Preference Reversal and Information Aggregation in Elections

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Abstract

I analyse informational efficiency of two-candidate elections where the utility of the voters depends on the realisation of an uncertain state variable. I show that large elections aggregate information for any voting rule in the unique equilibrium if and only if *everyone's* ranking over the alternatives changes in the same direction with a change in the state. This condition is similar to the "common values" assumption in Feddersen-Pesendorfer (1997). If there are any two groups of voters whose rankings change in opposite ways with a change in state, then for large classes of voting rules we always have equilibria with outcomes different from the full information outcome. I also show that the "common values" condition is hard to satisfy if the space of policies is multidimensional.

1 Introduction

Groups often choose by voting. Choosing the threshold of votes required by the winning alternative has two deep problems. The first concerns how best to aggregate preferences when voters have diverse preferences over alternatives. The second concerns how to aggregate information when voters have different information about some decision relevant aspect of the choice problem. This paper deals with *informational efficiency* of voting as a mechanism. In particular, we are interested in determining which of the plurality rules[1] aggregate information, and under what conditions.

If an individual is uncertain about some decision relevant feature of the environment - the *state* - then his preference over alternatives may depend on his information about the state. Consider the canonical example used in the literature to illustrate this. Suppose that

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[1]In this paper we focus only on elections with two given alternatives. The voting rules considered are plurality rules or q-rules, according to which the candidate getting more than q share of the votes wins the elections, where $q \in (0,1)$. We, however, denote a voting rule in this paper by θ.

a jury has to vote on whether to acquit or convict a defendant. If the evidence of guilt is overwhelming, then the jury members would want to convict him, otherwise they prefer to acquit him. In many cases like this, we do not know the state for sure and only have noisy evidence about it. If the state were known for sure, a given voting rule would result in a particular outcome in each state. Would the voting mechanism deliver the same outcome when the individuals are uncertain about the state? If it does, then individual uncertainty is irrelevant in the aggregate. This paper identifies the relationship between uncertainty and voter preferences under which information is aggregated.

The Condorcet Jury Theorem claims that when the electorate is large and everyone gets an independent private signal about the state that is correct with a probability greater than half, the majority rule always aggregates information. If everyone agrees on which alternative is "correct" for a given state and votes according to his signal, then the correct alternative almost always receives more than half the votes. Condorcet is, however, silent about whether information is aggregated when voters have different rankings over alternatives under full information. The main point of this paper is that the information aggregation property of large elections depends crucially on the relationship between preference heterogeneity among voters and the nature of uncertainty. In particular, non-aggregating outcomes occur if and only if the knowledge of a change in state induces some voters to switch their votes in favour of one alternative and some in favour of the other.

Over the last decade or so, the Jury Theorem has been subjected to renewed scrutiny. Earlier proofs[2] of the theorem relied on the "sincere voting" assumption, but Austen-Smith and Banks (1996) pointed out that one's vote matters only when he is "pivotal", i.e. when the others are tied or almost tied[3]. Conditioning on the event of being pivotal, one may find more information about the state, and may as a result vote against his signal in equilibrium[4]. Feddersen and Pesendorfer (1997), henceforth F-P, showed that with such strategic voting, under conditions of reasonable generality, we get full information aggregation for any voting rule. In their model, the state represents the "commonality" of preferences in the sense that for a given change in the state variable, everyone becomes more prone to voting for exactly one of the two alternatives. My paper may be treated as a generalization of F-P, where their assumption of "common values"[5] arises as a special case. The main result in this paper is that common values is not only sufficient, but also necessary for information to be fully

[2] For earlier proofs of this theorem using statistical arguments, see Berg (1992), Ladha (1992, 1993), Nitzan and Paroush (1985).

[3] See McLennan (1998) for a sophisticated enunciation of Condorcet's Jury Theorem and formal proof showing that if there exists an outcome that aggregates information with sincere voting, there exists a Nash equilibrium that does the same too.

[4] See Battaglini, Morton and Palfrey (2006) and Goeree and Yariv (2006) for experimental evidence that voters condition their decision on information about the state learnt from the event of being pivotal.

[5] F-P use the term "common values" in a sense slightly weaker than the way the term is defined for the first time in the context of auctions in Milgrom and Weber (1982). In the auction context, if the ranking of alternatives is the same for all individuals given a state, then we have common values. However, in this paper, we shall refer to the term in the sense used by F-P.

aggregated in all equilibria.

To understand the "common values" assumption, consider the canonical jury example where voters vary over what counts as "reasonable doubt". Individuals' preferences are similar in the sense that everyone wants to acquit for low levels of guilt and convict for high levels, but they vary with the precise level at which they switch from acquittal to conviction. So, as the level of guilt increases, more and more members favour the guilty verdict. However, the common values assumption may be violated in other situations. Consider the following example.

Suppose a country has so far been isolated and now is voting on whether to allow free trade by joining the WTO. Because of its isolation, it has developed both an industrial sector and an agricultural sector to suit its own consumption needs. If the country allows free trade, the sector in which it has comparative advantage will grow and the other will die. Assume that the voters do not know where the comparative advantage of the country lies. If the advantage lay in industry and this was commonly known, those engaged in the industrial sector would vote in favour of joining WTO while those in the agricultural sector would vote against, and conversely if the advantage lay in agriculture. Note here that it is as if there are two opposing interest groups. Their rankings over alternatives change with the state, but given a state, the ranking of each is opposed to that of the other. In other words, as the state changes, one set of types switch from status quo to the alternative, while another set switches the other way round. We call this type of situation one of non-common values. In contrast, in a common value situation, as the state changes, there is switching in only one direction.

For a second example consider an election with an incumbent candidate and a challenger. Assume that a candidate can only commit to his own most preferred point on the policy space. The incumbent's best point is known to be $\mathcal{Q}$, but there is some uncertainty about the location of the challenger, which can be one of two points: L or R. If L is to the left of $\mathcal{Q}$ and R to the right, then the extreme leftists prefer the challenger when he is located at L while the extreme rightists prefer the challenger when he is at state R. In that case, we are in a non-common values situation similar to the ones described above. However, if L is to the left of R, but both locations are to the left of $\mathcal{Q}$, then, for all practical purposes we have a leftist challenger and a rightist incumbent. As the challenger becomes more extremely leftist (state changes from R to L), he loses support of some of the moderates but does not have anyone new switching to him. Thus we are back in the common values situation.

It is worth noting that the common values situation prevails when the uncertainty is "small" in the sense that both of the possible positions of the challenger are on one side of the incumbent, of which one can be considered as extreme than the other. If the policies are constrained to be chosen on a line, then the extreme position is the one that is farther from the status quo. Thus everyone who prefers to vote for the challenger in the extreme state also prefers to vote for the challenger in the moderate state. If the voters care about more than one issue, i.e. if the policy space is multidimensional, it becomes hard to define what a more "extreme" position is. Unless both possible locations are on a ray originating from the status quo, the common values condition is violated if the space is large enough. Therefore, information is aggregated in a very rare set of situations if the policy space has more than one dimension. This also indicates that a higher dimensional policy space has aggregation properties that are fundamentally different from the situation when policy has only one dimension.

The main contribution of this paper is characterizing a necessary and sufficient condition for the voting outcome to be full information equivalent, and pointing out that it can be very hard to satisfy. The paper points out that the source of aggregation failure is a specific type of conflict of interest among groups of voters: reversal of rankings over alternatives with a change in state. It is worth noting that informational efficiency of voting depends only on the existence of opposed groups and not on the size of the conflicting groups, distribution of ideal points or the precision of signals.

More substantively, my paper indicates ways in which elections can fail to aggregate information. I show that there is always an equilibrium where almost everyone votes ignoring information: the larger group in favour of the status quo and the smaller one in favour of the alternative policy. Another finding is that, contrary to conventional wisdom, the set of voters whose voting decision is responsive to changes in information received (the set of "swing voters") can consist of groups of voters that vote in opposite ways. In certain cases, the responsive set can consist only of voters in the minority interest group. Also, the median voter may not vote responsively. These are different phenomena which may lead to the failure of information aggregation.

At a more technical level, the paper develops a methodology to identify all equilibria for each voting rule. This method can be used not only for unidimensional policy spaces, but for spaces of higher dimensions too. For the unidimensional case, in addition, this framework can identify the aggregation properties of each voting rule.

The rest of the paper is organized as follows. Section 2 discusses how this paper relates to the existing literature on voting and information aggregation. Section 3 provides an intuitive discussion of the model, shows how it works in an illustrative example, and discusses the main results. Section 4 sets up the formal model to be used throughout the rest of the paper. Section 5 discusses the benchmark common value case and Section 6 analyses the non-common values situation. Section 7 compares and contrasts the two settings and discusses the implications and an extension. Most of the proofs are relegated to the appendix.

2 Related Literature

Most of the early work on information aggregation in large electorates has assumed diversity in information but complete homogeneity in preferences among voters (e.g. Myerson (1998a, 1998b, 2000), Wit(1998), Meirowitz (2002)). These papers find positive results for most plurality rules with respect to information efficiency[6]. There are some notable exceptions that show a partial breakdown of aggregation. Feddersen and Pesendorfer (1998) shows that unanimity rule has higher probability of mistakes than other rules, Razin (2003) demonstrates that aggregation can break down if voters have a motivation signal their preference to the candidates through their votes, and Martinelli (2006) shows that aggregation may not happen if information is not cheap enough. Persico (2004) allows for investment in costly information acquisition and shows that information cannot be fully aggregated. There is also a literature on how the possibility of abstention affects aggregation. Feddersen and Pesendorfer (1996, 1999) show that if information is free to some non-partisan voters, information is aggregated even if abstention is allowed. It is worth noting that among these papers, only Feddersen and Pesendorfer (1997, 1999) look at a limited heterogeneity in preferences by assuming the common values condition while the others assume complete homogeneity.of voter preferences[7]

Kim (2005) looks at a non-common values setting with two groups of voters that have opposed ranking over the alternatives in each state. In this paper, the preferences within each group are identical, and information is fully aggregated for most voting rules as long as the.voter cares enough about mistakes in each state[8]. Kim and Fey (2006) works with a similar setting and finds that aggregation may break down when abstention is allowed. Meirowitz (2005) examines the issue of aggregation in the same setting with the addition of a communication stage. Oliveros (2005) examines the same issue in a slightly different setting, allowing for both abstention and costly information, and shows that there can be a breakdown of aggregation.

In order to demonstrate that the source of informational inefficiency in the democratic institution of voting lies in the relationship between uncertainty and voter preferences, I do not allow for abstention, communication or signaling motivation, and consider an environment with a more general correlation between the state variable and preferences. In the current setting, contrary to F-P, commonly perceived information about the true state need not lead to a common shift in induced preferences, and contrary to Kim, types with the same *ranking* over the alternatives under full information need not have the same *intensity* of preference for them, leading to different behaviour under uncertainty. The claim made here is that elections may involve voter preferences which look neither like adversarial committees nor like jury boards. My model is similar to Kim (2005) and Meirowitz (2005), in that I have two

[6]By taking a mechanism design perspective, Costinot and Kartik (2006) show that under completely homogeneous preferences, the designer's choice of optimal voting rule is independent of his beliefs about whether voters learn from being pivotal or not.

[7]Another paper that looks at a similar common values context where the members of the jury vary over what counts as "reasonable doubt" for acquittal is Li, Rosen and Suen (2001). While they examine aggregation failure in small committes, F-P shows that such conflict does not affect aggregation if the jury size is large.

[8]What is erroneously claimed as a failure of aggregation in Proposition 3 in Kim (2005) is really a full information equivalent outcome under what we call a $\mathcal{Q}$-trivial rule in this paper. We can indeed have a failure of aggregation in Kim's set-up if the *exogenous* utility loss from a mistake in one state is low enough for the majority group (see his Proposition 4).

competing groups of voters.with opposed preferences in either state, but since I use a spatial model of voting, I can allow for differing preferences within each group.

One interpretation of Condorcet Jury Theorem is that communication among voters is not necessary in large elections for the information problem to be solved. My paper indicates that if there is state-contingent conflict among voters, there are multiple equilibria, some or all of which lead to informationally inferior outcomes. One way to solve the problem may be to allow pre-game communication among voters. Note that since all members of each conflicting interest group have the same state-contingent rankings[9], and members each group would have an incentive to share information. Thus, I highlight the necessity for democratic deliberation[10] to improve election outcomes.

3 Discussion of the Model and Results

In this paper I develop a model that allows for both the common values and the non-common values situation depending on parameters, and compare equilibria that arise in the two cases. Voters have quadratic[11] preferences over a policy space, which is a compact subset of the real line. An extension of the model considering a policy space with multiple dimensions is discussed later. A voter's preference type is identified by his privately known bliss point on this space. There is a status quo $\mathcal{Q}$ whose location on the space is known. The location of the alternative policy $\mathcal{P}$ is unknown, and is considered the state variable. The state space is binary – $\mathcal{P}$ can be located at one of two given points on the policy space. Based on where these two points are located, we may have a non-common value or a common value set-up, as illustrated in the candidate competition example above. The voting rule θ is the share of votes requited for $\mathcal{P}$ to win. I study the limiting outcomes as the number of voters becomes large. Specifically, I look at whether there is an equilibrium which delivers an outcome different from the full information outcome in some state. If it does, I say that information is not aggregated.

The main result is that while in the common values case, information is aggregated with a very high probability in the unique limiting equilibrium for any voting rule, the property breaks down in the non-common values case. In the latter case, we have multiple limiting equilibria depending on beliefs, and for all economically important voting rules, one or more of the equilibria reach a "wrong" outcome with a very high probability. To understand why information aggregation can break down, note that, in equilibrium, there is always a set of types voting informatively while others ignore the signal. This set of responsive voters is usually (for example, in F-P) an interval of types, with types on either side of the interval voting in opposite directions uninformatively. Based on the full information outcome, there

[9] In this case, the condition for full revelation of information between any two members of the same group is satisfied according to Baliga and Morris (2002)

[10] See Austen-Smith and Feddersen (2005, 2006), Gerardi and Yariv (2006), Meirowitz (2005) for models of deliberation before voting. Goeree and Yariv (2006) demonstrates in an experimental setting that communication can improve outcomes.

[11] Quadratic utility functions are not necessary, all results go through with strictly convex preferences.

are two kinds of voting rules - the ones which, under full information, lead to different outcomes in the two states (the *consequential rules*), and those which lead to the same outcome in both states (the *trivial rules*).

For voting with consequential rules, we need the following to happen in equilibrium for information to aggregated:

1. The responsive types should be *influential*, i.e. the overall voting outcome should change as the responsive types vote differently in the different states. For a voting rule θ, this condition is satisfied if the θ-quantile type lies in the responsive set.[12]

2. The responsive types are *aligned* with society, i.e. they vote the same way as the full information mapping from the states to the outcomes demands. This always happens under common values, but under non-common values it happens only if the responsive types belong to the larger interest group.

On the other hand, for voting with trivial rules, we need the responsive types *not* to be influential for information to be aggregated.

Both conditions are satisfied under equilibrium in the common values situation, but in the non-common values situation, each can individually fail in the limiting equilibrium. More interestingly, the responsive set of types may itself be a disconnected set consisting of extreme types at either end of the space of ideal points.

To see how the model works, let us first consider the common values case. Assume that the policy space is $[-1, 1]$, $\mathcal{Q}$ is located at 0, and the policy $\mathcal{P}$ is located under states L and R at -0.8 and -0.2 respectively. A voter votes for whichever alternative is located closer to his bliss point. If the state is known to be L, everyone left of -0.4 votes for $\mathcal{P}$ and if it is known to be R, and everyone left of -0.1 votes for $\mathcal{P}$. Everyone else votes for $\mathcal{Q}$. Therefore, if the state is known, the types between -0.4 and -0.1 vote for $\mathcal{P}$ when it the state is moderate (R) and switch to $\mathcal{Q}$ when it is extreme (L). These are the types that have an incentive to change their vote based on information about the state. The responsive set is thus a subset of $[-0.4, -0.1]$ for any voting rule. Assume that 35% of the voters have bliss points left of -0.4, and 55% have bliss points to the left of -0.1. Under full information, for $\theta < 35\%$, $\mathcal{P}$ obtains as the winner under both states, for $35\% < \theta < 55\%$ we get $\mathcal{Q}$ under state L and $\mathcal{P}$ under state R, and for $\theta > 55\%$ we always get $\mathcal{Q}$. To follow what happens in equilibrium under incomplete information for different values of θ, we need to track the responsive set in equilibrium and whether the θ-quantile type belongs to this set.

[12] We define the θ-quantile type as the type which has exactly θ proportion of types below it, when the types are ranked in order of their bliss points.

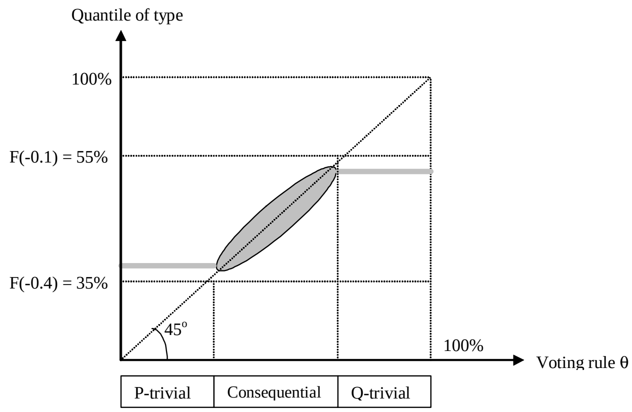

Figure 1: Responsive set in limiting equilibrium under common values

I show the responsive set in the limiting equilibrium for each voting rule in Figure 1. The voting rule θ is plotted on the horizontal axis and the quantile of types (considered from the left) on the vertical axis. For all $\theta < 35\%$ ($\mathcal{P}$-trivial rules), the θ-quantile type is to the left of -0.4, and the responsive set is stuck at a small interval of types just right of -0.4, so the responsive types are never influential. $\mathcal{P}$ gets slightly more than 35% votes in both states and wins. Similarly, for all $\theta > 55\%$ ($\mathcal{Q}$-trivial rules), the responsive set is a small set of types just left of -0.1, and $\mathcal{P}$ receives less than 55% share of votes in both states and loses. .For rules between 35% and 55% (consequential rules), the responsive set includes the θ-quantile type. Hence the responsive set is influential for these rules. Also, the responsive types are aligned with the society – they vote for $\mathcal{Q}$ when L and $\mathcal{P}$ when R. Therefore, information is aggregated in the common values set up under any voting rule θ short of unanimity.

Next, consider the case where the locations of the alternative at L and R are on two sides of $\mathcal{Q}$ - say at -0.8 and at 0.8 respectively. If the state is known, under L, only the types to the left of -0.4 (say 35% of the voters) vote for $\mathcal{P}$, while under R, only those to the right of 0.4 vote for $\mathcal{P}$ (say 55% of the voters). We shall call these two groups of types as the L-group and R-group respectively. Note that both these groups are sensitive to information about the state, while the others, i.e. the types in $(-0.4, 0.4)$ always vote for $\mathcal{Q}$. Under full information, here too, $\theta < 35\%$ is $\mathcal{P}$-trivial, $\theta > 55\%$ is $\mathcal{Q}$-trivial, while $\theta \in (35\%, 55\%)$ is consequential with $\mathcal{P}$ winning in state R and losing in state L. Note that the R-group which is the majority interest group, is aligned with society while the L-group is not.

If the common belief is that the responsive set will be in the L-group, then conditional on being pivotal, there is a larger probability that the state is L, i.e. the policy is at -0.8. Under state L, although the utility difference between $\mathcal{Q}$ and $\mathcal{P}$ is positive in the R- group and negative in the L-group, due to convex preferences, the absolute value of this difference is larger for the R-group since it is farther away from the location of the policy. As a result, under uncertainty, all the members of the R-group vote for $\mathcal{Q}$ without paying attention to the signal, while some in group L vote according to their signal – thus confirming the belief.

Similarly, if the responsive set is believed to lie in the R-group, members of the L-group vote uninformatively for $\mathcal{Q}$ while some in R-group have incentive to vote responsively. It is this phenomenon that gives us multiple equilibria based on the different beliefs.

Consider a $\mathcal{Q}$-trivial voting rule ($\theta > 55\%$). The policy $\mathcal{P}$ receives the largest share of votes when, conditional on being pivotal, virtually everyone believes that the state is R. Then, the entire R-group votes for $\mathcal{P}$. The policy therefore can receive at most 55% of the votes. Thus, Q-trivial rules always aggregate information.

Next, consider a consequential rule, say the majority rule ($\theta = 50\%$). There is indeed one equilibrium where the responsive set is in the R-group and contains the median voter. Since the R-group is aligned to the full-information mapping of states and the median is influential, information is aggregated in this equilibrium. However, there is another equilibrium where the responsive set is stuck at a small interval to the left of -0.4 in the L-group[13]. Here, in both states, the policy receives slightly less than 35% of votes – and therefore fails to win in either state. This equilibrium delivers the "wrong" outcome in state R since the responsive set is neither influential nor aligned. This equilibrium looks like "block voting" in that almost everyone votes disregarding the signal.

Third, consider a $\mathcal{P}$-trivial rule that is not too small, say 30%. In this case, there are two equilibria, one with responsive types either completely or mostly in the L-group and the other with responsive types either completely or mostly in the R-group. In both cases, the responsive set is influential, and therefore in each equilibrium, we get different outcomes in different states. Here, aggregation fails since the median voter is influential when he should not be.

For each voting rule, there is a third equilibrium where non-extreme types vote for the status quo and extreme types at either end vote in accordance with their information in opposite directions in such a way that the vote shares for the policy are almost equal in both states and constant with respect to the voting rule. This again leads to a failure of information aggregation except for very low $\mathcal{P}$-trivial rules.

One important thing to note here that is although the failure of aggregation happens only when there is noise in the signals, it does not depend on the level of noise. Thus, for a large set of rules in the non-common values situation, even with a very slight noise in the signals, we can get outcomes different from those under full information with a very high probability. This is because incomplete information equilibria result from voting strategies based on learning from the event of being pivotal. This is not the case under full information. When states are known, no one is pivotal, so we do not have the pivotal voting outcomes under full information.

Apart from this somewhat technical point, our major inference from the above analysis is the difference in information aggregation properties between the common values and the non-common values situation. The common values situation is essentially a discrete version of F-P

[13] Gul and Pesendorfer (2005) show aggregation failure with strategic candidates where there is an equilibrium with somewhat similar properties.

and hence we find full information aggregation in the unique equilibrium. In the non-common values situation, for consequential rules there is equilibrium that has the aggregation property and two others where we get the status quo in both states with a very high probability. And for all $\mathcal{P}$-trivial rules above some minimum, none of the three equilibria aggregates information. Thus the common values condition is necessary and sufficient for information aggregation.

How important is this non-common values condition empirically? Is it really prevalent enough for us to be concerned about the informational efficiency of voting as a mechanism? At least in the incumbent-challenger example, it appears a little far-fetched. Although we may not know the exact policy preferences of a challenger, we at least know whether he is to the right or to the left of the incumbent. Thus, we seem to be getting back to the common values world in the model with one-dimensional policy space. Hwoever, elections are often fought over many issues. Real policy spaces are multidimensional. I show in an extension that the above analysis holds true even in the case of a policy space with finitely many dimensions. Moreover, as indicated earlier, the common values condition is hard to obtain in a multidimensional context. Hence my claim is that there is a very real problem with voting as a mechanism of information aggregation.

4 The Set-up

Suppose there is an electorate composed of a finite number $(n+1)$ of people who are voting for or against a policy $\mathcal{P}$. If the policy gets more than a proportion θ of the votes[14], then $\mathcal{P}$ wins; otherwise the status quo $\mathcal{Q}$ wins. Assume that the policy space is $[-1,1]$. While $\mathcal{Q}$ is known to be located at 0, there is uncertainty about the location of the alternative $\mathcal{P}$. $\mathcal{P}$ is located at $L \in [-1,1]$ or $R \in [-1,1]$ with equal probability. The event that $\mathcal{P}$ is located at S, where $S \in \{L,R\}$ is referred to as state S. To give a natural meaning to the names of the state, we assume that $L < R$[15]. We also assume that the policy never coincides with the status quo, i.e both L and R are non-zero[16]. Each voter receives a private signal $\sigma \in \{l,r\}$ about the state. Signals are independent and identically distributed conditional on the state, with the distribution being:

$$\Pr(l|L) = \Pr(r|R) = q \in (\frac{1}{2}, 1)$$

Voters have single peaked preferences defined on the policy space. Every individual has a privately known bliss point x that is drawn independently from a commonly known distribution $F(\cdot)$ with support $[-1,1]$ and a density $f(\cdot)$. The utility from the alternative A, when

[14] To simplify the analysis, assume the tie breaking rule that if the policy receives exactly θ proportion of votes, the status quo wins.

[15] There is some loss of generality - by this assumption, we exclude the case that in the policy location is state invariant, i.e. $L = R$. Thus assuming $L < R$ is tantamount to assuming that there is always some uncertainty about the policy location.

[16] In other words, we assume that if the state were known, then there will always be a positive interval of types that would strictly prefer to vote for the policy in either state.

it is located at a, is given by:

$$U(x,A) = -(x-a)^2,\ A \in \{\mathcal{Q},\mathcal{P}\}$$

Given a draw of x and S, we define $v(x,S)$ as the difference in utility between the policy alternative and the status quo:

$$v(x,S) = U(x,\mathcal{P}) - U(x,\mathcal{Q}) = x^2 - (x-S)^2,\ S \in \{L,R\} \tag{1}$$

We shall use $v(x,S)$, the utilitity difference between the two alternatives as given by (1) for all further analysis. If the state S is known, a voter votes for $\mathcal{P}$ if and only if $v(x,S)$ is non-negative. If S is not known, a voter calculates the expected value of this function using the relevant probability distribution over states and votes $\mathcal{P}$ if the expectation is non-negative. Based on the location of the policy $\mathcal{P}$, we distinguish between two situations with the following very important condition.

Definition 1 *Define $\mathbb{P}(S)$ to be the set of types that (weakly) prefer the alternative policy to the status quo if they know that the state is S:*

$$\mathbb{P}(S) = \{x : v(x,S) \geq 0\}$$

$\mathbb{P}(S)$ *exhibits* common values *if* $\mathbb{P}(L) \subset \mathbb{P}(R)$ *or* $\mathbb{P}(L) \supset \mathbb{P}(R)$, *and* non-common values *otherwise.*

Denote $\mathbb{P}(L) \cap \mathbb{P}(R)$ as $\mathbb{P}_{LR}$, and notice that it can be empty. The set of types $\mathbb{P}_{LR}$ always votes for the policy irrespective of the state. They are *committed types,* or *type-$\mathcal{P}$ partisans* according to the nomenclature in Feddersen-Pesendorfer (1996)[17]. Now consider the sets $\mathbb{P}(L)\backslash\mathbb{P}_{LR}$ and $\mathbb{P}(R)\backslash\mathbb{P}_{LR}$. These are the *independent types*, as they change their vote based on the state. Definition (1) says that in a common values setting, all independent types switch their votes in the same direction ($\mathcal{P}$ to $\mathcal{Q}$ or $\mathcal{Q}$ to $\mathcal{P}$) when S changes. In a non-common values situation, some (a positive mesaure of) independents switch from $\mathcal{P}$ to $\mathcal{Q}$ and some from $\mathcal{Q}$ to $\mathcal{P}$ for a change in S.[18]

The intuition behind Definition (1) can be clarified by looking at the common values condition in F-P. Their assumption is that $v(x,S)$ is strictly increasing in S for every value of x. If $x \in \mathbb{P}(L)\backslash\mathbb{P}_{LR}$, then $v(x,L) > 0$, but $v(x,R) < 0$. And if $x \in \mathbb{P}(R)\backslash\mathbb{P}_{LR}$, the $v(x,L) < 0$, but $v(x,R) > 0$. If $\mathbb{P}(L) \subset \mathbb{P}(R)$ or $\mathbb{P}(L) \supset \mathbb{P}(R)$, exactly one of $\mathbb{P}(L)\backslash\mathbb{P}_{LR}$ or $\mathbb{P}(R)\backslash\mathbb{P}_{LR}$ is

[17] Given that the location of Q is known and $b_i \neq 0$, there is always an interval of types around 0 that are *Q-partisans.*

[18] One way to mathematically characterise the common and non-common value settings in the unidimensional case is as follows. Consider any two ideal points $x < x'$ lying in the interior of $[-1,1]$.Also note that $L < R$. Then, we have common values if $v(x,L)v(x',R) > v(x,R)v(x',L)$ and non-common values if $v(x,L)v(x',R) < v(x,R)v(x',L)$. It is very interesting to note the similarity with the affiliation property in Milgrom and Weber (1982). However, in this case, the common values condition is not equivalent to log supermodularity since the $v(\cdot,\cdot)$ can be negative.

empty. So, for all independent types, $v(x, L) - v(x, R)$ takes the same sign. In other words, for all independent types, the F-P condition holds, at least in the weak sense. That justifies the name "common values". On the other hand, if neither of $\mathbb{P}(L)\backslash\mathbb{P}_{LR}$ and $\mathbb{P}(R)\backslash\mathbb{P}_{LR}$ is empty, then we are considering independent types that do not satisfy the F-P condition. The ranking of alternatives for each of the two sets of types changes with the state, and the voters' preferences are opposed to each other in both states.

Remark 1 *If L and R have the same sign, we have a common values situation, and if they have different signs, we have a non-common values situation.*

Proof. In Appendix. ■

The intuition behind this remark is illustrated by the example in Section 1.

The equilibrium concept we employ is symmetric Bayesian Nash equilibrium in undominated strategies.

Given an individual's private information (bliss point x and signal σ), the strategy specifies a probability of voting for $\mathcal{P}$:

$$\pi(x, \sigma) : [-1, 1] \times \{r, l\} \to [0, 1]$$

Thus, under state S, the expected share of votes is:

$$t(S, \pi) = \int_{-1}^{1} \Pr(l|S)\pi(x, l)dF(x) + \int_{-1}^{1} \Pr(r|S)\pi(x, r)dF(x),\ \ S = L, R \tag{2}$$

Exapanding (2) we can write

$$\begin{array}{c} t(L, \pi) = q \int_{-1}^{1} \pi(x, l)dF(x) + (1-q) \int_{-1}^{1} \pi(x, r)dF(x) \\ t(R, \pi) = (1-q) \int_{-1}^{1} \pi(x, l)dF(x) + q \int_{-1}^{1} \pi(x, r)dF(x) \end{array}$$

Under a rule θ a voter is pivotal if $n\theta$ votes are cast for the policy $\mathcal{P}$ from among the remaining n voters. So, the probability of being pivotal under state S is given by[19]:

$$\Pr(piv|\pi, S) = \binom{n}{n\theta} (t(S, \pi))^{n\theta} (1 - t(S, \pi))^{n - n\theta},\ \ S = L, R \tag{3}$$

Note that (3) actually denotes a pair of equations, one for each state. Call these the *pivot equations*. Note that if $t(S, \pi) \in (0, 1)$ then $\Pr(piv|\pi, S) > 0$. I show later that in any equilibrium of the model, we must have $t(S, \pi) \in (0, 1)$. Assuming the belief on the state conditional on being pivotal is well defined, it is given by:

$$\beta(S|piv, \pi) = \frac{\Pr(piv|\pi, S)}{\Pr(piv|\pi, L) + \Pr(piv|\pi, R)},\ \ S = L, R \tag{4}$$

Since $\Pr(piv|\pi, S) > 0$ for both states, we have $\beta(S|piv, \pi) \in (0, 1)$. The strategies played in equilibrium determine the pivot probabilities in each state through (2) and (3). In

[19] For technical convenience, we assume that $n\theta$ is an integer.

return,the probability of state L conditional on being pivotal is determined by Bayes rule by (4). We call $\beta(L|piv,\pi)$ the *induced prior* and denote it as β_L. The posterior beliefs given a signal are:

$$\left.\begin{array}{l}\beta(L|piv,\pi,l)=\frac{q\beta_L}{q\beta_L+(1-q)(1-\beta_L)}\\ \beta(L|piv,\pi,r)=\frac{(1-q)\beta_L}{(1-q)\beta_L+q(1-\beta_L)}\\ \beta(R|piv,\pi,l)=\frac{(1-q)(1-\beta_L)}{q\beta_L+(1-q)(1-\beta_L)}\\ \beta(R|piv,\pi,r)=\frac{q(1-\beta_L)}{(1-q)\beta_L+q(1-\beta_L)}\end{array}\right\}\tag{5}$$

We refer to $\beta(L|piv,\pi,l)$ as p_l and to $\beta(L|piv,\pi,r)$ as p_r. Note that while both p_l and p_r are increasing functions of the induced prior, p_l is concave and p_r is convex throughout. This, coupled with their equality at the extreme values of β_L, i.e. $p_l=p_r=\beta_L$ at $\beta_L=0$ and $\beta_L=1$, implies that $p_l>p_r$ for all other values of β_L. Figure 2 graphs the posteriors as functions of the induced prior β_L.

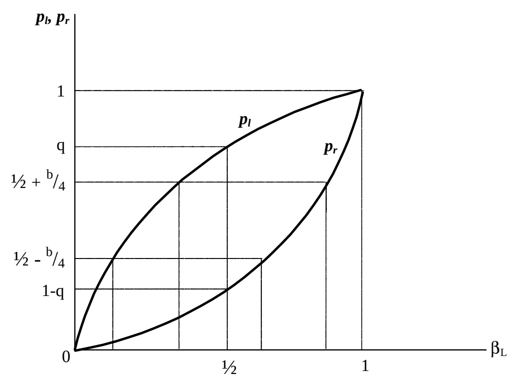

Figure 2: Posteriors as functions of the induced prior belief

Before identifying equilibrium strategies, we provide some important definitions.

Definition 2 *A voting strategy is a* cut-off *strategy if, given a signal and an induced belief, the type space* $[-1,1]$ *can be partitioned into exactly two intervals (one possibly empty) such that every type votes for* $\mathcal{Q}$ *in one interval and for* $\mathcal{P}$ *in the other. The cut-offs are said to be* ordered[20] *if, as the location of the cut-off changes due to changes in the induced belief, only the types to the left (or right) of the cut-off vote for* $\mathcal{P}$.

In other words, a voting strategy is a cut-off strategy if given $\sigma\in\{l,r\}$ and $\beta_L\in(0,1)$, there is some $x_\sigma(\beta_L)$ such that for any $x_1<x_\sigma(\beta_L)$ and $x_2>x_\sigma(\beta_L)$, the absolute value

[20] Note that the definition of ordering of cut-offs is different here from the one in F-P (page 1035) where ordering is defined based on whether cut-offs are monotonic in signals. Here, for any location of $\mathcal{P}$, we always have cut-offs monotonic in the signals. However, it is possible that for some values of the cut-off, those to the left of the cut-off vote for $\mathcal{P}$ while for other values of the cut-off, those to the right of the cut-off vote for $\mathcal{P}$. We distinguish those situations as unordered.

of $\pi(x_2, \sigma) - \pi(x_1, \sigma)$ is 1. If $x_\sigma(\beta_L) \in \{0, 1\}$, a cut off strategy requires that $\pi(x, \sigma)$ be 0 or 1 for all x. A cut-off strategy is said to be ordered if, given any two types x and x' with $x \neq x'$, the sign of $\pi(x, \sigma) - \pi(x', \sigma)$ is either always nonnegative or always nonpositive for any value of β_L.

The nature of the cut-off strategies varies on the basis of the possible locations of the uncertain alternative.

Definition 3 *If the vote of an individual with type x changes with the signal, i.e. if $\pi(x, l) \neq \pi(x, r)$, then type x is said to be* responsive*.*

Definition 4 *Suppose that given a consequential rule θ, under full information, $\mathcal{P}$ wins under state S. A type is said to be* aligned *if under full information, he prefers $\mathcal{P}$ in state S and $\mathcal{Q}$ in the other state, i.e. the state $\{L, R\} \backslash S$.*

The responsiveness and alignment conditions have been discussed in detail in the introduction.

5 Common values

We start by looking at the benchmark case with common values. As we shall see, this turns out to be the discrete version of the F-P model. A common values game is defined by its parameters $(F(\cdot), q, L, R, n, \theta)$. Since we shall look for the sequence of limiting equilibria of this game for all values of θ as $n \to \infty$, we define a common values setting by a collection $(F(\cdot), q, L, R)$.

5.1 Strategies and equilibria

Lemma 1 *In the common values case, all equilibrium strategies are ordered cut-off strategies.*

Proof. A voter with signal $\sigma, (\sigma \in \{l, r\})$ evaluates the state using the distribution $\beta(S|piv, \pi, \sigma)$ and votes for the policy if and only if the expected value of the function $v(\cdot, \cdot)$ is non-negative. Assume for now that $\beta(S|piv, \pi, \sigma)$ is well-defined. Define $x(p_\sigma)$ as the solution of the equation $E(v(x, S)|piv, \pi, \sigma) = 0$. Solving,

$$x(p_\sigma) = \frac{1}{2}\left(\frac{(L)^2 p_\sigma + (R)^2 (1 - p_\sigma)}{Lp_\sigma + R(1 - p_\sigma)}\right) \in \left[\frac{L}{2}, \frac{R}{2}\right]$$

Thus, $x(p_\sigma)$ always exists uniquely. Also, since $\frac{\partial Ev(x,S)}{\partial x} = 2(Lp_\sigma + R(1 - p_\sigma))$, $R > L > 0 \Rightarrow \frac{\partial Ev(x,S)}{\partial x} > 0 \Rightarrow Ev(x, S) > 0$ iff $x > x(p_\sigma)$. Similarly, if $L < R < 0$, $Ev(x, S) > 0$ iff $x < x(p_\sigma)$. This establishes the cut-off nature of strategies. Given L and R, the strategies do not depend on the precise location of $x(p_\sigma)$. If $L < R < 0$, types to the left of the cut-off $x(p_\sigma)$ vote for $\mathcal{P}$, while if $0 < L < R$, types to the right of the cut-off $x(p_\sigma)$ vote for $\mathcal{P}$. This proves the ordered nature of the cut-off strategies, and establishes the lemma, under the assumption that $\beta(S|piv, \pi, \sigma)$ is well-defined. ■

Denote $x(p_l)$ as x_l and $x(p_r)$ as x_r. The the cut-off strategies are given by (6) and (7):

$$\left\{\begin{array}{l} \pi(x,l)=\begin{cases} 1 \; if \; x \geq x_l \\ 0 \; otherwise \end{cases}, \\ \pi(x,r)=\begin{cases} 1 \; if \; x \geq x_r \\ 0 \; otherwise \end{cases} \end{array}\right\} \text{ when } R > L > 0 \tag{6}$$

$$\left\{\begin{array}{l} \pi(x,l)=\begin{cases} 1 \; if \; x \leq x_l \\ 0 \; otherwise \end{cases}, \\ \pi(x,r)=\begin{cases} 1 \; if \; x \leq x_r \\ 0 \; otherwise \end{cases} \end{array}\right\} \text{ when } L < R < 0 \tag{7}$$

Remark 2 *Note that for* $\beta_L = 1,\ x_r = x_l = \frac{L}{2}$, *and likewise for* $\beta_L = 0,\ x_r = x_l = \frac{R}{2}$.*Since* $\frac{dx(p)}{dp} = -\frac{1}{2}\left(\frac{(R-L)LR}{(Lp+R(1-p))^2}\right) < 0$, *and since* $p_l > p_r$ *for* $\beta_L \in (0,1)$, $x_r > x_l$ *for these values of* β_L.

Thus, for any induced prior, the strategies in the benchmark case are characterised by cutpoints x_l and x_r, with $x_l \leq x_r$. If $R > L > 0$, types $x < x_l$ always vote for $\mathcal{Q}$, types $x \in [x_l, x_r]$ vote for $\mathcal{P}$ if they get signal l and $\mathcal{Q}$ if they get signal r, and the types $x > x_r$ vote for $\mathcal{P}$ regardless of the signal. If $L < R < 0$, types left of x_l always vote for $\mathcal{P}$ and those right of x_r vote for $\mathcal{Q}$ while types in $[x_l, x_r]$ vote informatively (according to their signal). In either case, $[x_l, x_r]$ is the responsive set, while the other types vote according to their bias. Henceforth, we shall deal only with the case $L < R < 0$, noting that the other case is completely symmetric.

Note that the ordered cut-off nature of the strategies ensures that there will always be one and only one responsive interval. Also, irrespective of the location of the cutoffs, the responsive set is always *aligned* with the society. This means that whenever the responsive set is influential, information will be aggregated. Thus, for consequential rules, all we need to show for information aggregation is that in any limiting equilibrium, the responsive set is indeed influential. For this, we need monotonicity of the vote shares under both states, which is again ensured by the ordered nature of the cut off strategies. We define the probability of an individual voting for the alternative $\mathcal{P}$ given σ as z_σ, *i.e.* $z_\sigma \equiv \int_{-1}^{1} \pi(x,\sigma) dF$. For $L < R < 0$, we have from (7),

$$z_\sigma = F(x_\sigma),\ \sigma = \{l, r\}$$

Therefore, using (2) we write[21]:

$$\left.\begin{array}{l} t(L,\pi) = q z_l + (1-q) z_r = qF(x_l) + (1-q)F(x_r) \\ t(R,\pi) = (1-q) z_l + q z_r = (1-q)F(x_l) + qF(x_r) \end{array}\right\} \tag{8}$$

Note that since the cut-offs x_l and x_r are functions of the induced prior, the vote shares $t(L,\pi)$ and $t(R,\pi)$ are also functions of β_L. The following lemma examines how the vote share in each state changes as a function of the induced prior.

[21] For $0 < L < R$, we have $z_\sigma = G(x_\sigma)$, where $G(y) \equiv 1 - F(y)$, $y \in [-1, 1]$

Lemma 2 *The expected share of votes $t(S,\pi)$ in state S decreases strictly with the induced prior β_L from $F(\frac{R}{2})$ at $\beta_L = 0$ to $F(\frac{L}{2})$ at $\beta_L = 1$. Also, for all interior values of the induced prior, i.e. for all $\beta_L \in (0,1)$, $t(L,\pi) < t(R,\pi)$*[22].

Proof. By Remark 2, at $\beta_L = 0$, $z_l = z_r = F(\frac{R}{2}) \Rightarrow t(S,\pi) = F(\frac{R}{2})$ for $S \in \{L,R\}$. Similarly, at $\beta_L = 1$, $t(S,\pi) = F(\frac{L}{2})$ for $S \in \{L,R\}$. Also, since p_σ is a strictly increasing function of β_L, x_σ is decreasing in β_L by Remark 2. The full support assumption guarantees that $F(\cdot)$ is strictly increasing. Hence, $t(S,\pi)$ is strictly decreasing in β_L. For the second part of the lemma, note that

$$t(L,\pi) - t(R,\pi) = (2q-1)\left(F(x_l) - F(x_r)\right)$$

By remark 2 again, for $\beta \in (0,1)$, $F(x_l) - F(x_r) < 0$, and since $q > \frac{1}{2}$, we have $t(L,\pi) < t(R,\pi)$. ■

Lemma 2 states that as the induced prior probability of the state being L (conditional on being pivotal) increases, the expected share of votes for the alternative policy decreases under either state because the state L is deemed to be more "extreme". Informative voting by the responsive set ensures that the policy receives more votes in the "moderate" state (R) unless the prior is degenerate. Note also that at any induced prior, the difference in expected vote shares is increasing in the informativeness of the signal. The expected vote shares in the two states are plotted against the induced prior in figure 3.

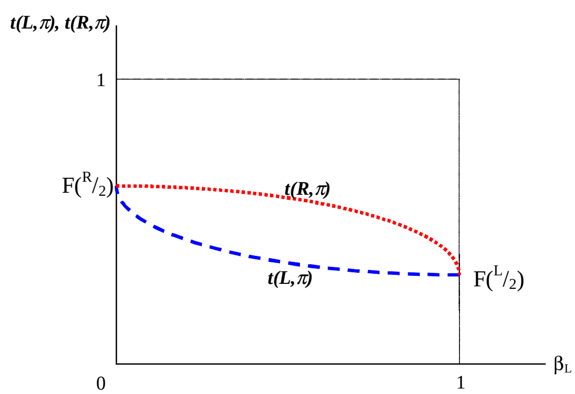

Figure 3: Vote shares in each state under common values when $L < R < 0$

Lemma 2 also ensures that since $t(S,\pi)$ lies strictly between 0 and 1, and $\beta(S|piv,\pi,\sigma)$ is always well-defined. Intuitively, since the types left of $\frac{L}{2}$ are $\mathcal{P}$-partisans and those to the right of $\frac{R}{2}$ are $\mathcal{Q}$-partisans, there is always a positive probability that any given type is pivotal. This finally proves Lemma 1.

[22] If $0 < L < R$, then both $t(L,\pi)$ and $t(R,\pi)$ increase strictly with the induced prior β_L from $F(\frac{R}{2})$ at $\beta_L = 0$ to $F(\frac{L}{2})$ at $\beta_L = 1$. Also, for all $\beta_L \in (0,1)$, $t(L,\pi) > t(R,\pi)$.

The following proposition guarantees the existence of an equilibrium of the common values voting game $(F(\cdot), q, L, R, n, \theta)$.

Proposition 1 *In the common values case, there exists a voting equilibrium π^* for every population size n and every voting rule $\theta \in (0,1)$, characterized by ordered cut-off strategies x_σ given by the solution of the equation $E(v(x_\sigma, s)|piv, \pi^*, \sigma) = 0$ for $\sigma = (l, r)$.*

Proof. For the proof of this proposition, we first note that the strategy for each voter can be denoted by two numbers x_l and x_r, both lying between $\frac{L}{2}$ and $\frac{R}{2}$. Thus the strategy space is a compact, convex and non-empty set $\left[\frac{L}{2}, \frac{R}{2}\right] \times \left[\frac{L}{2}, \frac{R}{2}\right]$. The rest of the proof follows from the proof of Proposition 1 in F-P. ■

To find the equilibrium of the model, what we essentially do is find a fixed point on the belief space $\beta_L \in [0,1]$. In other words, suppose everyone else holds some belief β_L, which determines two distributions $p_\sigma(\beta_L)$ according to (5) and correspondingly, the cut-off strategies $x_\sigma(\beta_L)$ according to (7). From the cut-off strategies, the expected shares of votes for the policies $t(S, \pi)$ in the two states $S = L, R$ is determined by (8). Given these shares, the number of players n and the voting rule θ, a player forms $\Pr(piv|\pi, S)$: probabilities of being pivotal in each state according to the pivot equations (3). These probabilities define belief $\widetilde{\beta_L}$ by (4) : in equilibrium, this $\widetilde{\beta_L}$ should be equal to the initial belief β_L. Thus the induced prior should have the rational expectations property in equilibrium. Note that

$$\frac{\Pr(piv|\pi, L)}{\Pr(piv|\pi, R)} = \frac{\beta(L|piv, \pi)}{\beta(R|piv, \pi)} = \frac{\beta_L}{1-\beta_L}$$

Thus, using the above and the pivot equations, the *equilibrium condition* can be simply stated as:

$$\frac{\beta_L}{1-\beta_L} = \frac{\Pr(piv|\pi, L)}{\Pr(piv|\pi, R)} = \left[\frac{(t(L,\pi^n))^\theta (1-t(L,\pi^n))^{1-\theta}}{(t(R,\pi^n))^\theta (1-t(R,\pi^n))^{1-\theta}}\right]^n \tag{9}$$

5.2 Limiting Equilibria under Common Values

In this section, we consider the properties of the voting equilibria as the electorate grows in size arbitrarily, keeping all other parameters of the model constant. Therefore, every quantity is superscripted by the number of voters n. The superscript will be suppressed when there is no ambiguity Suppose, given L, R and θ for some n, the equilibrium is π^n, and the cutoffs are x_σ^n. As long as common values assumption is satisfied, existence of equilibrium for any n implies the existence of a convergent subsequence with an accumulation point as $n \to \infty$. If a limit of this sequence exists, we call it π^0. By continuity arguments, as $x_\sigma^n \to x_\sigma^0$, $t(S, \pi^n)$, β_L^n, p_l^n,and p_r^n all converge to finite limits $t(S, \pi^0)$, β_L^0, p_l^0, and p_r^0 respectively along the sequence.

Rewriting the equilibrium condition:

$$\frac{\beta_L^n}{1-\beta_L^n} = \left[\frac{(t(L,\pi^n))^\theta (1-t(L,\pi^n))^{1-\theta}}{(t(R,\pi^n))^\theta (1-t(R,\pi^n))^{1-\theta}}\right]^n \quad \textit{for all } n \tag{10}$$

By Proposition 1, a solution to (10) exists for every n. From continuity, if a limit exists, we can also say that the above relation has to hold in the limit; call this the *limiting equilibrium condition.*

$$\frac{\beta_L^0}{1-\beta_L^0} = \lim_{n\to\infty}\left[\frac{\left(t(L,\pi^0)\right)^{\theta}\left(1-t(L,\pi^0)\right)^{1-\theta}}{\left(t(R,\pi^0)\right)^{\theta}\left(1-t(R,\pi^0)\right)^{1-\theta}}\right]^n \tag{11}$$

To avoid writing complicated expressions, we define:

$$\alpha_n = \frac{\left(t(L,\pi^n)\right)^{\theta}\left(1-t(L,\pi^n)\right)^{1-\theta}}{\left(t(R,\pi^n)\right)^{\theta}\left(1-t(R,\pi^n)\right)^{1-\theta}} \text{ and } \alpha_0 = \frac{\left(t(L,\pi^0)\right)^{\theta}\left(1-t(L,\pi^0)\right)^{1-\theta}}{\left(t(R,\pi^0)\right)^{\theta}\left(1-t(R,\pi^0)\right)^{1-\theta}}$$

Note that the vote shares $t(S,\pi^n)$ are functions of β_L^n. Next, we look at the properties of the limit, assuming existence for the time being. We later show that in the commmon values game, for any voting rule, there is only one accumulation point of π^n which must be the limit.

Lemma 3 *If* $\beta_L^0 \in (0,1),\ \alpha_0 = \lim_{n\to\infty} \alpha_n = 1$

Proof. See Appendix. Note that this lemma does not use the common values condition, so it is true of non-common values too. ■

Lemma 4 *If* $\beta_L^0 = 1$, *then* $x_\sigma^n \to \frac{R}{2}$ *from the left for* $\sigma = l, r$. *Similarly, if* $\beta_L^0 = 0$, *then* $x_\sigma^n \to \frac{L}{2}$ *from the right for* $\sigma = l, r$

Proof. Follows from continuity of x_σ^n in p_σ^n and of p_σ^n is β_L^n, along with Remark 2. ■

Note, as an aside to Lemma 4, that although under both signals the cutoffs converge to $\frac{R}{2}$ or $\frac{L}{2}$ if the induced prior converges to 1 or 0, by remark 2, we always have $x_l^n < x_r^n$. Thus, in the responsive set, the voters always vote for $\mathcal{Q}$ if they get moderate signal r and $\mathcal{P}$ if they get the extreme signal l. In the limit, the responsive interval is vanishingly small as the induced prior distribution converges to state R , grows for intermediate values of the prior, and again shrinks to a vanishing size as the distribution converges to a degenerate distribution at state L. Thus, given q, a level of precision of the signals, the difference between expected shares in the two states is low for extreme values of the induced prior and high for intermediate values.

Lemma 3 and Lemma 4 together imply that for any limiting induced prior, given a voting rule under any equilibrium, the vote shares in each state must be related in a certain way. This is stated in Proposition 2 below. According to Lemma 3, if α_n is bounded away from 1, then β_L^0 must be either 0 or1. Under conditions of Lemma 4, if β_L^n is indeed 0 (or 1), then the voters are almost sure of the state in which they are pivotal and vote as if under (almost) full information. Every type except those in a vanishing set votes uninformatively, and the vote shares under either state are the same in the limit. Thus, in equilibrium, we have $\alpha_0 = 1$ for all values of the induced prior.

Proposition 2 *In all limiting equilibria, we must have* $\alpha_0 = 1$*, i.e.*

$$\left(t(L,\pi^0)\right)^{\theta}\left(1-t(L,\pi^0)\right)^{1-\theta} = \left(t(R,\pi^0)\right)^{\theta}\left(1-t(R,\pi^0)\right)^{1-\theta}, i.e.\ \alpha_0 = 1$$

Proof. For any equilibrium with $\beta_L^0 \in (0,1)$, the proposition follows straightforwardly from Lemma 3. If $\beta_L^0 = 1$, the first part of Lemma 4 implies that

$$\begin{aligned} t(L,\pi^n) &= qF(x_l^n) + (1-q)F(x_r^n) \to qF(\frac{R}{2}) + (1-q)\,F(\frac{R}{2}) \to F(\frac{R}{2}) \\ t(R,\pi^n) &= qF(x_r^n) + (1-q)F(x_l^n) \to qF(\frac{R}{2}) + (1-q)\,F(\frac{R}{2}) \to F(\frac{R}{2}) \end{aligned}$$

$$\therefore \alpha_0 = \lim_{n\to\infty}\frac{\left(t(L,\pi^n)\right)^{\theta}\left(1-t(L,\pi^n)\right)^{1-\theta}}{\left(t(R,\pi^n)\right)^{\theta}\left(1-t(R,\pi^n)\right)^{1-\theta}} = \frac{\left(F(\frac{R}{2})\right)^{\theta}\left(1-F(\frac{R}{2})\right)^{1-\theta}}{\left(F(\frac{R}{2})\right)^{\theta}\left(1-F(\frac{R}{2})\right)^{1-\theta}} = 1\left(\because F(\frac{R}{2}) \in (0,1)\right)$$

If $\beta_L^0 = 0$, the proof follows in exactly the same way since the second part of Lemma 4 implies that then $t(S,\pi^n) \to F(\frac{L}{2})$ for $S \in \{L,R\}$. ■

Note that Proposition 2 is based on a necessary condition that must be true for a β_L^0 to which induced belief converges in the limiting equilibrium. It helps exclude certain voting rules that cannot support a given value of β_L in the limit. To do this formally, define $\Theta(\beta_L)$ as the set of voting rules that can support β_L as an induced belief in the *limiting equilibrium condition* (11) for *some* distribution of preferences in the cut-off equilibrium. To emphasize that $t(S,\pi)$ is a function of β_L, we write $t(S,\pi)$ as $t_S(\beta_L)$ for $S \in \{L,R\}$.

Lemma 5 *Under common values, (i) If* $\beta_L \in (0,1)$*, then* $\Theta(\beta_L)$ *is a strictly increasing function* $\theta^*(\beta_L)$*, with* $t_L(\beta_L) < \theta^*(\beta_L) < t_R(\beta_L)$*. (ii) Otherwise,* $\Theta(1) = \{\theta : \theta < F(\frac{L}{2})\}$*, and* $\Theta(0) = \{\theta : \theta > F(\frac{R}{2})\}$

Proof. In Appendix. ■

The first part of the lemma is almost a corollary of Proposition 2. For each interior value β_L of the induced prior, it identifies a unique θ as the only possible voting rule to support β_L in the limiting equilibrium. As long as the expected vote shares in the two states are different, the only voting rule that can satisfy Proposition 2 is one that lies strictly between the two shares. This has the implication that under one state the status quo wins, while in the other, the policy wins. If there are any equilibria with beliefs that place positive probability on both states, then the responsive set of types for these equilibria are always influential. The lemma also notes that such equilibria are possible only for consequential rules. The second part of the lemma says that the extreme beliefs can be supported only by extreme values of the voting rules.

Note that since $\theta^*(\beta_L)$ is strictly increasing, its inverse function $\beta_L^{-1}(\theta)$ exists for $\theta \in \left(F(\frac{L}{2}), F(\frac{R}{2})\right)$ and is strictly increasing. Thus, according to Lemma 5, for every θ, there is a unique β_L that can be supported as an induced prior in the limit, *for any distribution of* *types*. Call it $\beta(\theta)$. We can write:

$$\beta(\theta) = \begin{cases} 1 \; if \; \theta < F(\frac{L}{2}) \\ \quad \beta_L^{-1}(\theta) \; if \; \theta \in \left(F(\frac{L}{2}), F(\frac{R}{2})\right) \\ 0 \; if \; \theta > F(\frac{R}{2}) \end{cases} \tag{12}$$

We plot the correspondence $\Theta(\beta_L)$ along with the expected vote shares in each state against the induced prior in Figure 4.

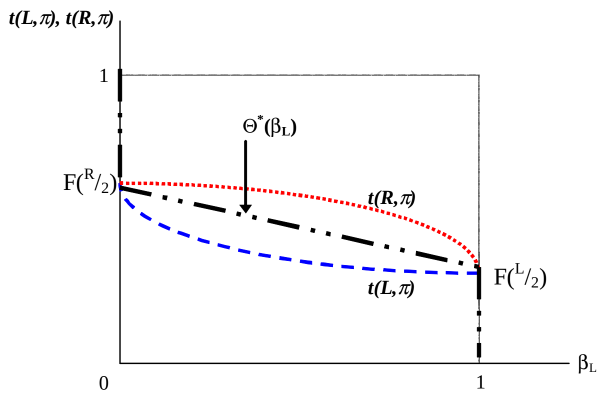

Figure 4: Correspondence $\Theta(\beta_L)$ under common values $(L < R < 0)$

The next theorem gives a characterization of cut-off equilibria in large populations for different voting rules under common values.

Theorem 1 *Assume L,R satisfy the common values condition, $F(\cdot)$ satisfies full support, and $q \in (\frac{1}{2}, 1)$. Fix a voting rule $\theta \in (0,1)$. Then there is a unique limiting equilibrium π^0 with ordered cut-off strategies and with the induced prior converging to β_L if and only if $\theta \in \Theta(\beta_L)$, or alternatively, if and only if $\beta_L = \beta(\theta)$.*

Proof. According Proposition 2, a voting equilibrium π^n with ordered cut-off strategies exists for a given θ, for any n. Since β_L lies in a compact set, there is an accumulation point π^a, given θ. We show in the appendix that this π^a is the limiting equilibrium π^0 given θ. Lemma 5 states that for any distribution of types, if a limit exists, there is a unique number $\beta(\theta)$ to which the induced prior converges in the limit along the sequence of equilibria under voting rule θ. ■

Note that once the limiting value of the induced prior β_L is established, the limiting posterior distributions p_σ, the limit cut-offs x_σ etc. are all determined from β_L. Thus this theorem describes all relevant information about strategies, vote shares and statewise outcomes in equilibria with a voting rule when the population size becomes large. Note that, by the Law of Large numbers, the actual vote shares are arbitrarily close to the expected vote shares[23]. From here onwards, I do not distinguish between the expected and actual, and just call it "vote share".

5.3 Outcomes and Information Aggregation

In the introduction we have informally discussed a classification of voting rules according to the outcomes produced under full information. Here we formalise the discussion, and then examine the information aggregation properties of each class of voting rules.

For the purposes of this paper define a social choice rule H as a function that maps a state to an outcome, i.e.

$$H \,:\, \{L, R\} \rightarrow \{\mathcal{P}, \mathcal{Q}\}$$

When $H(\cdot)$ is a constant function, i.e. when the planner wants the same outcome in both states, we call it a *trivial rule*. There are two trivial rules - one where the planner always wants the status quo to prevail ($H(L) = H(R) = \mathcal{Q}$), and the one that maps both states to the policy ($H(L) = H(R) = \mathcal{P}$). We call the first one *$\mathcal{Q}$-trivial* and the second one *$\mathcal{P}$-trivial rule*. If the function maps different states to different outcomes ($H(L) \neq H(R)$), we call it a *consequential rule*.

A voting rule is said to *correspond* to a particular social choice rule if *under full information*, the voting outcome is the same as the outcome determined by the social choice rule for any given state. A voting rule is said to *implement* the corresponding social choice rule $H(\cdot)$ if, for any $\epsilon > 0$, we can find a number N such that when the population size is larger than N, in either state the outcome of the voting game *under incomplete information* of the state is the same as the outcome determined by the social choice rule with a probability larger than $1 - \epsilon$. When a voting rule implements the corresponding social choice rule, then the voting rule is said to satisfy *full information equivalence*[24]. In other words, the voting game under incomplete information gives the same outcome that would have occurred if there were common knowledge of the state.

With full information, under state L, the policy would get $F(\frac{L}{2})$ share of votes; and similarly under state R, the policy would get $F(\frac{R}{2})$ share of votes. Therefore:

- Any voting rule $\theta < F(\frac{L}{2})$ corresponds to the $\mathcal{P}$-trivial rule, i.e. $\mathcal{P}$ wins under both states.
- Any voting rule $F(\frac{L}{2}) < \theta < F\left(\frac{R}{2}\right)$ corresponds to a consequential rule, i.e. $\mathcal{P}$ wins in state R and $\mathcal{Q}$ in state L[25].

[23] More specifically, given any $\epsilon > 0$ and $\delta > 0$, we can find some number N such that as long as the polupation size is larger than N, the actual vote share is within ϵ of the exoected share with a probability higher than $1 - \delta$.

[24] The concept of full information equivalence was formalised by F-P, and I adapt their definition to my setting.

[25] Note that the other consequential rule, i.e. $\{G(L) = \mathcal{P}, G(R) = \mathcal{Q}\}$ cannot be implemented under full information by the plurality rule with the common values case we are considering, i.e. $L < R < 0$.

- Any voting rule $\theta > F\left(\frac{R}{2}\right)$ corresponds to the $\mathcal{Q}$-trivial rule, i.e. outcome is status quo under both states.

We identiify each voting rule by the social choice rule it corresponds to.

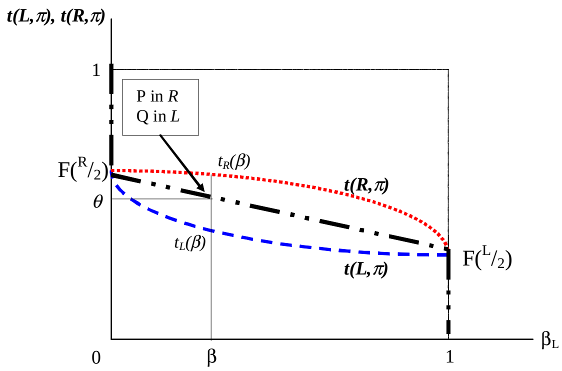

Figure 5(a): Outcome under a Consequential rule θ under common values *(L < R < 0)*

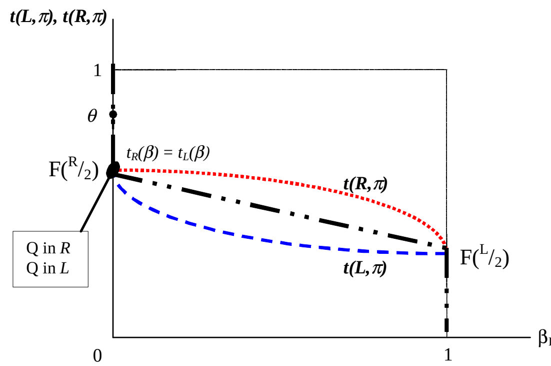

Figure 5(b): Outcome under a Q-trivial rule θ under common values *(L < R < 0)*

Figure 5(a), 5(b)

Theorem 2 *Under common values, any plurality rule* $\theta \in (0, 1)$ *satisfies full information equivalence for any distribution of types.*

Proof. In appendix. ∎

According to the theorem, under common values, any voting rule aggregates information. Since the vote shares in each state is between $F(\frac{L}{2})$ and $F\left(\frac{R}{2}\right)$, any trivial rule aggregates information. Essentially, the responsive types lying between $\frac{L}{2}$ and $\frac{R}{2}$ can never be influential with trivial rules. With $\mathcal{P}$-trivial rules, everyone is virtually sure that conditional on being pivotal, the state is L. In other words, under such a rule, being pivotal at state L (when $\mathcal{P}$ receives least votes) is infinitely more probable than being pivotal at state R. Similarly, with any $\mathcal{Q}$-trivial rule, one has far higher chance of being pivotal in state R (when $\mathcal{P}$ receives most votes) than in state L. I depict the outcome in the limiting equilibrium with a $\mathcal{Q}$-trivial rule in figure 5(a). On the other hand, for any consequential rule, the induced prior places positive probability on both states in the limit, and the responsive set is influential. Since the responsive types are aligned too, we have outcome $\mathcal{P}$ in state R and $\mathcal{Q}$ in state L almost surely, and hence we have information aggregation. The limiting equilibrium outcome with a consequential rule is depicted in figure 5(b).

6 Non-common values

Recall that a non-common values situation occurs if $L < 0 < R$. We now look at the strategies and equilibria in this situation and compare and contrast their properties with that of the benchmark common value model. Specifically, we show how voting can fail to aggregate information in the presence of heterogeneous groups with competing interests.

We shall simplify the model a bit and consider a slightly special case with $L = -b$ and $R = b > 0$. Note that this is not too strong an assumption as we are considering all possible distributions of voter ideal points. However, we need to make an additional assumption on the informativeness of the signals.

Assumption I (Informativeness): $\Pr(l|L) = \Pr(r|R) = q > \frac{1}{2} + \frac{b}{4}$

The full support assumption is heneceforth referred to as Assumption F. A non-common value setting is denoted by the collection $(F(\cdot), q, b)$. In the previous section we proved the F-P result. In this section, we shall use the same method to examine the non-common values situation.

6.1 Strategies and equilibria

A voter with signal $\sigma, (\sigma \in \{l, r\})$ evaluates the state using the distribution $\beta(S|piv, \pi, \sigma)$ and votes for $\mathcal{P}$ if and only if the expected value is non-negative. So, the condition for voting for the policy after having received σ is:

$$Ev(x, \sigma) \geq 0 \Rightarrow 2x(1 - 2p_\sigma) \geq b$$

Hence, the voter votes for $\mathcal{P}$ iff

$$1 \geq |x| \ \geq \frac{b}{2(1 - 2p_\sigma)} \tag{13}$$

Using (13), we can determine the cut-offs:

$$x_\sigma = \begin{cases} \min(1, \frac{b}{2(1-2p_\sigma)}), & 0 \leq p_\sigma < \frac{1}{2} \\ \max(-1, \frac{b}{2(1-2p_\sigma)}), & \frac{1}{2} \leq p_\sigma \leq 1 \end{cases} \tag{14}$$

Now, according to the above definitions of the cut-off, we get:

$$\pi(x, \sigma) = \begin{cases} \left.\begin{matrix} 1 \ for \ x \leq x_\sigma \\ 0 \ for \ x > x_\sigma \end{matrix}\right\} if \ \frac{1}{2} \leq p_\sigma \leq 1 \\ \left.\begin{matrix} 1 \ for \ x \geq x_\sigma \\ 0 \ for \ x < x_\sigma \end{matrix}\right\} if \ \ 0 \leq p_\sigma < \frac{1}{2} \end{cases} \tag{15}$$

Or alternatively, combining (14) and (15), we define the strategies in terms of p_σ as follows:

$$\pi(x, \sigma) = \left\{ \begin{matrix} \left.\begin{matrix} 1 \ for \ x \leq \frac{b}{2(1-2p_\sigma)} \\ 0 \ for \ x > \frac{b}{2(1-2p_\sigma)} \end{matrix}\right\} \ if \ p_\sigma \geq \frac{1}{2} + \frac{b}{4} \\ 0 \ for \ all \ x \, if \ p_\sigma \in \left(\frac{1}{2} - \frac{b}{4}, \frac{1}{2} + \frac{b}{4}\right) \\ \left.\begin{matrix} 1 \ for \ x \geq \frac{b}{2(1-2p_\sigma)} \\ 0 \ for \ x < \frac{b}{2(1-2p_\sigma)} \end{matrix}\right\} \ if \ p_\sigma \leq \frac{1}{2} - \frac{b}{4} \end{matrix} \right\}$$

Any equilibria must have strategies of the above form. Note that $p_\sigma \in [0,1] \Rightarrow -1 \leq 1-2p_\sigma \leq 1$ and so $x_\sigma \in [-1, -\frac{b}{2}] \cup [\frac{b}{2}, 1]$. Also, for all values of p_σ, $\pi(x,\sigma) = 0$ in the range $(-\frac{b}{2}, \frac{b}{2})$. Thus a voter with his bliss point in this range always votes for the status quo irrespective of the signal.

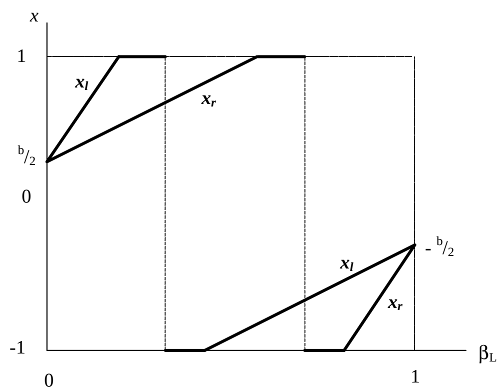

Figure 6: Cut-offs in a non-common value setting as functions of induced prior

Thus, although all equilibria must have cut-off strategies, the cut-offs are not ordered. The cutoffs as functions of the induced prior are plotted in Figure 6. From the figure we see the cut-off functions have a discontinuity that leads to a nonconvexity in the strategy space. When a cut-off is in $[-1, -\frac{b}{2}]$ (the L-group), the types to the left of the cut-off vote for $\mathcal{P}$, and when the cut-off lies in $[\frac{b}{2}, 1]$ (the R-group), types to the right of the cut-off vote for $\mathcal{P}$. This has several implications. First, the responsive types lying in these two groups would vote in opposite ways based on the same information since one of the groups is aligned with the society and the other is not. Second, in each state, the vote share is a non-monotonic function of the induced belief. Note that the monotonicity in vote shares was crucial for information aggregation with consequential rules in the common values case. Third, with unordered cut-offs, the existence of a well-defined induced prior is no longer trivial, and we need the informativeness assumption I on signals to guarantee that. Lastly, with a loss of the ordering property, uniqueness of the responsive set is no longer assured. This can give rise to a certain kind of equilibria that is not seen in the common values case, as we shall see in Proposition 4.

Recall that the probability of an individual voting for the alternative $\mathcal{P}$ given σ is z_σ, *i.e.* $z_\sigma \equiv \int_{-1}^{1} \pi(x,\sigma) dF$. In any equilibrium, we have:

$$z_\sigma = \begin{cases} F(x_\sigma) \ if \ x_\sigma \leq -\frac{b}{2} \\ 1 - F(x_\sigma) \ if \ x_\sigma \geq \frac{b}{2} \\ 0 \ otherwise \end{cases} \tag{16}$$

Although the definition of z_σ is different in the non-common values case, the vote shares in the two states in terms of z_σ are still given by equation (8):

$$t(L,\pi) = qz_l + (1-q)z_r$$
$$t(R,\pi) = (1-q)z_l + qz_r$$

Lemma 6 *In any equilibrium in the non-common values setting, the expected share of votes in any state lies strictly between* 0 *and* 1, *i.e.* $t(S,\pi) \in (0,1)$ *for* $S \in \{L,R\}$.

Proof. See Appendix. ■

Lemma 6 guarantees that the induced prior is indeed always well-defined. The expected share of people voting is less than unity because there is always a set of types close enough to 0 (between $-\frac{b}{2}$ and $\frac{b}{2}$) who vote for the $\mathcal{Q}$. On the other hand, the signal being informative enough (Assumption I) guarantees that the cut-offs are sufficiently distant when the induced priors are not overwhelmingly strong. In other words, if for one signal, no type votes for $\mathcal{P}$, there is an interior cut-off for the other signal. This ensures positive expected share for intermediate induced priors in each state. To see that from Figure 2, note that the range of β_L for which p_l lies between $\frac{1}{2} - \frac{b}{4}$ and $\frac{1}{2} + \frac{b}{4}$ lies entirely to the left of $\frac{1}{2}$, while the range of β_L for which p_r lies between $\frac{1}{2} - \frac{b}{4}$ and $\frac{1}{2} + \frac{b}{4}$ lies entirely to the right of $\frac{1}{2}$. This guarantees that, for any induced prior, at least one signal always leads to an interior cut-off.

Next, the existence of an equilibrium for the non-common value game $(F(\cdot), q, b, n, \theta)$.is proved. This is the analogous result to Proposition 1. Although the strategy set is non-convex and we cannot use a fixed point theorem to prove existence the way we did in the common values setting, we can still show the existence of a solution to equation (10), which is the *equilibrium condition.*

Remark 3 *In the non-common values case, there exists a voting equilibrium* π^* *for every population size* n *and every voting rule* $\theta \in (0,1)$. *The equilibrium is characterized by cut-off strategies* x_σ *given by the solution of* $E(v(x_\sigma, s)|piv, \pi^*, \sigma) = 0$ *for* $\sigma = (l, r)$.

Proof. From Lemma 5, we know that $t(S,\pi)$, is bounded by positive numbers both above and below. This implies that for any n, the right hand side of equation (10) is bounded above and below. However, as β_L goes from 0 to 1, the left hand side continuously increases from 0 to ∞. This guarantees the existence of a solution β_L^n to the equation, and hence existence. ■

We can immediately identify one particular equilibrium for the case with a distribution of types with density $f(\cdot)$ that is symmetric about 0.

Proposition 3 *For any* $F(\cdot)$ *for which the density* $f(\cdot)$ *is symmetric about* 0, *there is an equilibrium with* $x_l^* = -\frac{b}{2(2q-1)}$ *and* $x_r^* = -x_l^*$. *This is an equilibrium for all values of* $\theta \in (0,1)$ *and is independent of the number of voters* n.

Proof. Consider the situation where everyone else plays $x_\sigma = x_\sigma^*$, and $\sigma \in \{l, r\}$. Note that $x_l^* < -\frac{b}{2}$ and $x_r^* > \frac{b}{2}$. So, $z_l^* = F(x_l^*)$ and $z_r^* = 1 - F(x_r^*) = 1 - F(-x_l^*) = F(x_l^*) = z_l^*$, by symmetry of $f(\cdot)$. Therefore, $t(L,\pi) = t(R,\pi) = F(x_l^*)$ for each n, which implies that $\beta_L = \frac{1}{2}$ for every θ and n. Thus, the signals are fully informative, and we have $p_l = q$ and $p_r = 1 - q$. These, coupled with the Assumption I, imply that the best response to x_σ^* is indeed x_σ^*, which establishes the claim. ■

The proposition says that if the commonly held induced priors are uninformative, then sufficiently extreme types vote for the alternative $\mathcal{P}$ if and only if they get favourable signals, and everyone else votes uninformatively, disregarding their signal. There are a few things to be noted about the above equilibrium. First, this is the only "stable" equilibrium sequence in the sense that the strategies do not change with the number of players. Second, in this equilibrium, the expected vote share does not change with the state or the voting rule. If the required plurality for the policy to pass is higher than $F(x_\sigma^*)$, then the status quo always passes, and if the required share is lower than $F(x_\sigma^*)$, then the status quo always loses. If $\theta = F(x_\sigma^*)$, then we get either alternative (policy or status quo) with equal probability. As we shall see later in Section 6.3, this constitutes a failure of information aggregation. We note here that we do not even require the full force of symmetry of $f(\cdot)$ here. As long as we have $F\left(-\frac{b}{2(2q-1)}\right) = 1 - F\left(\frac{b}{2(2q-1)}\right)$, we shall have this equilibrium. We later establish that even if the distribution of ideal points is not symmetric, we always have an equilibrium at some belief β_L^* (not necessarily equal to $\frac{1}{2}$) that has the same vote share for each state and is independent of the voting rule.

Next, let us examine the vote share as a function of the induced prior in the non-common values set-up.

Lemma 7 *There exists some number β_L^* satisfying $0 < \beta_L^* < 1$ such that $\beta_L < \beta_L^*$, $t(R,\pi) > t(L,\pi)$, for $\beta_L > \beta_L^*$, $t(R,\pi) > t(L,\pi)$ and for $\beta_L = \beta_L^*$, $t(R,\pi) = t(L,\pi)$.*

Proof. See Appendix. ■

This lemma says that if the commonly held induced prior probability that one is pivotal at state L falls below a critical value β_L^*, then the expected vote share in favour of the policy in state L is higher than that in state R. If, on the contrary, the belief is higher than β_L^*, then the alternative $\mathcal{P}$ is expected to get a higher vote share in state R. However, given a state, the expected share of the votes in favour of the policy alternative increases as one gets more and more extreme beliefs, i.e. as one is surer and surer of the state in which one is pivotal. As the voters get more unsure about the state, only the very extreme types vote for the policy. Note that at β_L^*, we have $F(x_l) + F(x_r) = 1$, and under a symmetric distribution of types, $\beta_L^* = \frac{1}{2}$, and we have an equilibrium at $\beta_L = \frac{1}{2}$ according to Proposition 4. The expected share of votes under the two states in the non-common value situation (according to Lemma 7) as functions of the induced prior are shown in figure 7. To illustrate how the shares are constructed according to (8), we also show the functions z_l and z_r (i.e. the probability of voting for $\mathcal{P}$ on getting the signal l and r respectively) in the figure.

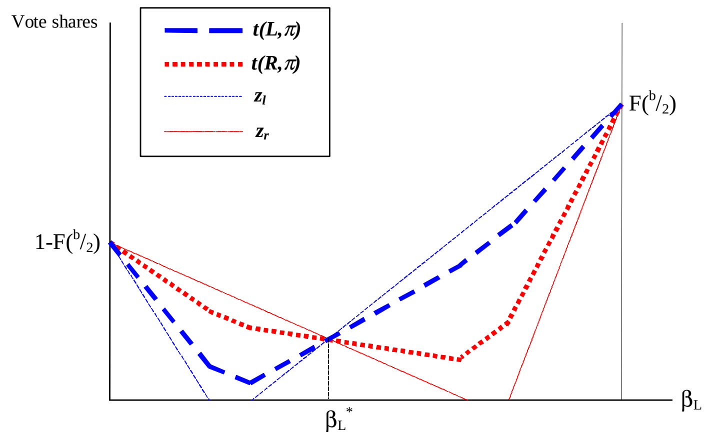

Figure 7: Construction of vote shares as functions of induced prior

6.2 Limiting equilibria in large elections

Given Proposition 3, equilibrium exists for every n. Therefore, we use the same notation as in Section 5.2. Since the cutoffs are bounded within a compact set, any sequence of x_σ^n will have a convergent subsequence. We look at such convergent subsequences x_σ^n as $n \to \infty$. We call an accumulation point of such a sequence of cutoffs as x_σ^0, and the resulting equilibrium as π^0. By the continuity arguments , as $x_\sigma^n \to x_\sigma^0$, $t(S, \pi^n)$, β_L^n, p_l^n,and p_r^n all converge to $t(S, \pi^0)$, β_L^0, p_l^0, and p_r^0 respectively along the subsequence. In this section we examine which outcomes can be supported in the limit.

The necessary conditions for the limit, the *limiting equilibrium condition* as identified in equation (11) remains exactly the same. Lemma 3 goes through without any change. Lemma 4 goes through too, with the slight modification that it is no longer true of all n, but it holds for large enough n. We state this in Lemma 8. For a suffieicntly large electorate, if the induced prior converges to 0 (1), both cut-offs are in the L-group (R-group).

Lemma 8 *If* $\beta_L^0 = 1$, (i) $\exists$ *some* m *such that* $x_l^n > x_r^n$ *for all* $n > m$; *and* (ii) $x_\sigma^n \to -\frac{b}{2}$ *from the left for* $\sigma = l, r$. *Similarly, if* $\beta_L^0 = 0$, (i) $\exists$ *some* m_1 *such that* $x_l^n > x_r^n$ *for all* $n > m_1$; *and* (ii) $x_\sigma^n \to \frac{b}{2}$ *from the right for* $\sigma = l, r$

Proof. See Appendix. ■

Proposition 2 now goes through in exactly the same form. The proof follows from Lemma 3 and Lemma 8 analogously. Thus, in the limiting equilibrium, we must have the same relationship between the shares and the voting rule in the common value case and the non-common value case. In other words, the local properties of the limiting equilibria are the same. Next, we examine which voting rules can be supported by a given value of the induced prior in the limit. We look for an equivalent of Lemma 5.

Lemma 9 *Under non-common values, (i)for $\beta_L \in (0, \beta_L^*) \cup (\beta_L^*, 1)$, $\Theta(\beta_L)$ is a continous function $\theta^*(\beta_L)$, with $t_L(\beta_L) < \theta^*(\beta_L) < t_R(\beta_L)$ for $\beta_L < \beta_L^*$, and $t_L(\beta_L) > \theta^*(\beta_L) > t_R(\beta_L)$ for $\beta_L > \beta_L^*$, (ii)Otherwise, $\Theta(1) = \{\theta : \theta > F(-\frac{b}{2})\}$, $\Theta(0) = \{\theta : \theta > 1 - F(\frac{b}{2})\}$ and $\Theta(\beta_L^*) = \{\theta : \theta \in (0, 1)\}$.*

Proof. In Appendix. ■

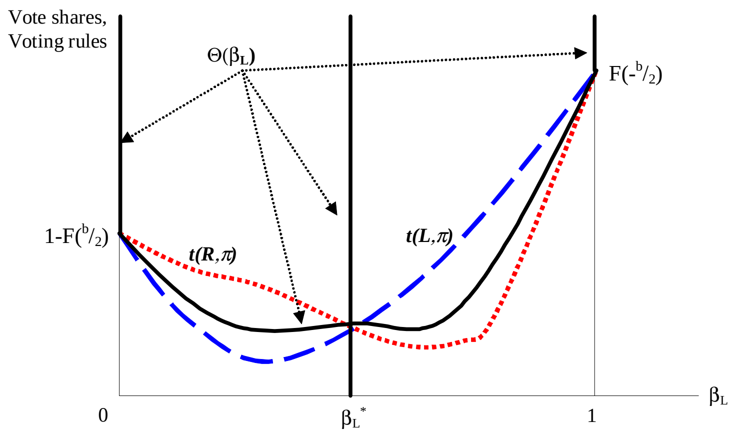

Figure 8: The correspondence $\Theta(\beta_L)$ under non-common values

The correspondence $\Theta(\beta_L)$ for the non-common values case, as inferred in Lemma 9, is depicted in figure 8. Note that in this case, if we invert the correspondence to get the supporting induced belief β_L for each voting rule θ, we no longer get a function $\beta(\theta)$ as defined in (12) in the common values case, but rather a correspondence.

Denote $t(L, \beta_L^*) = t(R, \beta_L^*)$ by z. Because of the non-monotonic vote share functions, for any voting rule $\theta > z$, there can be three different limiting equilibria. One equilibrium is an approximation to the symmetric equilibrium in Proposition 4. With the equilibrium belief at β_L^*, the vote shares are equal in both states and independent of the voting rule. For any consequential rule or a $\mathcal{P}$-trivial rule, information is not aggregated in this equilibrium. Of the two other equilibria, one has induced prior probability of state L less than β_L^* and has the responsive set of types entirely (or mostly) in the R-group. For voting rules less than $1 - F(\frac{b}{2})$, the responsive set in this equilibrium is influential, and $\mathcal{P}$ obtains in state R and $\mathcal{Q}$ in state L. For $\theta > 1 - F(\frac{b}{2})$, the responsive set in this equilibrium cannot be influential, and the status quo obtains in both states. Similarly, there is another equilibrium with induced prior belief greater than β_L^*, where the responsive set is entirely or mostly in the L-group.

Note that so far we have only claimed that for a given voting rule there *can* be three equilibria. The next theorem states that all the equilibria discussed above exist for any distribution of ideal points. Given an induced prior β_L, any voting rule that is not ruled out by the necessary condition (11) can indeed support a limiting equilibrium with beliefs converging to β_L.

Theorem 3 *For any $b > 0$ satisfying the non-common values condition, and for any q satisfying Assumption I and any distribution of preferences $F(\cdot)$ satisfying assumption F, given a voting rule θ, there is a limiting equilibrium π^0 with cut-off strategies and with the induced prior converging to β_L if $\theta \in \Theta(\beta_L)$*[26].

Proof. In appendix. ■

6.3 Voting rules and Information Aggregation

From Lemma 9, we can deduce possible outcomes for each value of the induced prior. All these outcomes occur almost surely, in the same way as in the common values case.

- For $\beta_L = 0$, the only possible outcome is $\mathcal{Q}$ under both states. Here, the responsive set is in the R-group but is not influential.
- For $\beta_L \in (\beta_L^*, 1)$, the only possible outcome is $\mathcal{Q}$ under state L and $\mathcal{P}$ under state R. Here, the responsive set is in the R-group and is influential.
- For $\beta_L = \beta_L^*$, the vote share in each state is fixed at z and the outcome depends on whether the voting rule is greater or less than z.
- For $\beta_L \in (0, \beta_L^*)$, the only possible outcome is $\mathcal{P}$ under state L and $\mathcal{Q}$ under state R. Here, the responsive set is in the L-group and is influential.
- For $\beta_L = 0$, the only possible outcome is $\mathcal{Q}$ under both states. Here, the responsive set is in the L-group but is not influential.

From here onwards, we assume with a slight loss of generality that $F(-\frac{b}{2}) > 1 - F(\frac{b}{2})$[27]. In other words, we assume that the L-group is the larger interest group, and hence the group that is aligned with the society. Therefore,

- Any voting rule $\theta < 1 - F\left(\frac{b}{2}\right)$ is $\mathcal{P}$-trivial
- Any voting rule $1 - F\left(-\frac{b}{2}\right) \leq \theta < F\left(\frac{b}{2}\right)$ is a consequential rule[28], i.e. the policy wins in state L and the status quo in state R.
- Any voting rule $\theta \geq F\left(-\frac{b}{2}\right)$ is a $\mathcal{Q}$-trivial rule.

For all $\mathcal{Q}$-trivial rules, the beliefs that can be supported in equilibrium are $\beta = \{0, \beta_L^*, 1\}$. Since the maximum share of received by the alternative $\mathcal{P}$ in any state is $F\left(-\frac{b}{2}\right)$, $\mathcal{Q}$-trivial

[26] This theorem requires an assumption that $\theta^*(\beta_L)$ is not constant over any range. We ignore that as a non-generic case.

[27] If $F(-\frac{b}{2}) = 1 - F(\frac{b}{2})$,then there are no consequential rules. $\theta = F(-\frac{b}{2})$ would implement a random social choice rule under full information if the L-group is the larger interest group.

[28] Note that the other consequential rule, i.e. $\{G(L) = Q, G(R) = P\}$ cannot be implemented under full information by the plurality rule

rules always aggregate information. Figure 9(a) depicts the limiting equilibria for a $\mathcal{Q}$-trivial rule.

For informaion to be aggregated under consequential rules, we need the responsive set to be influential and in the L-group. For these rules however, there is always one equilibrium with $\beta_L = 0$ where the responsive set in the R-group and is not influential. Hence we get $\mathcal{Q}$ in both states. In another equilibrium for these rules, $\beta_L = \beta_L^*$, and here too, we get $\mathcal{Q}$ in both states with a very high probability. However, there is a third equilibrium with induced prior converging to some belief in $(0, \beta_L^*)$ with the responsive set entirely in the L-group and influential. This equilibrium aggregates information. Figure 9(b) depicts all the possible limiting equilibria for a consequential rule.

For $\mathcal{P}$-trivial rules greater than z we have two equilibria with opposite outcomes in the different states: one with equilibrium induced prior in the set $(0, \beta_L^*)$ and the other in the set $(\beta_L^*, 1)$. The responsive sets are influential here when information aggregation requires that they not be so. So, for these voting rules we have no information-aggregating equilibrium. The third equilibrium has beliefs converging to β_L^*. Since at this belief, the vote share in both states is z, in this equilibrium we always get the status quo. Figure 9(c) shows the possible equilibria for one such rule. However, information is aggregated almost surely by the very low $\mathcal{P}$-trivial rules[29].

I summarise the inferences about information aggregation for different voting rules in a non-common value setting in the next theorem. I use the same definition of full information equivalence as in Section 5.3. We define an equilibrium as *non-information aggregating* when in at least one state, voting under incomplete information delivers an outcome different from the full-information outcome with a probability arbitrarily close to 1.

Theorem 4 *All (limiting) voting equilibria with $\mathcal{Q}$-trivial voting rules satisfy the full information equivalence property. For consequential rules, there is one equilibrium that satisfies full information equivalence and two that are non-information aggregating. For $\mathcal{P}$-trivial rules that are sufficiently large, all equilibria are non-information aggregating. All $\mathcal{P}$-trivial rules below some threshold aggregate onformation.*

The above theorem establishes the bias in favour of the status quo. Unless the required vote share for the policy to win is very low, competition between two groups along with risk aversion ensures that the status quo wins in at least one state. Note that the only voting rules for which information is aggregated in any equilibrium are all $\mathcal{Q}$-trivial rules and the very low $\mathcal{P}$-trivial rules.

[29]More speficically, the $\mathcal{P}$-trivial voting rules that aggregate information for sure for any distribution of preferences are those that are below the minimum share of votes received by $\mathcal{P}$ for any belief, i.e. those rules that satisfy $\theta < \min\{\min_{\beta_L} t(L, \pi), \min_{\beta_L} t(L, \pi)\}$. Equilibrium induced prior is β_L^* and equilibrium shares in both states are $z > \theta$ in the limit.

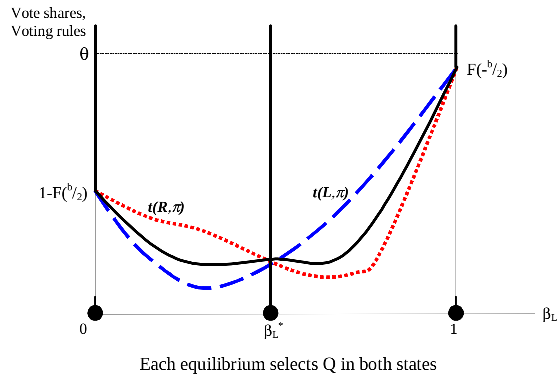

Figure 9(a): Equilibria under a $\mathcal{Q}$-trivial rule

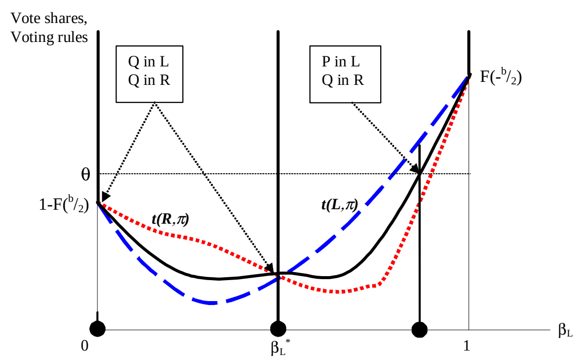

Figure 9(b): Equilibria under a consequential rule

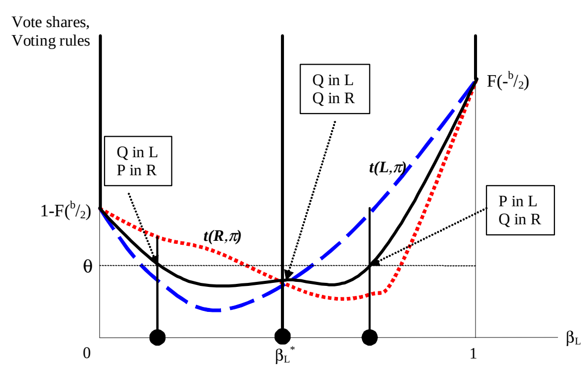

Figure 9(c): Equilibria under a large $\mathcal{P}$-trivial rule

7 Discussion and extension

The unidimensional spatial model brings into sharp relief the role of the common value assumption. It shows how this assumption ensures that the responsive set of voters always swing the election in the direction preferred by the majority, which is the full-information outcome. Under non-common values, we have two competing groups with opposed preferences, and it is possible that swing voters can belong to the minority group. Then, depending on the equilibrium, we may have a sure outcome different from the full information outcome. We also show that this competition effect dominates the efficiency-enhancing effect of increased precision of signals. In other words, the same results prevail even if the signals are almost fully informative.

Another issue that the non-common values framework demonstrates is how exactly information aggregation can fail in presence of competing interests. Such failure can happen in several ways: the misaligned set of types can be influential, the responsive set of types may be influential when they should not be, or there can be two disjoint sets of responsive voters under any rule, making the swing voters behave in different ways.

The non-common values model also presents the problem of multiple equilibria, with different equilibria giving completely different results. The multiplicity issue makes the role of beliefs crucial. The model endogenises the process of formation of beliefs about which types are going to be responsive to information. Information aggregation can fail because of "wrong" beliefs. For example, while a consequential rule needs the responsive set to be in the larger interest group, voters can believe that almost everyone is voting uninformatively, pretending that the state is known. However, a more serious failure is possible with everyone believing that the responsive set is in the minority interest group and is influential, leading to "wrong" outcome in both states. This requires us to look at a multidimensional policy space.

The framework discussed in the paper also throws light on a problem that is slightly different from the information aggregation issue. From the point of view of implementation one could look for a voting rule which would deliver two pre-specified outcomes in two states with a very high probability in all equilibria. For example, we might look for a voting rule that delivers the majority preferred outcome in both states. We show here that such a rule does not exist unless the pre-specified outcome is the same in both states.

Lastly, one might wonder how empirically relevant the non-common values condition is. In the unidimensional model, unless the uncertainty is somewhat extreme, we do not encounter the non-common value situation. For example, in most elections, it is known whether the challenger is to the left or right of the incumbent. However, in a multidimensional policy space, the common value assumption is much harder to justify. I claim that this framework can readily handle the extension to a multidimensional policy space, and the main conclusions carry over. In this paper, I only provide the intuition for this.

7.1 Multidimensional extension

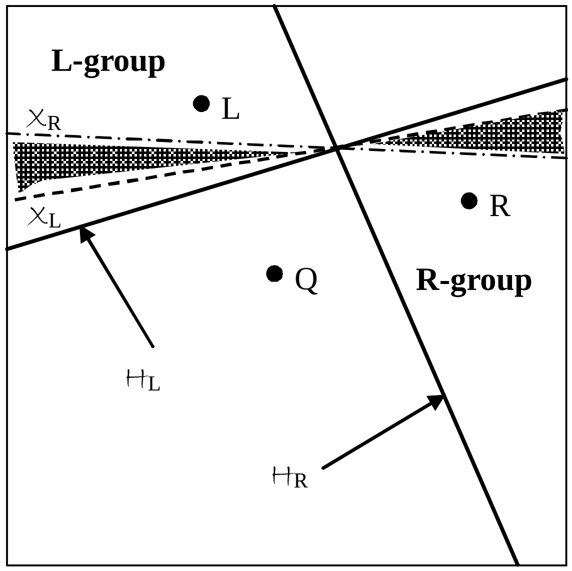

Figure 10: Cut-offs and responsive set in the multidimensional policy space

Think of the policy space as a many-dimensional cube, with each dimension being $[-1, 1]$. Suppose that the status quo $\mathcal{Q}$ is located at the origin, and the policy alternative $\mathcal{P}$ is located at two points L and R under states L and R respectively. Given a state S, a hyperplane $\mathcal{H}_S$ separates the cube into two parts, one containing the origin that supports $\mathcal{Q}$ and the other that supports $\mathcal{P}$ under full information. Just as described in Section 4, we can define as $\mathbf{P}(S)$ the set of types that prefer $\mathcal{P}$ in state S. The common value condition is exactly the same - that $\mathbf{P}(L)$ be included in $\mathbf{P}(R)$ or vice versa. Note that this is harder to satisfy. In particular, for a given location L, as the size of the cube increases, the set of locations R for which $\mathbf{P}(S)$ exhibits common values keeps shrinking and approches a ray connecting L with $\mathcal{Q}$ at the origin.

If the hyperplanes $\mathcal{H}_L$ and $\mathcal{H}_R$ are parallel, we are either in a common value situation or in a situation where there are two disjoint, completely opposed interest groups, much like the unidimensional non-common value situation. Otherwise, for a large enough policy cube, we have four sets: two of opposed independent types, one type committed to $\mathcal{P}$ under both states and one type committed to $\mathcal{Q}$ under both states. Denote the set of indpendents preferring $\mathcal{P}$ under L and $\mathcal{Q}$ under R by the L-group and the set of independents preferring $\mathcal{P}$ under R and $\mathcal{Q}$ under L as the R-group.

Suppose the hyperplanes $\mathcal{H}_L$ and $\mathcal{H}_R$ meet at a straight line $\mathcal{L}$[30]. Under uncertainty, given a signal σ, the "cut-offs" that separate those who vote for $\mathcal{P}$ from those who vote $\mathcal{Q}$ are hyperplanes $\mathcal{X}_\sigma$. As the induced prior changes from 0 to 1, $\mathcal{X}_\sigma$ rotates about $\mathcal{L}$[31], starting

[30] If the policy space is two-dimensional, the hyperplanes $\mathcal{H}_L$ and $\mathcal{H}_R$ will be straight lines and $\mathcal{L}$ will be a point.

[31] Using a simple geometric argument, it is easy to show that every type on $\mathcal{L}$ should be indifferent between $\mathcal{P}$ and $\mathcal{Q}$ for any beliefs. Hence the cut-offs should always contain $\mathcal{L}$.

at $\mathcal{H}_R$, and ending at $\mathcal{H}_L$. The strategy of a voter can also be described by the angle that each of the cut-off hyperplanes makes with the line $\mathcal{L}$. This is a compact set, and therefore, an equilibrium exists. If the hyperplanes $\mathcal{H}_L$ and $\mathcal{H}_R$ are parallel, then the cutoffs $\mathcal{X}_\sigma$ do not rotate, but translate from $\mathcal{H}_R$ to $\mathcal{H}_L$.Thus we can trace vote shares t_L and t_R in the two states as a function of the induced prior β_L. Once we have done that, the rest of the analysis is exactly as in the unidimensional model. Note here that the responsive set is always non-convex: it has two subsets, one of which lies in the L-group and the other in the R-group. Figure 10 demonstrates the cut-offs in the multidimensional policy space, with the responsive set being the shaded area.

In a common value setting, the vote shares in both states are monotonic functions of the induced prior. Thus all our results for this case hold. However, in the non-common value case, we do not necessarily have U-shaped share functions. The equilibria depend on the particular shape of the distribution of preferences. This makes generalised equilibrium characteristics and aggregation (or non-aggregation) results difficult to get in a multidimensional set-up. However, given a distribution of preferences we can use the limiting equilibirium conditions developed in this paper to identify all the possible voting equilibria for that particular case and make judgements about information aggregation properties of each voting rule.

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9 Appendix

9.1 Proof of Remark 1

Proof. Let us first look at the situation with $0 < L < R$. Here, $\mathbb{P}(R) = \{x : x \geq \frac{R}{2}\} \subset \{x : x \geq \frac{L}{2}\} = \mathbb{P}(L)$. Similarly, if we have $L < R < 0$, $\mathbb{P}(L) = \{x : x \leq \frac{L}{2}\} \subset \{x : x \leq \frac{R}{2}\} = \mathbb{P}(R)$. On the other hand, if $L < 0 < R$, $\mathbb{P}(L) = \{x : x \leq \frac{L}{2}\}$ and $\mathbb{P}(R) = \{x : x \geq \frac{R}{2}\}$, thus $\mathbb{P}(L) \cap \mathbb{P}(R) = \phi$. ■

9.2 Proof of Lemma 3

By hypothesis of the lemma, $\lim_{n\to\infty} \frac{\beta_L^n}{1-\beta_L^n} = \frac{\beta_L^0}{1-\beta_L^0}$ is a finite, positive number. Now suppose $\exists$ some $\varepsilon > 0$ such that $\alpha_n > 1 + \varepsilon$ for all n. Then $\frac{\beta_L^n}{1-\beta_L^n} = (\alpha_n)^n > (1+\varepsilon)^n \to \infty \; as \; n \to \infty$ which is a contradiction. On the other hand, suppose $\exists$ some $\varepsilon \in (0,1)$ such that $\alpha_n < 1 - \varepsilon$ for all n. Then $\frac{\beta_L^n}{1-\beta_L^n} = (\alpha_n)^n < (1-\varepsilon)^n \to 0 \; as \; n \to \infty$, which is again a contradiction.

9.3 Proof of Lemma 5

Proof. For part (i) of the lemma, since $\beta_L \in (0,1)$, Proposition 2 holds. Suppose $0 < y < x < 1$, and $f(z,\theta) = z^\theta (1-z)^{1-\theta}$, with both z and θ lying in $(0,1)$. Note that if we fix θ, the function $f(z,\theta)$ is continuous and single peaked in z with the peak lying at θ. From the properties of this function, we can show that for any$0 < y < x < 1$, there exists a unique $\theta^* s.t.\; f(x,\theta^*) = f(y,\theta^*)$, and $x < \theta^* < y$. To be specific, $\theta^* = \frac{\log \frac{1-y}{1-x}}{\log \frac{x(1-y)}{y(1-x)}}$. Also, if both x and y increase, θ^* must increase. Since $0 < F(\frac{L}{2}) < t_L(\beta_L) < t_R(\beta_L) < F(\frac{R}{2}) < 1$, taking $t_R(\beta_L) = x$ and $t_L(\beta_L) = y$ and noting that $t_R(\beta_L)$ and $t_L(\beta_L)$ are strictly increasing functions of β_L, part (i) of the Lemma is established.

For part (ii), note that for any n, by Remark 2, we have $x_l^n < x_r^n$. Since $z_\sigma^n = F(x_\sigma^n)$, we have $z_r^n > z_l^n > 0$. Define, for any n, $h^n = z_r^n - z_l^n > 0$. Substituting, we have: $t(R,\pi^n) = z_l^n + qh^n$, and $t(L,\pi^n) = z_l^n + (1-q)h^n$. Therefore:

$$\frac{1-\beta_L^n}{\beta_L^n} = \left[\frac{(t(R,\pi^n))^\theta (1-t(R,\pi^n))^{1-\theta}}{(t(L,\pi^n))^\theta (1-t(L,\pi^n))^{1-\theta}}\right]^n = \left[\frac{(z_l^n + qh^n)^\theta (1 - z_l^n - qh^n)^{1-\theta}}{\left(z_l^n + (1-q)h^n\right)^\theta \left(1 - z_l^n - (1-q)h^n\right)^{1-\theta}}\right]^n$$

If $\beta_L^0 = 0$ (or 1) , the left hand side of the above equation goes to infinity (or 0). This requires the term in the bracket large enough n to be greater (or less) than unity, or its logarithm to be positive (or negative). We can write,

$$\log \frac{(z_l^n + qh^n)^\theta (1 - z_l^n - qh^n)^{1-\theta}}{\left(z_l^n + (1-q)h^n\right)^\theta \left(1 - z_l^n - (1-q)h^n\right)^{1-\theta}} > 0 \Leftrightarrow \theta > \zeta(z_l^n, h^n) \ \forall n$$

where the function $\zeta(z_l^n, h^n)$ is defined as:

$$\zeta(z_l^n, h^n) \equiv \frac{-\log\left[\frac{1-z_l^n-qh^n}{1-z_l^n-(1-q)h^n}\right]}{\log\left[\frac{\left(z_l^n+qh^n\right)\left(1-z_l^n-(1-q)h^n\right)}{\left(z_l^n+(1-q)h^n\right)\left(1-z_l^n-qh^n\right)}\right]}$$

By Lemma 4, we know that for any sequence, with $\beta_L^0 \in \{0, 1\}$, $h^n \to 0^+$. Hence,

$$\lim_{h^n \to 0^+,\ z_l^n = t} \zeta(z_l^n, h^n) = \lim_{h^n \to 0^+,\ z_l^n = t} \left(\frac{-\log\left[\frac{1-z_l^n-qh^n}{1-z_l^n-(1-q)h^n}\right]}{\log\left[\frac{\left(z_l^n+qh^n\right)\left(1-z_l^n-(1-q)h^n\right)}{\left(z_l^n+(1-q)h^n\right)\left(1-z_l^n-qh^n\right)}\right]} \right) = \lim_{z_l^n = t} z_\sigma^n = t$$

By Lemma 4, if $\beta_L^0 = 0$, $t = F(\frac{R}{2})$, and $\theta > \zeta(z_l^n, h^n)\ \forall n \Rightarrow \theta > \lim_{h^n \to 0^+,\ z_l^n = t} \zeta(z_l^n, h^n) = F(\frac{R}{2})$. Similarly, if $\beta_L^0 = 1$, $t = F(\frac{L}{2})$, and $\theta < \zeta(z_l^n, h^n)\ \forall n \Rightarrow \theta < \lim_{h^n \to 0^+,\ z_l^n = t} \zeta(z_l^n, h^n) = F(\frac{L}{2})$. ■

9.4 Proof of Theorem 1

Here we only show that the only accumulation point is also the limit. For this, it is enough to show that given $\theta \in \Theta\left(\beta_L^0\right)$, for any neighbourhood ϵ of β_L^0, there is some large enough N, such that β_L^n in the equilibrium sequence must lie within the nighbourhood for all values of $n > N$.

First consider $\beta_L^0 \in (0, 1)$.Suppose the accumulation point is not the limit, and there is an infinite equilibrium subsequence β_L^m of the sequence β_L^n, such that for any $\epsilon > 0$, there is some M so that for all values of m larger than M, β_L^m lies outside $\left(\beta_L^0 - \epsilon, \beta_L^0 + \epsilon\right)$. Since even this subsequence must have an accumulation point, it must be either 0 or 1. But, by the second part of Lemma 5, since the limiting equilibrium condition must hold for accumulation points too, there cannot be an accumulation point for θ in $\Theta\left(\beta_L^0\right)$ at 0 or 1. Hence there is no such infinite subsequence.

The proof for $\beta_L^0 \in \{0, 1\}$ is similar.

9.5 Proof of Theorem 2

Proof. Theorem 1 guarantees existence of limiting equilibrium for all θ.

Consider $\theta < F\left(\frac{L}{2}\right)$. By Lemma 2, $t(S, \pi^n) > F\left(\frac{L}{2}\right)\ \forall n$ for $S = L, R$. Let $\delta = F\left(\frac{L}{2}\right) - \theta$. By Law of large numbers, given ϵ we can find N such that actual share of votes $\tau(S,\pi^n, \theta)$ under rule θ in any state S is greater than $F\left(\frac{L}{2}\right) - \delta > \theta$ for any $n > N$ with a probability larger than $1 - \epsilon$. Thus, under both states, $\mathcal{P}$ wins with a probability larger than $1 - \epsilon$.

Since $t(S,\pi^n) < F\left(\frac{R}{2}\right)\ \forall n \forall S$, by the same logic as above, any $\mathcal{Q}$-trivial rule aggregates information too.

Consider a consequential rule θ, for which the only equilibrium induced prior in the limit is $\beta_L^{-1}(\theta)$. By Lemma 5, $t_L\left(\beta_L^{-1}(\theta)\right) < \theta < t_R\left(\beta_L^{-1}(\theta)\right)$.

Also, for any consequential rule θ, we can find a positive number η such that $F\left(\frac{L}{2}\right) + \eta < \theta < F\left(\frac{R}{2}\right) - \eta$. By Lemma 5, we can find a similar number $\kappa > 0$ such that $\kappa < \beta_L^{-1}(\theta) < 1 - \kappa$. Now, by Lemma 1 and Lemma 2, we can find some $\lambda > 0$ such that $t_R\left(\beta_L^{-1}(\theta)\right) - t_L\left(\beta_L^{-1}(\theta)\right) > \lambda$. Now, from Proposition 2, we can derive θ from $t_R\left(\beta_L^{-1}(\theta)\right)$ and $t_L\left(\beta_L^{-1}(\theta)\right)$ and can find another number $\mu > 0$ such that $t_L\left(\beta_L^{-1}(\theta)\right) + \mu < \theta < t_R\left(\beta_L^{-1}(\theta)\right) - \mu$. Since t_R, t_L and θ^* are all continuous functions of β_L, we can find a number $\xi > 0$ such that for a range $\left(\beta_L^{-1}(\theta) - \xi, \beta_L^{-1}(\theta) + \xi\right)$ around $\beta_L^{-1}(\theta)$, $t_L - \frac{\mu}{2} < \theta < t_R + \frac{\mu}{2}$. Given ξ,we can find M_1 such that $\beta_L^n \in \left(\beta_L^{-1}(\theta) - \xi, \beta_L^{-1}(\theta) + \xi\right)$ in any π^n whenever $n > M_1$. Now consider $\delta = \min\left((t_R\left(\beta_L^{-1}(\theta) - \xi\right) + \frac{\mu}{2} - \theta,\ \theta - t_L(\beta_L^{-1}(\theta) + \xi) - \frac{\mu}{2}\right)$. By Law of large numbers, given ϵ we can find M_2 such that actual share of votes under rule θ under state R, $\tau(R,\pi^n,\theta)$ is less than $t_R\left(\beta_L^{-1}(\theta) - \xi\right) + \frac{\mu}{2} - \delta < \theta$ for any $n > M_2$ and the actual share under state L, $\tau(L,\pi^n,\theta)$ is greater than $t_L\left(\beta_L^{-1}(\theta) + \xi\right) + \frac{\mu}{2} - \delta > \theta$ for any $n > M_2$ with a probability larger than $1 - \epsilon$. Set $N = \max(M_1, M_2)$ and we are done. ■

9.6 Proof of Lemma 6

Proof. If $x_\sigma \le -\frac{b}{2}$, $z_\sigma = F(x_\sigma) \le F(-\frac{b}{2})$ since $F(\cdot)$ is nondecreasing. If on the other hand, $x_\sigma \ge \frac{b}{2}$, $z_\sigma = 1 - F(x_\sigma) \le 1 - F(\frac{b}{2})$. Thus, for $\sigma \in \{l, r\}$, $z_\sigma \le \max\left(F(-\frac{b}{2}), 1 - F(\frac{b}{2})\right)$.Therefore,

$$t(S,\pi) \le q \max(z_l, z_r) + (1-q)\max(z_l, z_r) = \max(z_l, z_r) \le \max\left(F(-\frac{b}{2}), 1 - F(\frac{b}{2})\right) < 1$$

The last inequality in the chain is guaranteed by assumption F. To show $t(S,\pi) > 0$, it is sufficient to show that both z_l and z_r cannot be 0 simultaneously. From assumption F and the definition of x_σ, $z_\sigma = 0 \Rightarrow p_\sigma \in [\frac{1}{2} - \frac{b}{4}, \frac{1}{2} + \frac{b}{4}]$. We show that both p_l and p_r cannot be simultaneouly in this range. We start by noting that p_l and p_r increase in tandem, since both increase with β_L. When $p_l = q$, $\beta_L = \frac{1}{2}$. So, $p_r = 1 - q$. By the above positive relationship, $p_l < q \Rightarrow p_r < 1 - q$ and $p_r > 1 - q \Rightarrow p_l > q$. Note that by Assumption I, $q > \frac{1}{2} + \frac{b}{4}$ and $1 - q < \frac{1}{2} - \frac{b}{4}$.Hence,

$$p_l \in [\frac{1}{2} - \frac{b}{4}, \frac{1}{2} + \frac{b}{4}] \Rightarrow p_r < 1 - q < \frac{1}{2} - \frac{b}{4} \ \textit{and}\ p_r \in [\frac{1}{2} - \frac{b}{4}, \frac{1}{2} + \frac{b}{4}] \Rightarrow p_l > q > \frac{1}{2} + \frac{b}{4}$$

■

9.7 Proof of Lemma 7

Proof. At $\beta_L = 0$, $x_l = x_r = \frac{b}{2} \Rightarrow z_l = z_r = 1 - F\left(\frac{b}{2}\right)$. Now, consider the interval of β_L such that p_l lies in $(0, \frac{1}{2} + \frac{b}{4}]$. In this interval, $x_l \in (\frac{b}{2}, 1] \cup \{-1\} \Rightarrow z_l = 1 - F(x_l)$. Also, in this interval of β_L, $p_r < \frac{1}{2} - \frac{b}{4} \Rightarrow x_r \in (\frac{b}{2}, 1) \Rightarrow z_r = 1 - F(x_r) > 0$, by assumptions F and I. For values of β_L such that $x_l \le 1$, $x_r < x_l \Rightarrow z_l = 1 - F(x_l) < 1 - F(x_r) = z_r$, again by assumption F. For values of β_L such that $x_l = -1$, $z_l = 1 - F(-1) = 0 < z_r$. Thus, over this entire interval $z_r > z_l$. Note also that over this set of values of β_L, z_r is strictly decreasing, while z_l first strictly decreases and then stays at 0. For β_L such that $p_l = \frac{1}{2} + \frac{b}{4}$, $z_r = \overline{z_r}$, say. In the same way, consider the interval of β_L such that p_r lies in $[\frac{1}{2} - \frac{b}{4}, 1]$. Here, by the same token, $z_r < z_l$ except for $\beta_L = 1$ where $z_l = z_r = F\left(-\frac{b}{2}\right)$. z_l increases strictly from $\overline{z_l} > 0$ to $F\left(-\frac{b}{2}\right)$ over this interval, while z_r is initially 0 and then strictly increases.

Now, consider the remaining interval of β_L which is $\left(p_l^{-1}(\frac{1}{2} + \frac{b}{4}), p_r^{-1}(\frac{1}{2} - \frac{b}{4})\right)$. That this is a valid nonempty interval is guaranteed by assumption I. In this interval, $x_r \in (\frac{b}{2}, 1]$, and x_r increases with β_L. Thus, $z_r = 1 - F(x_r)$ is a strictly falling continuous function, going from $\overline{z_r} > 0$ to 0 over this interval. Similarly, z_l strictly and continuously increases from 0 to $\overline{z_l} > 0$. Therefore, there exists a unique β_L^* in this interval where $z_l = z_r$. This implies that at β_L^*, $t(L, \pi) = t(R, \pi)$. For all $\beta_L < \beta_L^*$, $z_l < z_r \Rightarrow t(L, \pi) = qz_l + (1-q)z_r < qz_r + (1-q)z_l = t(R, \pi)$. Similarly, for $\beta_L > \beta_L^*$, where $z_l > z_r$, we have $t(L, \pi) > t(R, \pi)$. ■

9.8 Proof of Lemma 8

Proof. We prove the result for the case $\beta_L^0 = 1$, the other one follows symmetrically. First we look at how $\frac{p_l}{p_r}$ changes with β_L.

$$\frac{p_l}{p_r} = \left(\frac{q}{1-q}\right)\left(\frac{q\beta_R + (1-q)\beta_L}{q\beta_L + (1-q)\beta_R}\right) = \left(\frac{q}{1-q}\right)\left(\frac{q + (1-q)\alpha}{q\alpha + (1-q)}\right),$$

where $\alpha = \frac{\beta_L}{\beta_R}$. Therefore, we have:

$$\frac{d}{d\beta_L}\left(\frac{p_l}{p_r}\right) = \frac{d\alpha}{d\beta_L} \cdot \frac{d}{d\alpha}\left(\frac{p_l}{p_r}\right) = \frac{1}{(1-\beta_L)^2}\left(\frac{q}{1-q}\right)\frac{(1-q)^2 - q^2}{(q\alpha + (1-q))^2} < 0$$

At $\beta_L = 1$, we have $p_l = p_r = 1$. Thus, for $\beta_L \in [0, 1)$, we always have $p_l > p_r$ by the above strictly monotonic relationship. Since $\beta_L^0 = 1 \Rightarrow p_r^n \to 1$, by continuity we can find some m large enough such that for all $n > m$, we have $p_r^n > \frac{1}{2} + \frac{b}{4}$. Since $p_l^n > p_r^n$, for all $n > m$, $p_l^n > \frac{1}{2} + \frac{b}{4}$ too. Since we always have $\beta_L^n < 1$, $p_\sigma^n < 1$. Therefore, for all $n > m$, both x_l^n and x_r^n lie in the open interval $(-1, -\frac{b}{2})$. Also, $p_l^n > p_r^n \Rightarrow x_l^n > x_r^n$ for all $n > m$. This proves part (i). Part (ii) follows trivially from $p_\sigma^n \to 1$. ■

9.9 Proof of Lemma 9

Proof. Part (i) follows from Lemma 5 and Lemma 7.

For part (ii), we first consider the case with $\beta_L^0 = 1$. By Lemma 8, we know that for any such sequence, $x_\sigma^n \to \left(-\frac{b}{2}\right)^-$ for $\sigma = \{l, r\}$, and $x_l^n > x_r^n$ for all large enough n. For large enough n, $p_\sigma^n > \frac{1}{2} + \frac{b}{4} \Rightarrow z_\sigma^n = F(x_\sigma^n) \Rightarrow z_l^n > z_r^n > 0$ and $z_\sigma^n \to F(-\frac{b}{2})$. Define $h^n = z_l^n - z_r^n \to 0^+$. Substituting, we have: $t(L, \pi^n) = z_r^n + qh^n$, and $t(R, \pi^n) = z_r^n + (1-q)h^n$. Therefore:

$$\frac{\beta_L^n}{1-\beta_L^n} = \left[\frac{(t(L,\pi^n))^\theta (1-t(L,\pi^n))^{1-\theta}}{(t(R,\pi^n))^\theta (1-t(R,\pi^n))^{1-\theta}}\right]^n = \left[\frac{(z_r^n + qh^n)^\theta (1 - z_r^n - qh^n)^{1-\theta}}{(z_r^n + (1-q)h^n)^\theta (1 - z_r^n - (1-q)h^n)^{1-\theta}}\right]^n$$

If $\beta_L^0 = 1$, the left hand side of the above equation goes to infinity. This requires the term in the bracket large enough n to be greater than unity, or its logarithm to be positive .

For the case with $\beta_L^0 = 0$, we again use Lemma 8 which tells us that $x_\sigma^n \to \left(\frac{b}{2}\right)^+$ for $\sigma = \{l, r\}$, and $x_l^n > x_r^n$ for all large enough n.We also know that for large enough n, $p_\sigma^n > \frac{1}{2} - \frac{b}{4} \Rightarrow z_\sigma^n = 1 - F(x_\sigma^n) \Rightarrow z_r^n > z_l^n > 0$ and $z_\sigma^n \to 1 - F(-\frac{b}{2})$. Define $h^n = z_r^n - z_l^n \to 0^+$. Substituting, we have: $t(R, \pi^n) = z_r^n + qh^n$, and $t(L, \pi^n) = z_r^n + (1-q)h^n$. Therefore:

$$\frac{\beta_L^n}{1-\beta_L^n} = \left[\frac{(t(L,\pi^n))^\theta (1-t(L,\pi^n))^{1-\theta}}{(t(R,\pi^n))^\theta (1-t(R,\pi^n))^{1-\theta}}\right]^n = \left[\frac{(z_r^n + qh^n)^\theta (1 - z_r^n - qh^n)^{1-\theta}}{(z_r^n + (1-q)h^n)^\theta (1 - z_r^n - (1-q)h^n)^{1-\theta}}\right]^{-n}$$

Since the LHS goes to 0 in the limit, the term within the bracket in the RHS has to be greater than 1. Thus we have the exact same situation as in the proof of Lemma 5, and therefore, we need.

$$\log \frac{(z_r^n + qh^n)^\theta (1 - z_r^n - qh^n)^{1-\theta}}{(z_r^n + (1-q)h^n)^\theta (1 - z_r^n - (1-q)h^n)^{1-\theta}} > 0 \Leftrightarrow \theta > \zeta(z_r^n, h^n) \ \forall n$$

where the function $\zeta(z_l^n, h^n)$ is defined as in the proof of lemma 5.

By Lemma 4, if $\beta_L^0 = 0$, $t = 1 - F(\frac{b}{2})$, and $\theta > \zeta(z_l^n, h^n) \ \forall n \Rightarrow \theta > \lim_{h^n \to 0^+,\ z_l^n = t} \zeta(z_l^n, h^n) = 1 - F(\frac{b}{2})$. Similarly, if $\beta_L^0 = 1$, $t = F(-\frac{b}{2})$, and $\theta > F(-\frac{b}{2})$.

For $\beta_L^0 = \beta_L^*$, from Proposition 4, no value of θ can be ruled out. ■

9.10 Proof of Theorem 3

This is a proof by construction. Consider any non-common value setting $(F(\cdot), q, b)$. We show that every $\beta \in [0,1]$ can be supported by any $\theta \in \Theta(\beta)$ for any $F(\cdot)$ satisfying full support.

Define the function

$$f_n(\beta, \theta) = \frac{1}{1 + \left[\frac{t_R(\beta)^\theta (1-t_R(\beta))^{1-\theta}}{t_L(\beta)^\theta (1-t_L(\beta))^{1-\theta}}\right]^n}$$

If given (n, θ) we can show that there is some fixed point β_n of the function $f_n(\beta, \theta)$, then that β_n is the solution to the equilibrium condition (10), proving that π^n exists for that θ. We prove Theorem 3 by showing that for any $\theta \in \Theta(\beta^0)$, there is a sequence of fixed points of beliefs β_n such that $\beta_n \to \beta^0$ as $n \to \infty$. We prove this separately for different values ranges of β^0.

Case 1 $\beta^0 \in (0, \beta_L^*) \cup (\beta_L^*, 1)$

Proof. First, consider some β^0 the range of beliefs $(0, \beta_L^*) \cup (\beta_L^*, 1)$. By Lemma 9, in this range, $\Theta(\beta_L)$ is a continuous function $\theta^*(\beta_L)$. Since F admits a pdf f, $\theta^*(\beta_L)$ is differentiable too. Thus, there exists a neighbourhood $(\beta^0 - \epsilon, \beta^0 + \epsilon)$ where $\theta^*(\beta_L)$ is either only increasing, only decreasing or constant.

Suppose first that $\theta^*(\beta_L)$ is decreasing in $(\beta^0 - \epsilon, \beta^0 + \epsilon)$.

Now, for $\beta \in (\beta^0, \beta^0 + \epsilon)$, we must have $f_n(\beta, \theta^*(\beta^0)) \to 0$ as $n \to \infty$.

On the other hand, for $\beta \in (\beta^0 - \epsilon, \beta^0)$, we must have $f_n(\beta, \theta^*(\beta^0)) \to 1$ as $n \to \infty$.

Thus, for δ small enough, there must exist some m such that $f_n(\beta + \epsilon, \theta^*(\beta^0)) < \delta$ and $f_n(\beta - \epsilon, \theta^*(\beta^0)) > 1 - \delta$ for all $n > m$. In particular, choose $\delta < \epsilon$.

Thus, for all $n > m$, if $f_n(\beta, \theta^*(\beta^0))$ is plotted against β, it intersects the 45^0 line for some $\beta \in (\beta^0 - \epsilon, \beta^0 + \epsilon)$, which is the fixed point of the function. Call it β_n. To be specific, β_n is the solution of $f_n(\beta, \theta^*(\beta^0)) = \beta$, and for all $n > m$, $\beta_n \in (\beta^0 - \epsilon, \beta^0 + \epsilon)$.

Thus, there exists a sequence β_n such that for any $\epsilon > 0$ small enough, there is some m such that for all $n > m$, $f_n(\beta_n, \theta^*(\beta^0)) = \beta_n$ and $\left|\beta_n - \beta^0\right| < \epsilon$.

If $\theta^*(\beta_L)$ is increasing in $(\beta^0 - \epsilon, \beta^0 + \epsilon)$, then we can prove the theorem in an analogous way.

However, if $\theta^*(\beta_L)$ is constant in the range $(\beta^0 - \epsilon, \beta^0 + \epsilon)$, the theorem may not hold. However, this case requires that $t_L(\beta_L)$ increases (decreases) and $t_L(\beta_L)$ decreases (increases) so as to keep $\frac{t_R(\beta)^\theta(1-t_R(\beta))^{1-\theta}}{t_L(\beta)^\theta(1-t_L(\beta))^{1-\theta}}$ constant over the range. We ignore this case as non-generic. ■

Case 2 $\beta^0 \in \{0, \beta_L^*, 1\}$

Proof. Consider the case $\beta^0 = 0$. Note that $\Theta(0) = \{\theta : \theta > 1 - F(\frac{R}{2})\}$.

Select $\epsilon > 0$ small enough such that $t_R(\beta_L) > t_L(\beta_L)$ in the range $\beta_L \in (0, 2\epsilon)$. Choose $\delta < \epsilon$. By Case 1, for voting rule $\theta^*(\epsilon)$ there exists a sequence of equilibria β_n such that $f_n(\beta_n, \theta^*(\epsilon)) = \beta_n$ and $\beta_n \in (\epsilon - \delta, \epsilon + \delta)$ for n large enough. This implies $\beta_n < 2\epsilon$ for all n large enough.

Now consider the sequence β_n such that $f_n(\beta_n, \theta^*(\epsilon)) = \beta_n$. For $\theta > 1 - F(\frac{R}{2}) > \theta^*(\epsilon)$,we must have $f_n(\beta_n, \theta) < \beta_n$. Now, consider the function $f_n(\beta, \theta) - \beta$. At $\beta = \beta_n$, the function is negative while at $\beta = 0$, the function is positive due to the boundedness of the shares. Since $f_n(\beta, \theta)$ is continuous, there is some $0 < \beta_n' < \beta_n < 2\epsilon$ such that $f_n(\beta_n', \theta) = \beta_n'$.

Thus, given $\theta \in \Theta(0)$, for any ϵ small enough, there exists a sequence β_n'.such that $f_n(\beta_n', \theta) = \beta_n'$ and $\left|\beta_n' - 0\right| < 2\epsilon$ for all n large enough.

In the same way, we can prove the theorem for $\beta^0 = 1$.

Next, consider the case with $\beta^0 = \beta_L^*$. Note that $\Theta(\beta_L^*) = (0, 1)$. To show existence of a limiting equilibrium for $\theta < t_L(\beta_L^*)$, use the neighbourhood $(\beta_L^* - \epsilon, \beta_L^*)$ to the left of β_L^*, and to show existence of a limiting equilibrium for $\theta > t_L(\beta_L^*)$, use the neighbourhood $(\beta_L^*, \beta_L^* + \epsilon)$ to the right of β_L^* and apply the same method. ■