximize the objective function in problem R subject only to the feasibility constraint F, it is sufficient to solve the optimal partitioning problem

$$\max_{\mathcal{S} \in \mathcal{C}} \sum_{i \in N} u_i(S_i, t_i) \tag{4}$$

for each $t \in T$. If $\mathcal{S}(t)$ denotes a solution, then defining $q^*(\mathcal{S}(t), t) = 1$ for all t yields an optimal solution to problem R without the monotonicity constraint M. Of course, q^* need not be a solution to problem R since it need not satisfy M.

Definition 3 *The optimal combinatorial mechanism design problem is regular if q^* satisfies M whenever*

$$\mathcal{S}(t) \in \arg\max_{\mathcal{S} \in \mathcal{C}} \sum_{i \in N} u_i(S_i, t_i)$$

for each t and $q^*(\mathcal{S}(t), t) = 1$ for each t.

This means that, as a consequence of Proposition 4, a solution to problem P in the regular case can be found by first computing an optimal solution to 4, then defining $q^*(\mathcal{S}(t), t) = 1$ for all t as above and finally computing the payments according to the formula in Proposition 4. In the next two sections, we identify two related but different classes of problems for which reasonably tractable sufficient conditions for regularity can be identified.

5 Regularity and Supermodularity

In this section we will be interested in identifying sufficient conditions for regularity for a certain class of valuation functions. In order to describe this class and state our results we need several definitions.

Definition 4 *A map* $\mu : 2^\Omega \times [a,b] \to \Re$ *has nondecreasing differences if for each* $(S,t),(S',t') \in 2^\Omega \times [a,b]$ *such that* $t \le t'$ *and* $S \subseteq S'$

$$\mu(S,t') - \mu(S,t) \le \mu(S',t) - \mu(S',t)$$

and μ *has strictly increasing differences the inequality is strict whenever* $t < t'$ *and* $S \subset S'$.[4]

Definition 5 *A map* $\mu : 2^\Omega \times [a,b] \to \Re$ *is supermodular if for each* $(S,t),(S',t') \in 2^\Omega \times [a,b]$

$$\mu(S,\alpha) + \mu(S',\alpha') \le \mu(S \cup S', \max\{\alpha,\alpha'\}) + \mu(S \cap S', \min\{\alpha,\alpha'\})$$

and μ *is strictly supermodular if the inequality is strict whenever* $(S,\alpha) \neq (S',\alpha')$.

Definition 6 *A map* $\sigma : 2^\Omega \to \Re$ *is supermodular on* 2^Ω *if for each* $S, S' \subseteq \Omega$

$$\sigma(S) + \sigma(S') \le \sigma(S \cup S') + \sigma(S \cap S')$$

and σ *is strictly supermodular if the inequality is strict whenever* $S \neq S'$.

Definition 7 *The objects in* Ω *are complements if for each* i *and* t_i, $v_i(\cdot,t_i)$ *is supermodular.*

The following is a standard result in the theory of supermodular functions.

Proposition 5 *A map* $\mu : 2^\Omega \times [a,b] \to \Re$ *is (strictly) supermodular if and only if it has (strictly increasing) nondecreasing differences and* $\mu(\cdot,t)$ *is (strictly) supermodular for all* $t \in [a,b]$.

Proof. See Topkis (1998). ■

Now consider the class of problems which satisfy the following condition.

Condition A For each i, v_i has nondecreasing differences.

Note that under our assumption of differentiability v_i has nondecreasing differences if and only if the map $S \mapsto \partial_2 v_i(S,t_i)$ is isotone for each t_i.

Proposition 5 implies that if all valuations are supermodular, then Condition A is satisfied. However Condition A does not imply that the objects are complements: let $\Omega = \{\omega_1,\omega_2\}$ and $T_i = [0,1]$ and consider the valuation functions defined by: $v_i(\emptyset,t_i) = 0$, $v_i(\{\omega_1\},t_i) = t_i/2$ and $v_i(\{\omega_2\},t_i) = v_i(\Omega,t_i) = t_i$.

We can now identify sufficient conditions for regularity.

Proposition 6 *Suppose that for each* i

- ⋆ $u_i(\cdot,t_i)$ *is supermodular for each* t_i, *and*

[4] We use $\subseteq$ for weak and $\subset$ for strict set inclusion.

$\star\star$ *$u_i(\cdot,\cdot)$ has strictly increasing differences.*

Then for every selection

$$(S_1(t), ..., S_n(t)) \in \arg\max_{\mathcal{S}\in\mathcal{C}} \sum_{i\in N} u_i(S_i, t_i),$$

$t_i \mapsto S_i(t_i, t_{-i})$ is weakly expanding. If, in addition, Condition A is also satisfied, then the optimal combinatorial mechanism design problem is regular.

Proof. See the Appendix. ■

Note that Proposition 6 relies on two types of hypotheses. Condition A is an assumption about the underlying valuations v_i while conditions $\star$ and $\star\star$ are assumptions about the virtual valuations u_i. The proof proceeds in two steps. In the first step, we use conditions $\star$ and $\star\star$ to show that the optimal allocation exhibits a very strong "monotonicity" property with respect to agents' types. To be more precise, we show that all selections $(S_1(t), ..., S_n(t))$ that solve 4 have the property that for each i and t_{-i}, the set valued maps $t_i \mapsto S_i(t_i, t_{-i})$ are weakly expanding, i.e., $t_i < t_i'$ implies that $S_i(t_i, t_{-i}) \subseteq S_i(t_i', t_{-i})$. In the second step, we define $q^*(\mathcal{S}(t), t) = 1$ for all t. We then use this noncontracting property of the optimal allocation, in conjunction with Condition A, to show that q^* satisfies M, thus establishing regularity.

As a corollary to Proposition 6 we get the following result stated in terms of supermodularity properties only.

Corollary 3 *If each v_i is supermodular and each u_i is strictly supermodular, then the optimal combinatorial mechanism design problem is regular.*

Proof. If each v_i is supermodular, then Condition A is satisfied and u_i is strictly supermodular if and only if $u_i(\cdot, t_i)$ is strictly supermodular and $u_i(\cdot,\cdot)$ has strictly increasing differences. Now the result follows from an application of the Proposition. ■

Valuations will satisfy Condition A in the leading special case in which, for each i, $v_i(S, t_i) = w_i(S)t_i$ for some strictly supermodular $w_i : 2^\Omega \to T$. In this case, the problem is regular if the hazard rates of type distributions are nondecreasing. To see this note that the virtual valuation of agent i is given by

$$u_i(S, t_i) = w_i(S)[t_i - \frac{1 - F_i(t_i)}{f_i(t_i)}]$$

and is strictly supermodular if $t_i \mapsto t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$ is strictly increasing. This is, in turn, implied by nondecreasing hazard rates.

A special case where Condition A is satisfied has been analyzed by Levin (1997) who considers the problem of finding the optimal mechanism to allocate two complementary goods. His methodology is different from Myerson's. Instead of reformulating the optimal mechanism design problem, he uses a direct argument tailored to the two object case. Without invoking the machinery of supermodular optimization, he employs an exhaustive analysis of all possible cases to prove that the optimal allocation has the noncontracting property described above in our discussion of Proposition 6. It is important to point out that the assumptions that Levin made in his two object model coincide exactly with the assumptions of Proposition 6 when specialized to a two object problem. By applying supermodularity methods to the reformulated problem, we can generalize Levin's results to any number of objects.

6 Regularity and Single Crossing

The following example motivates the family of valuations that we analyze in this section.

Example 1 *Suppose that $v(S,t) = w(S)t$ where w is a strictly submodular isotone function. Then $u(S,t) = w(S)(t - \frac{1-F(t)}{f(t)})$ and $u(\cdot,t)$ is not supermodular and Proposition 6 does not apply.*

In this section we study regularity for valuation functions that satisfy the following condition.

Condition B For each i, there exist $w_i : 2^\Omega \to \Re, r_i : 2^\Omega \to \Re$ and $g_i : \Re \times T_i \to \Re$ such that

1. for each (S,t_i), $v_i(S,t_i) = g_i(w_i(S),t_i) + r_i(S)$, and
2. $g_i(\cdot,\cdot)$ has the *single crossing* property, i.e., if $z \le z'$ and $t_i \le t_i'$, then

$$g_i(z,t_i') - g_i(z,t_i) \le g_i(z',t_i') - g_i(z',t_i)$$

For future reference, note that under our assumption of differentiability of valuations in types, g_i has the single crossing property if and only if $z \mapsto \partial_2 g_i(z,t_i)$ is nondecreasing for all t_i and that g_i would have the *strict single crossing* property if the displayed inequality held strictly whenever $z < z'$ and $t_i < t_i'$. The single crossing property will have the role previously played by the isotonicity of $S \mapsto \partial_2 v_i(S,t_i)$ in Section 5.

Now let $\gamma_i : \Re \times T_i \to \Re$ be given by

$$\gamma_i(\alpha,t_i) = g_i(\alpha,t_i) - \partial_2 g_i(\alpha,t_i)\frac{1-F_i(t_i)}{f_i(t_i)}.$$

Recalling 4 we obtain

$$u_i(S,t_i) = \gamma_i(w_i(S),t_i) + r_i(S)$$

for each i, S and t_i. Now we can identify sufficient conditions under which the optimal mechanism design problem will be regular.

Proposition 7 *Suppose that γ_i has the strict single crossing property for each i. Then for every selection*

$$(S_1(t),...,S_n(t)) \in \arg\max_{\mathcal{S}\in\mathcal{C}} \sum_{i\in N} u_i(S_i,t_i),$$

$t_i \mapsto w_i(S_i(t_i,t_{-i}))$ is nondecreasing for each i and t_{-i}. If, in addition, Condition B is satisfied, then the optimal combinatorial mechanism design problem is regular.

Proof. See the Appendix. ■

Example 2 (Example 1 continued) *Suppose, once again, that $v(S,t) = w(S)t$ for some strictly submodular and isotone w. Now let $g : \Re \times T \to \Re$ be given by $g(z,t) = zt$ so that $v(S,t) = w(S)t = g(w(S),t)$ and Condition B is satisfied. We have*

$$u(S,t) = \gamma(w(S),t) = w(S)(t - \frac{1-F(t)}{f(t)})$$

and the problem is regular if the hazard rate is nondecreasing.

The proof of Proposition 7 is similar to that of Proposition 6 and again we proceed in two steps. In the first step, we use the strict single crossing property of the γ_is to establish a different kind of "monotonicity" property with respect to agents' types. To be more precise, we use the strict single crossing property to show that all selections $(S_1(t),...,S_n(t))$ which solve 4 have the property that for each i and t_{-i}, the real valued function $t_i \mapsto w_i(S_i(t_i,t_{-i}))$ is nondecreasing. In the second step, we define $q^*(S(t),t) = 1$ for all t. We then use this monotonicity property of the optimal allocation , in conjunction with Condition B, to show that q^* satisfies M, thus establishing regularity.

The monotonicity property of the optimal allocation established in Proposition 6 (i.e. that $t_i \mapsto S_i(t_i,t_{-i})$ is "weakly expanding") and the monotonicity property of the optimal allocation established in Proposition 7 (i.e. that $t_i \mapsto w_i(S_i(t_i,t_{-i}))$ is nondecreasing) are related but are not nested. If w_i is isotone and $t_i \mapsto S_i(t_i,t_{-i})$ is weakly expanding, then $t_i \mapsto w_i(S_i(t_i,t_{-i}))$ is nondecreasing. However for Proposition 7 to apply optimal allocations need not be expanding in types. We illustrate this in the next example.

Example 3 *Consider a single agent mechanism design problem with $v(S,t) = w(S)t + r(S)$ and let t be distributed uniformly over $[0,1]$. Then*

$$u(S,t) = w(S)(2t-1) + r(S).$$

Let $\Omega = \{\omega_1,\omega_2\}$ and let the following table give the values of the functions:

	$\emptyset$	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_1,\omega_2\}$
w	0	0	1	1
r	0	1	0	-2

Note that Condition A and Condition B are both satisfied in this example. Observe that

$$S(t) \in \arg\max_{S \subseteq \Omega} u(S,t) \Rightarrow S(t) = \begin{cases} \{\omega_1\} & \text{if } t < 1/2 \\ \{\omega_2\} & \text{if } t > 1/2 \end{cases}$$

However it is easy to see that $t \mapsto w(S(t)$ is nondecreasing so the problem is regular.

7 Applications

7.1 Mechanism design with a single object

Consider, once again, the single object mechanism design problem analyzed by Myerson (1981). To map this model into ours, let $\Omega = \{\omega_1\}$ and $v_i(S,t_i) = w_i(S)t_i$ where for each i

$$\begin{aligned} w_i(\Omega) &= 1, \\ w_i(\emptyset) &= 0. \end{aligned}$$

Then Condition A is trivially satisfied.[5] Now note that virtual valuations will be such that

$$\begin{aligned} u_i(\Omega,t_i) &= t_i - \frac{1-F_i(t_i)}{f_i(t_i)} \\ u_i(\emptyset,t_i) &= 0 \end{aligned}$$

and therefore $u_i(\cdot,t_i)$ is supermodular for each t_i. Furthermore u_i has strictly increasing differences if for each $t_i < t_i'$

$$[t_i' - \frac{1-F_i(t_i')}{f_i(t_i')}] - [t_i - \frac{1-F_i(t_i)}{f_i(t_i)}] > 0$$

and this happens if

$$\frac{1-F_i(t_i)}{f_i(t_i)} \geq \frac{1-F_i(t_i')}{f_i(t_i')}.$$

Thus we have the following corollary to Proposition 6.

Corollary 4 *(Myerson, 1981) If $\Omega = \{\omega_1\}$ and $v_i(S,t_i) = w_i(S)t_i$ where $w_i(\Omega) = 1$ and $w_i(\emptyset) = 0$ for each i, then the optimal mechanism design problem is regular if the distributions of types have nondecreasing hazard rates.*

Note that we could alternatively use the Proposition 7 to specialize to a single object mechanism design problem. Suppose that $\Omega = \{\omega_1\}$ and that for each i, $w_i(\emptyset) = 0$ and $w_i(\Omega) = 1$ and $g_i(z,t_i) = zt_i$ so that $v_i(S,t_i) = g_i(w_i(S),t_i)$. Note that g_i satisfies the single crossing property and Condition B is satisfied. Now the preceding result becomes a corollary to Proposition 7.

[5] In fact, for any model in which valuations are affine in types and isotone in sets received, Condition A is automatically satisfied.

7.2 Mechanism design with scalar measure valuations

Agents have *scalar measure valuations* if for each i, there exists numbers w_i^k, $k = 1, 2, ..., |\Omega|$ and a map $g_i : \Re \times T_i \to \Re$ such that $v_i(S, t_i) = g_i(\sum_{k \in S} w_i^k, t_i)$. If g_i has the single crossing property, then Condition B is satisfied with $w_i(S) = \sum_{k \in S} w_i^k$. [6]

As an example consider Branco's (1996) analysis of a model in which the principal will allocate several identical objects among the agents. Suppose that $w_i^k = 1$ for each i and k so that $w_i(S) = |S|$. According to Proposition 7, the sufficient conditions for regularity are Condition B and the strict single crossing property of virtual valuation u_i defined in the usual way.

Suppose that the marginal valuation of agent i for the kth object, given by $g_{ik} : T_i \to \Re$ where

$$g_{ik}(t_i) := g_i(k, t_i) - g_i(k - 1, t_i).$$

Branco assumes that each g_{ik} is nonnegative valued, increasing and differentiable. Note that if each g_{ik} is increasing, then Condition B is satisfied. Moreover if each marginal virtual valuation $u_{ik} : T_i \to \Re$ defined by

$$u_{ik}(t_i) := u_i(k, t_i) - u_i(k - 1, t_i)$$

is increasing, then both hypotheses of Proposition 7 are satisfied and regularity is obtained.

Besides these, Branco's assumptions also include concavity of the virtual valuations which facilitates the solution of regular problems. In fact the solution to the optimal partitioning problem 4 is remarkably simple if each u_i only depends on the cardinality of S_i and if each u_i is increasing and concave in the cardinality of S_i. The principal will allocate each object to the agent with the highest marginal virtual valuation for that object to solve the partitioning problem. This process is known as the greedy algorithm. Branco's assumptions guarantee that the greedy algorithm can be used in his model.

7.3 Valuations with unit demand

Suppose that for each i there exists $c_i : \Omega \to \Re$ such that

$$v_i(S, t_i) = g_i(\max_{k \in S} c_i(k), t_i).$$

Hence we can take $w_i(S) = \max_{k \in S} c_i(k)$ for each i and S. Now if each g_i has the necessary increasing differences property, then these valuations satisfy Condition B.

Valuations of this form are useful when agents value each object in Ω but can make use of only one object and when there is also free disposal in the sense that agents can dispose of the objects they receive for which their valuation is not maximal.

[6] See Lehmann et al. (2006) for a discussion of valuations based on an underlying measure when X is a continuum. In our case with finite X, w_i^k is simply the measure of object k for agent i and $w_i(S)$ is the resulting measure of set S.

7.4 Chopstick preferences

Suppose that $|\Omega| = 3$ and $v_i(S, t_i) = w_i(S)t_i$ where $w_i(S) = 1$ if $|S| \geq 2$ and $w_i(S) = 0$ otherwise. These valuations satisfy Condition B and for each i, S and t_i

$$u_i(S, t_i) = w_i(S)[t_i - \frac{1 - F_i(t_i)}{f_i(t_i)}].$$

Since $\gamma_i(z, t_i) = z[t_i - \frac{1-F_i(t_i)}{f_i(t_i)}]$ for each $z \in \Re$ and $t_i \in T_i$ we can find that the problem is regular if the hazard rates are nondecreasing.

8 Discussion

In this section, we will shortly discuss two key aspects of our characterization of problem P in comparison with the literature.

8.1 Characterizing incentive compatibility

For notational simplicity, consider the single agent, single object version of our model. The valuation of the agent is, then, a map $v : T \to \Re$ which is absolutely continuous, differentiable and weakly increasing. If (q, x) is a mechanism, then $q : T \to [0, 1]$ defines the probability with which the agent is allocated the object. Then $U(t) = q(t)v(t) - x(t)$. In this setup a mechanism satisfies M if for each t and t',

$$(v(t) - v(t'))q(t') \leq \int_{t'}^{t} q(z)v'(z)dz$$

and it satisfies E if for each t and t'

$$U(t) = U(t') + \int_{t'}^{t} q(z)v'(z)dz$$

provided that the integrals are well defined. We know from the analysis above that a mechanism is incentive compatible if and only if it satisfies M and E.

Now consider the following condition on mechanisms (q, x).

Subgradient condition A mechanism (q, x) satisfies the subgradient condition if for each t, $q(t)v'(t)$ is a subgradient of U at t i.e., for each t'

$$U(t') \geq U(t) + (t' - t)q(t)v'(t).$$

It is a well known result that if v is convex, then incentive compatibility implies that the subgradient condition is satisfied. If v is actually linear in t, then a mechanism is incentive compatible if and only if it satisfies the subgradient condition. This characterization does not hold when v is convex but not linear in t. For example when v is convex, we need nothing less than conditions M and E to characterize incentive compatibility.

Example Let $T = [0, 1/2]$ and let $v(t) = t^2$. For each $t \in T$, let $q(t) = 1 - t$ and $x(t) = -t^3/3$. Then $U(t) = t^2 - (2/3)t^3$ and since $U''(t) = 2 - 4t \geq 0$ on T so U is convex on T. Furthermore $q(t)v'(t) = (1-t)2t = U'(t)$. Therefore, $q(t)v'(t)$ is a subgradient of U at t, for all $t \in T$. This mechanism is not incentive compatible: let $t = 0$ and $t' = 1/2$. Then $U(0) = 0 < 1/24 = q(1/2)v(0) - x(1/2)$.

In fact the insufficiency of the subgradient condition for incentive compatibility does not depend on the assumption that v is increasing in t.

Example Let $v(t) = 1/t$, $T = [1/2, 1]$ and $(q(t), x(t)) = (t^2, 2t)$. Then $U(t) = -t$ is linear and therefore convex and $U'(t) = -1 = q(t)v'(t)$ so the subgradient condition is satisfied. Incentive compatibility fails because $U(1) = -1 < -3/4 = q(1/2)v(1) - x(1/2)$.

It is easy to check that M is not satisfied in these examples.

8.2 Monotonicity

In our analysis we used monotonicity condition M which is somewhat cryptic. Consider the following alternate monotonicity condition which is more transparent.

Definition 8 *A mechanism (q, x) satisfies M' if for each i, t_i, t_i'*

$$\int_{T_{-i}} \sum_{\mathcal{S}} [v(S_i, t_i) - v_i(S_i, t_i')][q(\mathcal{S}, t_i, t_{-i}) - q(\mathcal{S}, t_i', t_{-i})] dF_{-i}(t_{-i}) \geq 0$$

Like M, condition M' is also a generalization of the monotonicity notion used by Myerson (1981) to a multi-object setting. It is easy to verify that M implies M'. Although M and M' are equivalent when there is a single object, so far we could not establish this equivalence in our combinatorial setting or find an example that shows that M' is weaker than M. We leave this for future work.

9 Conclusion

In this paper we analyzed a model of multiobject mechanism design with one dimensional private information. Using techniques developed by Myerson (1981) we reformulated the revenue maximizing mechanism design problem of the principal using a monotonicity constraint. Next, we studied conditions under which the problem will be regular, i.e., the monotonicity constraint will not be binding. We showed that for two large classes of valuations, sufficient conditions for regularity can be obtained. The first class consists of valuations that satisfy our Condition A. All supermodular valuations belong to this class as well as many other interesting valuation functions. The second class consists of valuations that satisfy Condition B. For both classes, sufficient conditions for regularity are statements about the behavior of virtual valuations.

There are several potential directions for future work.

The first direction is the identification of other classes of valuation functions for which similar analyses can be carried out. The two classes we worked with offered very convenient conditions under which a solution to 4 will satisfy the monotonicity constraint. In the first class of valuations that satisfy Condition A, the theory of maximization of supermodular functions on lattices suggested that supermodularity properties of the maps u_i would be sufficient for regularity. In the second class of valuations satisfying Condition B, it is the single crossing property of the maps u_i that would be sufficient for regularity. In both classes regularity implies that the monotonicity condition can be dropped in the solution. Analysis of other classes of valuations in terms of sufficient conditions for regularity is an interesting area of further research.

The second direction is regarding the analysis of irregular problems. A very useful aspect of the notion of regularity used by Myerson (1981) is that its converse also lends itself to a tractable analysis. This is mainly caused by the fact that in the simple model of Myerson, regularity is defined only in terms of the distributions of agents' types. In our model sufficient conditions for regularity include conditions on valuation functions themselves. Hence the interesting question about the analysis of irregular problems may be far too complicated. One exception is the case of valuations which are affine in types. In particular if $v_i(S, t_i) = w_i(S)t_i$ for some $w_i : 2^\Omega \to \Re$, then Myerson's treatment of the irregular single-object mechanism design problems can easily be generalized to several objects.

The third direction concerns the interpretation of the optimal mechanism in terms of a practical economic mechanism, e.g., an auction. Myerson's optimal mechanism has an auction interpretation if all agents are symmetric. If it exists, such an interpretation in our model must depend on a description of the solution to the associated set partitioning problem. As noted above, there is no known procedure to solve this problem and this indicates that an auction interpretation to the optimal combinatorial mechanism design problem may not exist.

Finally, our model assumes that agents care only about their own private information. In an interdependent values model, each agent is also interested in other agents' types. Relaxing the private values assumption and moving towards interdependent valuations is a promising area of future research.

10 Appendix

10.1 Proof of Proposition 3

Proof. First note that by a simple integration by parts argument we can get for any i and t_{-i}

$$\int_{a_i}^{b_i}\int_{a_i}^{t_i}\sum_{\mathcal{S}}\partial_2 v_i(S_i,y)q(\mathcal{S};y,t_{-i})dydF_i(t_i)$$
$$=\int_{a_i}^{b_i}\sum_{\mathcal{S}}\partial_2 v_i(S_i,t_i)\frac{1-F_i(t_i)}{f_i(t_i)}q(\mathcal{S};t_i,t_{-i})dF_i(t_i).$$

Now, using condition E write, for any i and t_i

$$\int_{T_{-i}} x_i(t_i,t_{-i})dF_{-i}(t_{-i})$$
$$=\int_{T_{-i}}\sum_{\mathcal{S}} v_i(S_i,t_i)q(\mathcal{S};t_i,t_{-i})dF_{-i}(t_{-i})-U_i(a_i)$$
$$-\int_{T_{-i}}\int_{a_i}^{t_i}\left[\sum_{\mathcal{S}}\partial_2 v_i(S_i,y)q(\mathcal{S};y,t_{-i})dy\right]dF_{-i}(t_{-i}).$$

Taking expectations over t_i we get

$$\int_T x_i(t)dF(t)$$
$$=\int_T\sum_{\mathcal{S}} v_i(S_i,t)q(\mathcal{S};t)dF(t)-U_i(a_i)$$
$$-\int_{T_{-i}}\int_{a_i}^{b_i}\int_{a_i}^{t_i}\sum_{\mathcal{S}}\partial_2 v_i(S_i,y)q(\mathcal{S};y,t_{-i})dydF_i(t_i)dF_{-i}(t_{-i}).$$

Using the equation derived above and summing over i, we get the result. ■

10.2 Proof of Proposition 4

Proof. Suppose that q solves R and x is defined as hypothesized. Then (q,x) is incentive compatible since for each i and t_i

$$\begin{aligned} U_i(t_i) &= \int_{T_{-i}}\left[\sum_{\mathcal{S}} v_i(S_i,t_i)q(\mathcal{S};t_{-i},t_i)-x_i(t_{-i},t_i)\right]dF_{-i}(t_{-i}) \\ &= \int_{T_{-i}}\int_{a_i}^{t_i}\sum_{\mathcal{S}}\partial_2 v_i(S_i,y)q(\mathcal{S},y,t_{-i})dydF_{-i}(t_{-i}) \end{aligned}$$

so that

$$U_i(t_i) = U_i(t_i') + \int_{T_{-i}} \int_{t_i'}^{t_i} \sum_{\mathcal{S}} \partial_2 v_i(S_i, y) q(\mathcal{S}, y, t_{-i}) dy dF_{-i}(t_{-i})$$

for each i, t_i and t_i'. Now incentive compatibility follows from the equation above and the fact that q is feasible in R. It is also easy to see that (q, x) is individually rational since $U_i(a_i) = 0$ and $U_i(t_i) \geq U_i(a_i)$ for each i and t_i. Now suppose $(\overline{q}, \overline{x})$ is another feasible mechanism for problem P. Let $\overline{U}_i : T_i \to \Re$ be defined for $(\overline{q}, \overline{x})$ the way U_i is defined for (q, x). Then

$$\begin{aligned}
\int_T \sum_i \overline{x_i}(t) dF(t) &= \int_T [\sum_i \sum_{\mathcal{S}} u_i(S_i, t_i) \overline{q}(\mathcal{S}; t)] dF(t) - \sum_i \overline{U_i}(a_i) \\
&\leq \int_T [\sum_i \sum_{\mathcal{S}} u_i(S_i, t_i) \overline{q}(\mathcal{S}; t)] dF(t) \\
&\leq \int_T [\sum_i \sum_{\mathcal{S}} u_i(S_i, t_i) q(\mathcal{S}; t)] dF(t) \\
&= \int_T \sum_i x_i(t) dF(t)
\end{aligned}$$

and we are done. ■

Remark: As noted in the text, there is a sense in which this implication can be reversed. To see this suppose that (q, x) solves P. Then, by Lemmas 1 and 3, (q, x) also solves

$$\max_{(q,x)} \int_T [\sum_i \sum_{\mathcal{S}} u_i(S_i, t_i) q(\mathcal{S}; t)] dF(t) - \sum_i U_i(a_i)$$
$$\text{subject to } F, E, M \text{ and } U_i(a_i) \geq 0 \text{ for each } i.$$

Since x only appears in $\sum_i U_i(a_i)$ in the objective function, we must have $U_i(a_i) = 0$ and

$$U_i(t_i) = \int_{a_i}^{t_i} \left[\int_{T_{-i}} \sum_{\mathcal{S}} \partial_2 v_i(S_i, y) q(\mathcal{S}; y, t_{-i}) dF_{-i}(t_{-i}) \right] dx,$$

which gives

$$\begin{aligned}
&\int_{T_{-i}} x_i(t) dF_{-i}(t_{-i}) \\
&= \int_{T_{-i}} \left[\sum_{\mathcal{S}} v_i(S_i, t_i) q(\mathcal{S}, t_i, t_{-i}) - \int_{a_i}^{t_i} \sum_{\mathcal{S}} \partial_2 v_i(S_i, y) q(\mathcal{S}, y, t_{-i}) dy \right] dF_{-i}(t_{-i}).
\end{aligned}$$

Now q must also solve

$$\max \int_T [\sum_i \sum_{\mathcal{S}} u_i(S_i, t_i) q(\mathcal{S}; t)] dF(t) \text{ subject to } F \text{ and } M$$

and the result follows.

10.3 Proof of Proposition 6

The proof will follow from several lemmas.

Define for each i the map $\varphi_i : 2^\Omega \times T \to \Re$ given by

$$\varphi_i(S, t_i, t_{-i}) = u_i(S, t_i) + \max\{\sum_{j \neq i} u_j(A_j, t_j) : A_j \text{ are disjoint and } \cup_j A_j \subseteq \Omega \backslash S\}.$$

Lemma 1 *If $u_i(\cdot, \cdot)$ exhibits strictly increasing differences, then so does $\varphi_i(\cdot, \cdot, t_{-i})$ for every t_{-i}.*

Proof. Pick i, t_{-i}, $t_i < t_i'$ and $S \subset S'$. Then

$$\begin{aligned} \varphi_i(S', t_i', t_{-i}) - \varphi_i(S', t_i, t_{-i}) &= u_i(S', t_i') - u_i(S', t_i) \\ &> u_i(S, t_i) - u_i(S, t_i) \\ &= \varphi_i(S, t_i', t_{-i}) - \varphi_i(S, t_i, t_{-i}). \end{aligned}$$

which is what we needed to show. ■

Lemma 2 *If $h_i : 2^\Omega \to \Re$ is supermodular for each $i = 1, ..., k$ and $H_k : 2^\Omega \to \Re$ is defined by*

$$H_k(S) = \left\{ \max \sum_{i=1}^k h_i(A_i, t_i) : A_i \text{ are disjoint and } \bigcup_{i=1}^k A_i \subseteq S \right\}$$

then H_k is also supermodular.

Proof. The proof will proceed by induction on k.

Note that H_1 is supermodular since if $H_1(S) = h_1(A)$ and $H_1(S') = h_1(A')$ then $A \cup A' \subseteq S \cup S'$ and $A \cap A' \subseteq S \cap S'$ and

$$\begin{aligned} H_1(S) + H_1(S') &= h_1(A) + h_1(A') \\ &\leq h_1(A \cup A') + h_1(A \cap A') \\ &\leq H_1(S \cup S') + H_1(S \cap S'). \end{aligned}$$

Now suppose that H_{k-1} is supermodular and note that

$$H_k(S) = \max_{A \subseteq S} \{h_k(A) + H_{k-1}(S \backslash A)\}.$$

Since $A \mapsto H_{k-1}(A)$ is supermodular so is $A \mapsto H_{k-1}(S \backslash A)$. The sum of supermodular functions is supermodular and therefore so is $A \mapsto h_k(A) + H_{k-1}(S \backslash A)$. Then H_k is also supermodular. ■

Lemma 3 *If each $u_i(\cdot, t_i)$ is supermodular, then each $\varphi_i(\cdot, t)$ is supermodular.*

Proof. By the previous lemma and the definition of φ_i, $\varphi_i(\cdot, t)$ is the sum of two supermodular functions and is therefore supermodular. ■

Lemma 4 *If $(S_1(t), ..., S_n(t)) \in \arg\max_{(S_1,...,S_n)\in\mathcal{C}} \sum_i u_i(S_i, t_i)$, then for each i, $S_i(t) \in \arg\max_{S\in 2^\Omega} \varphi_i(S, t_i, t_{-i})$.*

Proof. The result follows from the fact that $S_k(t) \cap S_j(t) = \emptyset$ if $k \neq j$ and that for each i and S

$$\begin{aligned} \sum_{j\in N} u_j(S_j(t), t_j) &= u_i(S_i(t)) + \sum_{j\neq i} u_j(S_j(t), t_j) \\ &\geq \varphi_i(S, t). \end{aligned}$$

■

Lemma 5 *For each i and t_{-i} if $\varphi_i(\cdot, \cdot, t_{-i})$ has strictly increasing differences and if $\varphi_i(\cdot, t_i, t_{-i})$ is supermodular for each t_i, then any selection*

$$S(t) \in \arg\max_{S\in 2^\Omega} \varphi_i(S, t)$$

is such that $t_i \mapsto S(t_i, t_{-i})$ is weakly expanding.

Proof. Pick $i, t_{-i}, t_i < t'_i$ and

$$\begin{aligned} S_i(t_i, t_{-i}) &\in \arg\max_{S\in 2^\Omega} \varphi_i(S, t_i, t_{-i}) \\ S_i(t'_i, t_{-i}) &\in \arg\max_{S\in 2^\Omega} \varphi_i(S, t'_i, t_{-i}) \end{aligned}$$

and therefore if $S_i(t'_i, t_{-i}) \neq S_i(t'_i, t_{-i}) \cup S_i(t_i, t_{-i})$ then

$$\begin{aligned} 0 &\geq \varphi_i(S_i(t'_i, t_{-i}) \cup S_i(t_i, t_{-i}), t'_i, t_{-i}) - \varphi_i(S_i(t'_i, t_{-i}), t'_i, t_{-i}) \\ &> \varphi_i(S_i(t'_i, t_{-i}) \cup S_i(t_i, t_{-i}), t_i, t_{-i}) - \varphi_i(S_i(t'_i, t_{-i}), t_i, t_{-i}) \\ &\geq \varphi_i(S_i(t_i, t_{-i}), t_i, t_{-i}) - \varphi_i(S_i(t'_i, t_{-i}) \cap S_i(t_i, t_{-i}), t_i, t_{-i}) \\ &\geq 0 \end{aligned}$$

which is an impossibility. We conclude, then, that $S_i(t_i, t_{-i}) \subseteq S_i(t'_i, t_{-i})$ which finishes the proof. ■

Proof of Proposition 6 Now for all t, pick any selection

$$\mathcal{S}(t) = (S_i(t))_{i\in N} \in \arg\max \sum_{i\in N} u_i(S_i, t_i)$$

and let

$$q^*(\mathcal{S}(t), t) = 1.$$

By the lemmas, for each i and t_{-i}, $t_i \mapsto S_i(t_i, t_{-i})$ is weakly expanding. Pick $i, t_i' < t_i$. Then $S_i(t_i', t_{-i}) \subseteq S_i(y, t_{-i})$ for all $t_i' \le y \le t_i$ and consequently $\partial_2 v_i(S_i(t_i', t_{-i}), y) \le \partial_2 v_i(S_i(y, t_{-i}), y)$ by Condition A. Then

$$\begin{aligned}
& \int_{T_{-i}} \sum_{\mathcal{S}} q^*(S, t_{-i}, y)[v_i(S_i, t_i) - v_i(S_i, t_i')]dF_{-i}(t_{-i}) \\
= & \int_{T_{-i}} [v_i(S_i(t_{-i}, t_i'), t_i) - v_i(S_i(t_{-i}, t_i'), t_i')]dF_{-i}(t_{-i}) \\
\le & \int_{t_i'}^{t_i} \int_{T_{-i}} \partial_2 v_i(S_i(t_{-i}, t_i'), y)dF_{-i}(t_{-i})dy \\
= & \int_{t_i'}^{t_i} \int_{T_{-i}} \partial_2 v_i(S_i(t_{-i}, y), y)dF_{-i}(t_{-i})dy \\
= & \int_{t_i'}^{t_i} \int_{T_{-i}} \sum_{\mathcal{S}} q^*(S, t_{-i}, y)\partial_2 v_i(S_i, y)dF_{-i}(t_{-i})dy.
\end{aligned}$$

A parallel argument for $t_i \le t_i'$ finishes the proof. ■

10.4 Proof of Proposition 7

Proof. We will first show that $t_i \mapsto w_i(S_i(t_i, t_{-i}))$ is nondecreasing for each i and t_{-i}. To see this choose i, t_{-i} and $t_i' < t_i$. Let $\mathcal{S}(t_{-i}, t_i')$ and $\mathcal{S}(t_{-i}, t_i)$ be optimal partitions that maximize the sum of virtual valuations when the type vectors are (t_{-i}, t_i') and (t_{-i}, t_i) respectively. Suppose, toward a contradiction, that $w_i(S_i(t_{-i}, t_i')) > w_i(S_i(t_{-i}, t_i))$. By strictly increasing differences, we have, for all t_{-i}

$$\begin{aligned}
& u_i(S_i(t_{-i}, t_i'), t_i) - u_i(S_i(t_{-i}, t_i'), t_i') \\
& = \gamma_i(w_i(S_i(t_{-i}, t_i')), t_i) - \gamma_i(w_i(S_i(t_{-i}, t_i')), t_i') \\
& > \gamma_i(w_i(S_i(t_{-i}, t_i)), t_i) - \gamma_i(w_i(S_i(t_{-i}, t_i)), t_i') \\
& = u_i(S_i(t_{-i}, t_i), t_i) - u_i(S_i(t_{-i}, t_i), t_i')
\end{aligned}$$

and therefore

$$\begin{aligned}
& u_i(S_i(t_{-i}, t_i'), t_i) - u_i(S_i(t_{-i}, t_i'), t_i') + u_i(S_i(t_{-i}, t_i), t_i') \\
& > u_i(S_i(t_{-i}, t_i), t_i).
\end{aligned}$$

Adding and subtracting $\sum_{j\neq i}[u_j(S_j(t_{-i},t_i'),t_j)+u_j(S_j(t_{-i},t_i),t_j)]$ we get

$$\begin{aligned}
&\sum_i u_i(S_i(t_{-i},t_i'),t_i) - \left[u_i(S_i(t_{-i},t_i'),t_i') + \sum_{j\neq i} u_j(S_j(t_{-i},t_i'),t_j)\right] \\
&+ \left[u_i(S_i(t_{-i},t_i),t_i') + \sum_{j\neq i} u_j(S_j(t_{-i},t_i),t_j)\right] \\
&> \sum_i u_i(S_i(t_{-i},t_i),t_i) \\
&\geq \sum_i u_i(S_i(t_{-i},t_i'),t_i)
\end{aligned}$$

where the last inequality follows from the optimality of $\mathcal{S}(t_{-i},t_i)$ for type vector (t_{-i},t_i). Cancelling terms we get

$$\begin{aligned}
&u_i(S_i(t_{-i},t_i),t_i') + \sum_{j\neq i} u_j(S_j(t_{-i},t_i),t_j) \\
&> u_i(S_i(t_{-i},t_i'),t_i') + \sum_{j\neq i} u_j(S_j(t_{-i},t_i'),t_j)
\end{aligned}$$

which is a contradiction to the optimality of $\mathcal{S}(t_{-i},t_i')$ for (t_{-i},t_i'). So $t_i \mapsto w_i(S_i(t_i,t_{-i}))$ must be nondecreasing.

Now we will show that if $t_i \mapsto w_i(S_i(t_i,t_{-i}))$ is nondecreasing, then q^* satisfies M. Choose i and $t_i' < t_i$. We have

$$\begin{aligned}
& \int_{T_{-i}} \int_{t_i'}^{t_i} \sum_{\mathcal{S}} \partial_2 g_i(w_i(S_i),z) q^*(\mathcal{S};z,t_{-i}) dz dF_{-i}(t_{-i}) \\
= & \int_{T_{-i}} \int_{t_i'}^{t_i} \partial_2 g_i(w_i(S_i(t_{-i},z),z) dz dF_{-i}(t_{-i}) \\
\geq & \int_{T_{-i}} \int_{t_i'}^{t_i} \partial_2 g_i(w_i(S_i(t_{-i},t_i'),z) dz dF_{-i}(t_{-i}) \\
= & \int_{T_{-i}} [g_i(w_i(S_i(t_{-i},t_i'),t_i) - g_i(w_i(S_i(t_{-i},t_i'),t_i')] \, dF_{-i}(t_{-i}) \\
= & \int_{T_{-i}} \sum_{\mathcal{S}} [g_i(w_i(S_i),t_i) - g_i(w_i(S_i),t_i')] \, q^*(\mathcal{S};t_i',t_{-i}) dF_{-i}(t_{-i})
\end{aligned}$$

which is exactly the requirement in problem R. Similarly if $t_i' > t_i$. This finishes the proof. ■

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