Imminent Nash Implementation*

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Abstract

We introduce delay in simultaneous-move mechanisms. Delay is infinitesimal in equilibrium, hence the name: imminent implementation. We show that mechanisms with delay implement rules that are not implementable without delay in Nash equilibrium and its refinements. We obtain a version of the monotonicity condition that is necessary and sufficient for imminent implementability. If the domain includes all possible preferences, we find that the only unanimous functions that are implementable are dictatorial, as in Nash implementation. This result stands in sharp contrast to the result of a lottery-based “virtual implementation” that nearly all SCRs can be virtually implemented. Suppose that a virtual environment is restricted to the lotteries between an outcome that is fixed in advance and an arbitrary pure outcome. With this restriction, virtual and imminent environments are isomorphic. Furthermore, mechanisms with delay exist in practice. As an example, we consider legislative procedures, which allow the obstruction of bills. We conjecture that these procedures persist because they are particular instances of mechanisms that use delay.

JEL classification: D78, C72, D60, D71, D72

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1 Introduction

Suppose that the society knows outcomes that it wants to achieve in certain situations. Typically, the outcomes are different in different states of the world: one when it rains and another when it shines. If every member of society honestly reports what the state of the world is, the socially desirable outcome is known and delivered to society. This outcome, although socially desirable, may not be the best outcome for a particular individual in a given state: an umbrella maker may like rain every day. Such an individual could try to obtain her preferred outcome by misrepresenting the state of the world: reporting rain when the sun is shining. If someone misrepresents, the report is not unanimous and the true state is not known; alternatively, agents may agree on a coordinated lie about the state. If so, what outcome should be chosen? The implementation problem that we study in this paper is the design of a mechanism or an institution that circumvents these incentives to misrepresent and delivers the “right” outcome in every state.

This design problem has applications in many fields of economics. The definition of a social choice rule – the outcomes that the society deems desirable – is general enough to embrace a variety of economic problems, from negotiation, regulation and provision of public goods to constitutional design and optimal taxation. It is therefore important to delineate the boundaries of institutional design, that is, to characterize the social choice rules (SCRs), which can be implemented. That characterization is the goal of implementation theory.

A person’s incentives to misrepresent typically will depend on the actions of other members of society. Thus, understanding whether a particular institution implements a given SCR requires predicting how individuals interact with each other. The behavioral assumption, or “solution concept,” that has the least predictive power is that an individual picks the dominant action: the action that gives her the highest payoff against any possible strategy of other individuals.

Although dominance is an attractive solution concept, it allows the implementation of few SCRs. If a SCR defined on the domain that includes all possible strict preferences – “full domain” – and satisfies some additional requirements, it is implementable only if it is dictatorial (Gibbard 1973, Satterthwaite 1975). That is, there exists an individual that gets her most preferred outcome in every state. If the designer can rule out certain preferences, a natural assumption for most applications, dominant strategy mechanisms still exist only for “rather special economic environments” (Dasgupta, Hammond, and Maskin 1979, p.186).

\footnote{The SCR should be “onto” with the range that contains at least three alternatives}
To achieve implementation in more general environments we need to abandon dominant strategies and consider the equilibrium behavior of agents. We use Nash equilibrium: the requirement that each agent chooses the strategy that gives the highest payoff against the equilibrium strategies of other agents. We restrict our attention to simultaneous-move mechanisms in environments with complete information.

A necessary and almost sufficient condition for implementation in Nash equilibrium is “monotonicity” (Maskin 1999). The condition requires that, if an outcome $a$ belongs to the social choice in state $\theta$ and does not in state $\phi$, there should be a “preference reversal” around $a$: there exists an agent and an outcome that is no better than $a$ in state $\theta$ for the agent, yet better than $a$ in state $\phi$.

Although monotonicity is permissive, it is far from trivial: many interesting and important SCRs are not Nash implementable. Examples include King Solomon’s dilemma (Glazer and Ma 1989, Moore 1992), plurality rule when there are more than two alternatives (Abreu and Sen 1990), strong Pareto optimal correspondence when there are indifferences (Palfrey and Srivastava 1991). Monotonicity is almost sufficient: additional conditions are needed for the sufficiency result (Maskin 1999, Moore and Repullo 1990).

The limitations of Nash implementation are most striking on the full domain of strict preferences. Similarly to dominant strategy implementation, if there are at least three outcomes, any unanimous and monotonic social choice function is dictatorial (Serrano 2004).

One way to circumvent those limitations is to further strengthen a solution concept. This has proven to be a very fruitful path. For simultaneous mechanisms, Palfrey and Srivastava (1991) show that almost any social choice rule that satisfies (a weak version of) “no veto power” could be implemented in undominated Nash equilibrium. If one allows sequential announcements and employs subgame perfection, Moore and Repullo (1988) and Abreu and Sen (1990) demonstrate that the conditions for implementation become substantially weaker.

Another approach, which we take in this paper, is to expand the set of outcomes $A$ and implement a given social choice rule approximately. Expansion can happen along different dimensions. One such dimension is lotteries. Introduced by Abreu and Sen (1991) and Matsushima (1988) and called virtual implementation, the approach replaces $A$ by the lotteries over it, $\Delta A$, and requires that an outcome prescribed by a SCR is implemented with an arbitrarily high probability. Abreu and Sen and Matsushima show that virtual implementation is very permissive – with three or more agents, any SCR could be implemented.

We consider a different extension: time. We replace $A$ by $A \times \mathbb{R}_+$, where
elements of $\mathbb{R}_+$ are interpreted as a delay in the delivery of a physical allocation $a$. The designer is required to implement the physical allocation prescribed by the SCR exactly, but is allowed to delay the delivery of that allocation. The delay is infinitesimal in the equilibrium, hence justifying the name: imminence implementation.

We characterize imminently implementable SCRs. We then assume that time is continuous and that a SCR to be implemented prescribe no delay. We break the necessary and sufficient condition into three parts, each of which explain how adding delay aids in relaxing a necessary condition for Nash implementation, Maskin monotonicity. We also demonstrate that the SCRs mentioned above, the King Solomon’s dilemma, plurality rule, and strong Pareto correspondence, are imminently implementable when certain assumptions on time preferences are imposed. These rules are not Nash implementable and some are not implementable in the refinements of Nash equilibrium either.

We also show that the condition that makes monotonicity sufficient, “no veto power”, is vacuously satisfied in an environment with delay if we assume that at least two agents are “impatient.” Therefore monotonicity becomes not only necessary but also sufficient, thus providing the full characterization in the case when there are at least three agents.

The conditions for imminent implementation remain restrictive on the full domain of preferences. The results for Nash implementation extend to imminent implementation: only “dictatorial” SCRs are implementable. The lottery-based virtual implementation approach does not encounter this problem, as (almost) any SCR is virtually implementable. Therefore there is a gap between the sets of imminently implementable and virtually implementable rules. We then ask the question: what is responsible for such a gap?

We argue that the introduction of lotteries allow for three modifications to the original Nash implementation problem, while the introduction of delay only allows for two of them.

In virtual implementation, the first modification is that the set of outcomes, $A$, becomes dense. The second is, because of the dense set, that the designer can approximate a SCR. The third is that “mixing” of two outcomes is introduced to achieve the approximation. The mix – the lottery which results in $a$ with probability $p$ and in $b$ with probability $(1-p)$, $pa + (1-p)b$ – inherits the preference reversals of the two parent outcomes, $a$ and $b$, yet makes it possible to deliver $a$ with arbitrarily high probability.

In contrast, imminent implementation also makes an environment dense and this allows us to approximate a SCR. Yet, there is no mixing. That results in stronger conditions that an imminently implementable SCR should satisfy.
We formalize this intuition by constructing an isomorphism between the environment with delay, $A \times \mathbb{R}_+$ and a modified environment with lotteries, $\Delta^d$. In $\Delta^d$ the randomization can happen only between an outcome that is fixed and the same for all states, $d$, and an arbitrary pure outcome. That is, any lottery should take the form \( pa + (1 - p)d \).

We argue that mixing, although it brings very powerful results, may not be a desirable mode of approximation of a SCR. An outcome of the mechanism is an ex-post realization of the lottery and, “[w]hile [the definition of virtual implementation] may seem like an innocuous definition, it may be that two lotteries that are $\epsilon$ close to each other are not even roughly equivalent to a society that cares about ex-post realizations” (Jackson 2001). A vivid example of problems mixing may cause is the virtual implementation of King Solomon’s dilemma. King Solomon, faced with two women claiming a single child, wants to give the child to the true mother, not an impostor. Using the virtual approach, King Solomon would be able to implement his objective with probability $(1 - \epsilon)$, yet with probability $\epsilon$ the baby has to be cut in half even though King Solomon knows who the true mother is (Serrano 2004). That is, baby-cutting is happening on the equilibrium path. We suspect that society is unlikely to find such a mechanism acceptable, as players are punished even when they report truthfully. Even though the outcome that virtual implementation uses for preference reversal need not be bad, that outcome is typically not the part of the social choice rule.

In contrast, on the time-augmented space the mechanism that implements the King Solomon dilemma (Artemov 2006) only requires a small delay in allocating the child. There is no ex-post stage: in every “realization” of a mechanism the outcome is different from the original one by at most $\epsilon$-delay.

Another potential problem of virtual implementation is its dependence on the linear structure of von Neumann-Morgenstern preferences. (Jackson 2001, p. 677) argues that “[o]nce one allows for slight amount of non-linearity in preferences over lotteries, then the crossing conditions are not automatically satisfied, and monotonicity once again becomes restrictive.” Although the degree of dependence is not, in fact, known, this sort of criticism is not applicable to imminent implementation, as it does not exploit the linearity of the outcome space.

One appealing feature of imminent implementation is that the mecha-

\footnote{Bochet and Maniquet (2006) consider a related but more general problem. They allow several outcomes to form an “admissible support” and study the implementation problem for the resulting environment.}

\footnote{If the implementation problem is repeated infinitely, the strength of virtual implementation is regained (Chambers 2004), but so is the dependence on the linear structure of the outcome space.}
nisms that delay outcomes can be found in real life. Examples of delaying an outcome include requesting additional information, or forming a committee; we suppose everyone has his or her favorite example of a seemingly unnecessary delay.

We consider one such example in detail. Explicit, non-informational delays – obstruction of legislation – are embedded in the procedural rules of legislative bodies. Why obstruction is not abolished or controlled more tightly? We argue that embedding delays may expand the set of rules that are implementable. In particular, we construct a stylized example where the availability of obstructionist tactics in majority voting allows to implement a cardinal rule: a majority choose to side with a minority because of the intensity of the minority’s preferences.

The next section contains the notation, assumptions on time preferences and definitions. In Section 3 we provide the characterization of imminently implementable SCRs. We illustrate the necessary and sufficient condition by considering the SCR that prescribe no delay and pinning down the specific factors for why Maskin monotonicity can be relaxed. We then give examples of SCR that are not implementable in Nash equilibrium and its refinements but may be imminently implementable. In Section 4 we prove that on the full domain of preferences the unanimous and monotonic rules are “dictatorial”, similarly to the result for Nash implementation. In Section 5 we construct an isomorphism between the time-augmented environment and the environment with lotteries, suitably restricted. In section 6 we apply our framework to explain legislative delay. The last section contains a short summary of the results and a discussion of possible extensions.

2 Preliminaries

There is a finite set $N = \{1, \ldots, n\}$ of agents and a known set $A$ of social alternatives. Let the set $A \times \mathbb{R}_+$ be the set of social alternatives augmented by time. We interpret an element $(a, t) \in A \times \mathbb{R}_+$ as an alternative $a$ delivered with a delay $t$. $\Delta(A)$ is the set of all probability distributions over $A$.

Let $\Theta$ be the set of possible states. Associated with each $\theta \in \Theta$, there is a preference profile $\succeq^\theta = (\succeq^\theta_1, \ldots, \succeq^\theta_n)$, where $\succeq_i^\theta$ represents agent $i$’s preference ordering over $A \times \mathbb{R}_+$. We assume throughout the paper that preference are complete and transitive.

2.1 Assumptions on time preferences

We assume that the time structure is continuous at infinity and stationary. The first assumption, continuity at infinity states that an agent is indifferent
between outcomes that are never delivered.

**Continuity at Infinity Assumption.** For all \( i \in N, a, b \in A : (a, \infty) \sim_i (b, \infty) \).

The second assumption asserts that time preferences are stationary: there is no reversal in preferences between two outcomes if both outcomes are deferred or advanced by the same amount of time. That is, if an agent prefers an apple today to two apples tomorrow, the agent would also prefer an apple in one year to two apples in one year plus one day.

**Stationarity Assumption.** For all \( a, b \in A, t, s, \tau \in \mathbb{R}_+ \), \((a, t) \succeq_i (b, s)\) if and only if \((a, t + \tau) \succeq_i (b, s + \tau)\)

This assumption is similar to the stationarity assumption \( A5 \) in Fishburn and Rubinstein (1982). In their environment, adding \( A5 \) to the set of assumptions that has already guaranteed the existence of a continuous utility function allows the preferences to be represented by a function with exponential discounting.

Although stationarity assumption caused controversy in literature, the time-consistent model provides a useful benchmark. Time-inconsistent preferences provide more opportunities to find reversals in preferences that are necessary for implementation and, therefore, mechanisms should be easier to construct.

Stationarity implies that the direction of discounting cannot change: it is not possible that an agent prefers a more delayed outcome to a less delayed outcome over one range of delays and less delayed outcome to a more delayed outcome over some other range of delays. This allow us to classify the outcomes into *good*, which become worse if delayed, *bad*, which become better if delayed, and *time-neutral*, which evaluation does not change with time.

**Definition 1.** For an agent \( i \) the outcome \( a \in A \) is

- *good* if \( \forall t, \delta \in \mathbb{R}_{++} : (a, t) \succ_i (a, t + \delta) \)
- *bad* if \( \forall t, \delta \in \mathbb{R}_{++} : (a, t + \delta) \succ_i (a, t) \)
- *time-neutral* if it is neither *good* nor *bad*.

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4An example about apples above is from Thaler (1981), who argues that such an assumption is not plausible. See Frederick, Loewenstein, and O’Donoghue (2002) for a survey.
Continuity at infinity, in turn, implies that good, bad and time-neutral outcomes are separated: a good outcome, no matter how much it is delayed, is always better than a bad outcome.

**Proposition 1.** For any agent \( i \in N \) and any \( a, b, c, d \in A \) such that \( a \) is good, \( b, c \) are time-neutral and \( d \) is bad for \( i \), and for any \( t, t' \in \mathbb{R}_+ \).

1. \( (a, t) \succ_i (b, t') \)
2. \( (b, t) \sim_i (c, t') \)
3. \( (c, t) \succ_i (d, t') \)

**Proof.** Suppose \( \exists t, t' \in \mathbb{R}_+ \) such that \( (b, t') \succeq_i (a, t) \). For any \( \bar{t} > t \), \( (b, \bar{t}) \sim_i (a, \bar{t}) \). Therefore, \( (b, \infty) \succeq_i (b, \bar{t}) \succ_i (a, \bar{t}) \succ_i (a, \infty) \), a contradiction to continuity at infinity. Thus, \( (a, t) \succ_i (b, t') \).

Suppose \( \exists t, t' \in \mathbb{R}_+ : (b, t) \succ_i (c, t') \). This implies \( (b, \infty) \sim_i (b, t') \succ_i (c, \infty) \), a contradiction to continuity at infinity. Thus, \( (b, t) \sim_i (c, t') \).

The proof of 3 is analogous to the proof of 1. \( \square \)

### 2.2 Imminent Implementation: Definition

A social choice rule (SCR) \( F \) is a mapping from the set of states \( \Theta \) to the set of outcomes \( \Omega \): \( F : \Theta \rightarrow \Omega \). \( \Omega \) may represent the set of pure outcomes \( A \), time-augmented outcomes \( A \times \mathbb{R}_+ \) or a simplex of lotteries \( \Delta(A) \). A social choice function (SCF), denoted by \( f \), is a single-valued rule.

An arbitrary set \( M_i \) is an individual message space of agent \( i \); we denote a particular message of an agent \( i \) by \( m_i \). The Cartesian product of individual message spaces \( M_i \) is the message space, denoted by \( M \). Each element is denoted by a profile \( (m_1, \ldots, m_n) \in M \) and \( m_{-i} = (m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_n) \) denotes messages by all agents but an agent \( i \). A mechanism, \( \Gamma = (M, g) \), consists of the message space and an outcome function, \( g : M \rightarrow \Omega \).

We assume that every agent knows the state \( \theta \in \Theta \), but the outcome function (“the rules of the game”) cannot depend on the state. The designer’s task is to construct a mechanism that implements SCR \( F \); that is, to find an appropriate message space \( M \) and an outcome function \( g \), such that in each state \( \theta \) the agents, behaving according to a solution concept \( S \), send messages that result in the outcomes \( F(\theta) \) prescribed by the SCR in that state and in only those outcomes. Formally, denoting by \( g(S(M, g, \theta)) \) the outcomes of the game given the preferences in the state \( \theta \), we have the following definition:

**Definition 2.** A SCR \( F \) is implemented by a mechanism \( \Gamma = (M, g) \) via solution concept \( S \) if \( g(S(M, g, \theta)) = F(\theta) \) for all \( \theta \in \Theta \).
The SCR that can be implemented by a mechanism \((M, g)\) is called \(S\)-implementable.

In this paper we consider implementation mechanisms that use simultaneous announcements. We also assume that the appropriate behavior model is Nash equilibrium.\(^5\)

**Definition 3.** A *Nash equilibrium* of the mechanism \(\Gamma\) at state \(\theta \in \Theta\) is a strategy profile \(m^* = (m^*_1, \ldots, m^*_n)\) such that \(\forall i \in N, \forall m_i \in M_i : g(m^*) \geq_{i}^{\theta} g(m_i, m^*_{-i})\)

An \(\epsilon\)-delayed SCR \(F^\epsilon\) is the SCR that deliver the *same allocations* with a delay no more than \(\epsilon\).

**Definition 4.** A SCR \(F^\epsilon : \Theta \rightarrow A \times \mathbb{R}_+\) is \(\epsilon\)-delayed with respect to \(F : \Theta \rightarrow A \times \mathbb{R}_+\) if for any \(\theta, \forall (a, t) \in F(\theta), \exists t' \in [t, t + \epsilon]\) such that \((a, t') \in F^\epsilon(\theta)\) and \(\forall (a, t') \in F(\theta), \exists t \in [t' - \epsilon, t'] \cap \mathbb{R}_+\) such that \((a, t) \in F(\theta)\).

If for any \(\epsilon\) it is possible to find an \(\epsilon\)-delayed Nash implementable SCR \(F^\epsilon\), a SCR \(F\) is called *imminently implementable*.

**Definition 5.** A social choice correspondence \(F : \Theta \rightarrow A \times \mathbb{R}_+\) is *imminently Nash implementable* if \(\forall \epsilon > 0\) there exists an \(F^\epsilon : \Theta \rightarrow A \times \mathbb{R}_+, \epsilon\)-delayed with respect to \(F\), which is Nash implementable.

This definition diverges from the definition of a Nash implementable SCR, applied to the set \(A \times \mathbb{R}_+,\) in two ways. First, the assumptions *continuity at infinity* and *stationarity* impose additional structure on \(A \times \mathbb{R}_+\). Second, the designer is allowed to approximate a SCR by delaying outcomes.

## 3 Imminent Implementation: Characterization and Examples

In this section we establish the condition that is necessary and sufficient for imminent implementation if \(N \geq 3\). We then illustrate this condition by showing how it helps to implement a SCR \(F : \Theta \rightarrow A \times \{0\}\) when we assume that time is “continuous.” Further we provide examples of SCRs that are imminently implementable but not implementable in Nash equilibrium and its refinements.

\(^5\)We rule out mixed strategies in the definition of Nash equilibrium. Mixed strategies can be accounted for if we use mechanisms with integer games. We ignore issues associated with ruling out mixed strategies or using integer games as they are equally valid in this paper and in the general Nash implementation problem. See Jackson (2001) for more details.
The “imminent monotonicity” condition tell us that if an outcome \((a, t)\) is the social choice in state \(\theta\) but not in \(\phi\), then either \((a, t)\) is a limit point of outcomes that are the social choice in state \(\phi\) or there is a preference reversal for some agent \(i\) between \((a, t)\), perhaps infinitesimally delayed, and some other outcome in \(A \times \mathbb{R}_+\).

**Imminent Monotonicity.** If \(F\) is imminently monotonic if for every \(\phi \in \Theta\) such that \((a, t) \notin F(\phi)\), and for every \(\epsilon > 0\), either of the two conditions holds:

1. there exists \(t' \in [t - \epsilon, t + \epsilon] \cap \mathbb{R}_+\) such that \((a, t') \in F(\phi)\)
2. there exist \(\bar{t} \in \mathbb{R}_+\) and \(t' \in [t, t + \epsilon]\), such that
   \[(a, t') \succeq_i^\phi (b, \bar{t}), (b, \bar{t}) \succ_i^\phi (a, t')\]  

**Theorem 1.** If \(F\) is imminently implementable, it is imminently monotonic.

**Proof.** Suppose \(\exists \theta, \phi \in \Theta\) such that \((a, t) \in F(\theta), (a, t) \notin F(\phi)\), and the condition 2 of imminent monotonicity does not hold. That is, \(\exists \bar{\epsilon} > 0\) such that \(\forall i \in N, \bar{t} \in \mathbb{R}_+, t' \in [t, t + \bar{\epsilon}]\)

\[(a, t') \succeq_i^\phi (b, \bar{t}) \text{ implies } (a, t') \succeq_i^\phi (b, \bar{t}).\]  

Note that since this condition holds for \(\bar{\epsilon}\), it also holds for any \(\epsilon < \bar{\epsilon}\) and any \(t' \in [t, t + \epsilon]\).

Since \(F\) is imminently implementable, there exists a Nash implementable \(F^*\), such that there exists \(t' \in [t, t + \epsilon]\) such that \((a, t') \in F^*(\theta)\). Since \(F^*\) is Nash implementable, there exists a mechanism \(\Gamma\) and a profile \(m^*\) such that \(g(m^*) = (a, t')\) and \(m^*\) is a Nash equilibrium in state \(\theta\).

We argue that \(m^*\) is a Nash equilibrium in state \(\phi\). Suppose it is not. Then there exists a deviation \(m_i\) of agent \(i\); that is, \(m_i\) is such that \(g(m_i, m_{-i}^*) = (b, \bar{t})\) such that \((b, \bar{t}) \succ_i^\phi (a, t')\). We argue that \((b, \bar{t}) \succ_i^\phi (a, t')\) in state \(\theta\). Indeed, if \((a, t') \succeq_i^\phi (b, \bar{t})\) then it follows from 2 that \((a, t') \succeq_i^\phi (b, \bar{t})\), a contradiction. Since \(g(m_i, m_{-i}^*) = (b, \bar{t}) \succ_i^\phi (a, t') = g(m^*), m_i\) is a profitable deviation in state \(\theta\), thus \(m^*\) is not a Nash equilibrium in \(\theta\), a contradiction. Therefore, we conclude that \(m^*\) is a Nash equilibrium in state \(\phi\).

Since \(m^*\) is a Nash equilibrium in state \(\phi\) and a mechanism \(\Gamma\) Nash implements \(F^*\), it follows that \(g(m^*) = (a, t') \in F^*(\phi)\). It then follows from the fact that \(F^*\) is \(\epsilon\)-delayed with respect to \(F\) that there exists an outcome \((a, \tilde{t}) \in F(\phi)\) such that \(\tilde{t} \in [t' - \epsilon, t'] \cap \mathbb{R}_+ \subset [t - \epsilon, t + \epsilon] \cap \mathbb{R}_+\). As this condition holds for any \(\epsilon \leq \tilde{\epsilon}\), it constitutes part 2 of imminent monotonicity condition.
Imminent monotonicity is also sufficient if there are at least three agents and if the environment satisfies “impatience.” Impatience is satisfied if in any state \( \theta \) there are at least two individuals for whom a time-neutral outcome is not top-ranked.

**Definition 6.** An environment \( \mathcal{E} \) satisfies impatience if, \( \forall \theta \in \Theta, \exists i, j \in N, i = j \) such that for any outcome \((a, 0)\) that is time-neutral for \( i \), there exists \((b, t) \in A \times \mathbb{R}_+\) such that \((b, t) >_\theta^i (a, 0)\), and for any outcome \((a, 0)\) that is time-neutral for \( j \), there exists \((c, t') \in A \times \mathbb{R}_+\) such that \((c, t') >_j^i (a, 0)\).

We next show that in the environment which satisfies impatience, the “no veto power” condition is vacuously satisfied by SCRs with delay. No veto power and monotonicity are sufficient for Nash implementation; thus imminent monotonicity is sufficient for imminent implementation in the environment that satisfy impatience. Before stating this result formally, we formulate no veto power for the \( A \times \mathbb{R}_+ \) outcome space.

**Definition 7.** A SCR satisfies no veto power if, in any state \( \theta \in \Theta \), whenever there exists \((a, t)\) such that \( \exists j \in N \), \( \forall i \in N, i = j \), \( \forall (b, t') : (a, t) \succeq_i^j (b, t') \), it follows that \((a, t) \in F(\theta)\)

**Theorem 2.** In the environments with \( N \geq 3 \) that satisfy impatience, an imminent monotonic SCR is imminently implementable.

To prove the theorem, we construct a Maskin monotonic SCR that is \( \epsilon \)-delayed with respect to a given SCR and which does not deliver outcomes at time zero. We then restrict the environment to the one where there are no outcomes delivered at time zero \( (A \times \mathbb{R}_+) \). No veto power condition is vacuously satisfied on that domain. Therefore, a mechanism exists that implements the given SCR with an arbitrarily small delay. Such a mechanism implements the given SCR in the environment where outcomes can be delivered at time zero.

**Proof.** Consider a SCR \( F \) which satisfies imminent monotonicity. Fix some \( \epsilon \). Consider a SCR \( \hat{F} \), such that for any \( \theta \in \Theta \), \( \hat{F}(\theta) \) is the closure of \( F(\theta) \). That is, \( \forall \theta : (a, t) \in \hat{F}(\theta) \) if and only if there is a sequence \( \{(a, t_m)\} \in F(\theta) \) such that \((a, t_m) \rightarrow_m \rightarrow_\epsilon (a, t)\).

Construct a SCR \( \hat{F} \) from \( \hat{F} \) by delaying the outcomes of \( \hat{F} \) by exactly \( \frac{\epsilon}{2} \). That is, for any \( \theta \), \((a, t) \in \hat{F}(\theta) \) if and only if \((a, t + \frac{\epsilon}{2}) \in \hat{F}(\theta)\).

Note that \( \hat{F} \) does not prescribe zero delay in any state. Consider therefore an environment with the set of outcomes \( A \times \mathbb{R}_+ \) and a SCR \( \hat{F} : \Theta \rightarrow A \times \mathbb{R}_+ \) such that \( \forall \theta : \hat{F}(\theta) \equiv \hat{F}(\theta) \). \( \hat{F} \) satisfies no veto power condition. Indeed, since the environment satisfies impatience, there exist two individuals, \( i, j \in \ldots \)
imminently implementable. If $c$ is good, then $(c, \frac{t}{2}) \succ_i (c, t)$. If $c$ is bad, then $(c, 2t) \succ_i (c, t)$. Therefore $(c, t)$ is not top-ranked for $i$. Thus $i$ and, analogously, $j$ do not have top-ranked outcomes. Therefore, the no veto power condition is vacuously satisfied.

We next show that Maskin monotonicity of $F$ is implied by imminent monotonicity of $F$. Suppose that $(a, t) \in F(\theta), (a, t) /\in F(\phi)$. By the definition of $F$, since $(a, t) \in F(\theta), (a, t + \frac{\epsilon}{2}) \in \tilde{F}(\theta)$. In state $\phi$ two cases are possible: either $(a, t) \in F(\phi)$, $F(\phi)$ is a closure of $F(\phi)$, or $(a, t) /\in F(\phi)$. If $(a, t) \in \tilde{F}(\theta)$, it implies that $(a, t + \frac{\epsilon}{2}) \in \tilde{F}(\theta)$. Therefore, $(a, t + \frac{\epsilon}{2})$ is both in $\tilde{F}(\theta)$ and in $\tilde{F}(\phi)$, thus Maskin monotonicity is vacuously satisfied for $(a, t + \frac{\epsilon}{2})$.

If $(a, t) /\in \tilde{F}(\theta)$, then the condition of imminent monotonicity cannot hold. Therefore the condition of imminent monotonicity holds for $F$. That is, there exists $i \in N$, $t' \in [t, t + \frac{\epsilon}{2}), (b, \bar{t}) \in A \times \mathbb{R}_+$ such that $(a, t') \succ^\theta_i (b, \bar{t}), (b, \bar{t}) \succ^\phi_i (a, t')$. Then, observing that $(t + \frac{\epsilon}{2} - t') > 0$ and using stationarity assumption on time preferences, we conclude that $(a, t + \frac{\epsilon}{2}) \succ^\theta_i (b, \bar{t} + t + \frac{\epsilon}{2} - t'), (b, \bar{t} + t + \frac{\epsilon}{2} - t') \succ^\phi_i (a, t + \frac{\epsilon}{2})$. Therefore, Maskin monotonicity condition holds for $(a, t + \frac{\epsilon}{2}) \in \tilde{F}(\theta)$ in that case as well.

We have shown that $\tilde{F}$ satisfies both no veto power and Maskin monotonicity on $A \times \mathbb{R}_+$. Therefore, there exists a mechanism $\Gamma$ that Nash implements $\tilde{F}$. As $\tilde{F} \equiv \tilde{F}$, $\Gamma$ also Nash implements $\tilde{F}$ on $A \times \mathbb{R}_+$; the outcomes $A \times \{0\}$ are irrelevant for the mechanism.

By construction, $\tilde{F}$ is $\epsilon$-delayed with respect to the original SCR $F$. Indeed, if $(a, t) \in F(\theta)$, then $(a, t + \frac{\epsilon}{2}) \in \tilde{F}(\theta)$. If $(a, t') \in \tilde{F}(\theta)$, then either $(a, t' - \frac{\epsilon}{2}) = (a, t) \in F(\theta)$ or $(a, t' - \frac{\epsilon}{2}) \in \tilde{F}(\theta) \setminus F(\theta)$. If the former case holds, it immediately follows that $t \in [t' - \epsilon, t']$. If the latter, there exists $(a, t'') \in F(\theta)$ such that $t'' \in [t - \frac{\epsilon}{2}, t + \frac{\epsilon}{2}]$; therefore, $t'' \in [t' - \epsilon, t']$.

We have therefore shown that for any $\epsilon > 0$ there is a SCR $\tilde{F}$ which is Nash implementable and which is an $\epsilon$-delayed with respect to $F$. Therefore, $F$ is imminently implementable.

3.1 Implementation of rules that prescribe outcomes to be delivered at time zero

In the previous section we have obtained the condition that characterizes imminently implementable rules. In this section we assume that time is “continuous” and illustrate the characterization condition by considering three cases when the social choice rule that prescribe the delivery of an outcome
in time zero becomes implementable when we allow delay.

The first case explains how to exploit the change in preference over an outcome that is better than the social choice outcome in both states. Such a reversal is unusable for Nash implementation without delay. This case is a re-formulation of Maskin’s monotonicity to the outcome space \( A \times \mathbb{R}_+ \), with the time structure imposed on it. The second shows that the violations of Maskin’s monotonicity caused by indifferences in preferences are not present in the imminent framework. The third demonstrates that if there is a reversal in the direction of time discounting between two states – an outcome that becomes worse with time in one state but becomes better with time in another – then it can also aid in constructing a mechanism with delay.

These three cases exhaust all possibilities for preference reversals that mechanisms with delay can exploit implementing SCR that prescribe the delivery of outcomes at time zero.

Definition 8. The time is \textit{continuous} if, for any agent \( i \), any state \( \theta \), any \((a, \bar{t})\) and any sequence \( \{(a, t_n)\}, t_n \to_n \to^* \), such that
\[
\forall n : (a, t_n) \succeq^\theta_i (b, \bar{t}), \text{ it follows that } (a, t^* ) \succeq^\theta_i (b, \bar{t}),
\]
and
\[
\forall n : (b, \bar{t}) \succeq^\theta_i (a, t_n), \text{ it follows that } (b, \bar{t}) \succeq^\theta_i (a, t^* ).
\]

Proposition 2. Suppose that a SCR \( F : \Theta \to A \times \{0\} \) is imminently implementable and time is \textit{continuous}. Then for any \((a, 0)\) such that \((a, 0) \in F(\theta)\), \((a, 0) \notin F(\phi)\), there is an agent \( i \in N \) such that one of the following holds:

1. There exists an outcome \((b, \bar{t}) \in A \times \mathbb{R}_+\) such that \((a, 0) \succeq^\theta_i (b, \bar{t}), (b, \bar{t}) \succ^\phi_i (a, 0)\).

2. \( a \) is good for \( i \) in both \( \theta \) and \( \phi \) and there exists \( b \in A \), such that \((a, 0) \succ^\theta_i (b, 0), (b, 0) \sim^\phi_i (a, 0)\).

3. \( a \) is not good for \( i \) in \( \theta \) and good for \( i \) in \( \phi \).

Note that 1 does not imply 3. In 3, an outcome \( a \) is good in \( \phi \), while in 1 if \( a = b \), \( a \) is necessarily bad in \( \phi \). We also note that if a SCR satisfies 1, delay on the equilibrium path is not necessary; in contrast, if a SCR satisfies 2 or 3 but not 1, delay is used in the equilibrium.

\[6\] A similar observation has been recently made by Benoit and Ok (2004) and explored further by Sanver (2006) in the context of monetary awards. They show that allowing the designer to give a monetary reward to an agent resolves a problem that indifferences are causing to implementing mechanisms.
3.2 Imminently implementable rules that are not implementable in Nash equilibrium refinements

In this section we give two examples to show that the set of imminently implementable rules is not a subset of rules implementable in UNE and SPE. The SCR in the first example, strong Pareto correspondence, is not implementable in undominated Nash equilibrium (UNE) and subgame perfect equilibrium (SPE). It is however implementable with delay with time preferences that exclude time neutral outcomes but otherwise arbitrary. The SCR is not implementable in UNE and SPE because of the indifference of one of the agents over all the alternatives in one of the states. Similar example with less degenerate preferences can be found in Palfrey and Srivastava (1991). Their SCR is UNE but not SPE implementable; this rule is imminently implementable as well.

Example 1 (Strong Pareto correspondence). Suppose there are three agents and two states, $\theta$ and $\phi$. The preference of the agents are given in the table below, where higher outcomes are preferred to lower outcomes and two outcomes on the same row mean indifference between the two.

<table>
<thead>
<tr>
<th>Agent 1</th>
<th>Agent 2</th>
<th>Agent 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta, \phi$</td>
<td>$\theta, \phi$</td>
<td>$\theta, \phi$</td>
</tr>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$b$</td>
</tr>
<tr>
<td>$b$</td>
<td>$b$</td>
<td>$a$</td>
</tr>
</tbody>
</table>

The strong Pareto correspondence, which we denote by $\Pi$, is $\Pi(\theta) = \{a, b\}, \Pi(\phi) = a$. This correspondence is not UNE implementable and, therefore, not implementable in SPE and Nash equilibrium. It does not satisfy property $Q$, a necessary condition for UNE implementation.

Definition 9. $F$ satisfies Property $Q$ if for any $\theta, \phi$, if $x \in F(\theta), x \notin F(\phi)$, then either

1. there exists $i$ and $a, b \in A$ with $a \succ_i^{\theta} b$ and $b \succeq_i^{\phi} a$ and $c, d \in A$ with $c \succ_i^{\phi} d$

or

2. there exists $i$ and $a, b \in A$ with $a \succeq_i^{\theta} b$ and $b \succ_i^{\phi} a$

Neither condition 1 nor condition 2 of this definition can be satisfied by $\Pi$, thus $\Pi$ is not UNE implementable. It is, however, imminently implementable if outcome $b$ is not time-neutral. If outcome $b$ is time neutral in $\phi$ and

---

This is not a particular instance of the environment where Pareto correspondence is implementable. In fact, if $N \geq 3$, and time-neutral outcomes are excluded, any strong Pareto correspondence is imminently implementable.
cardinal preferences of agents 1 and 2 do not change, Π is be not imminently implementable.

If b is good in φ for agent 3, then for any ε > 0, there is t' ∈ (0, ε], small enough so that (b, t') ≽^θ_i (a, 0). Then, as t' > 0, (a, 0) ≽^φ_i (b, t'). Thus, part 2 of imminent monotonicity is satisfied. No veto power is satisfied by Π without impatience assumption and N = 3; therefore, Π is imminently implementable if b is good.

If b is bad in φ for agent 3, then a is also bad in φ. Let ¯t > 0 be such that (b, 0) ≽^θ_i (a, ¯t). Then (a, ¯t) ≽^φ_i (b, 0). These two preference reversals hold for any ε as we do not delay (b, 0) ∈ Π(θ); that is, the condition t' ∈ [0, ε] is trivially satisfied. We then have the preferences as required by part 2 of imminent monotonicity. As N = 3 and Π satisfies no veto power, Π is imminently implementable if b is bad.

The delay can be used not only when indifferences cause violations of monotonicity. In the example below all preferences are strict. The SCR is not SPE implementable, and, consequently, not Nash implementable. It is imminently implementable but only if we assume specific class of time preferences.

Example 2 (Plurality rule, Abreu and Sen (1990)). Suppose there are three outcomes, \( A = \{a, b, c\} \), three agents and two states, \( \theta \) and \( \phi \). Agent 1’s and 2’s preferences are identical in both states. Agent 3 prefers c to b in one state and has an opposite preference in the other. Their preferences are described by the table below, where higher outcomes are preferred to lower outcomes:

<table>
<thead>
<tr>
<th>Agent 1 (θ, φ)</th>
<th>Agent 2 (θ, φ)</th>
<th>Agent 3 (θ, φ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>a</td>
</tr>
</tbody>
</table>

The social choice rule picks the majority winner; in case of a tie, it picks the entire set. That is, \( F(\theta) = \{a, b, c\}, F(\phi) = b \). This rule is not SPE implementable (Abreu and Sen 1990). Indeed, note that for \( a \in F(\theta), a \notin F(\phi) \) a necessary condition for SPE implementation, condition \( \alpha \), is not satisfied. Condition \( \alpha \) requires that there exists an agent \( j(0) \in N \) such that an outcome \( b \) or \( c \) is at least as good as \( a \) in state \( \theta \) (part i of condition \( \alpha \)) plus \( a \) is not the maximal element for \( j(0) \) in state \( \phi \) (part iv of condition \( \alpha \)). The only agents that satisfy the former is agent 1, but \( a \) is maximal for agent 1 in \( \phi \). Thus, the plurality rule in not SPE implementable.

If we account for preferences over time, this rule may become implementable. Suppose that the correct description of the preferences of agent
3 is given below, where two outcomes on the same row mean indifference between the two and where preferences over outcomes \((z,t)\) not described in the table are consistent with stationarity and otherwise arbitrary:

<table>
<thead>
<tr>
<th>Agent 3</th>
<th>(\theta)</th>
<th>(\phi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((c,0))</td>
<td>((b,0))</td>
<td></td>
</tr>
<tr>
<td>((c,\tau)) ((b,0))</td>
<td>((b,t)) ((c,0))</td>
<td></td>
</tr>
<tr>
<td>((c,\tau + \tau')) ((b,\tau')) ((a,0))</td>
<td>((b,t + t')) ((c,t')) ((a,0))</td>
<td></td>
</tr>
</tbody>
</table>

where \(\tau' < t + t'\). The preferences of agents 1 and 2 over outcomes \((z,t)\) are arbitrary.

Consider a SCR on \(A \times \mathbb{R}_+\), \(\hat{F} : \Theta \rightarrow A \times \mathbb{R}_+\), such that \(\hat{F}(\theta) = \{(a,0), (b,0), (c,0)\}\), \(\hat{F}(\phi) = (b,0)\). We next show that \(\hat{F}\) is Maskin monotonic. For \((a,0) \in \hat{F}(\theta), (a,0) \notin \hat{F}(\phi)\) the outcome \((b,\tau')\) provides the reversal: \((a,0) \sim^\theta (b,\tau')\), while \((b,\tau') \succ^\phi (a,0)\). For \((c,0) \in \hat{F}(\theta), (c,0) \in \hat{F}(\phi)\), the reversal is provided by \((b,0)\): \((c,0) \succ^\theta (b,0), (b,0) \succ^\phi (c,0)\). Thus, \(\hat{F}\) is Maskin and, consequentially, imminently monotonic.

\(\hat{F}\) also satisfies no veto power as \(\hat{F}(\phi) = (b,0)\) and \((b,0)\) in state \(\phi\) is the only outcome which is top ranked for two people. Thus, \(N = 3\), \(\hat{F}\) is Maskin monotonic and satisfies no veto power thus, \(\hat{F}\) is imminently implementable. Note that the delay is not used in the equilibrium. Time is used to provide the opportunity for deviation for agent 3 in state \(\phi\), yet not to provide such an opportunity in state \(\theta\); to achieve this, the outcome \(b\) has to be sufficiently delayed. Note also that we do not need to assume impatience to conclude that the SCR is imminently implementable because the SCR satisfies no veto power without impatience assumption, as in the first example.

We note, however, that if \(\tau' \geq t + t'\), \(\hat{F}\) is neither Nash nor imminently implementable, provided cardinal preferences of agents 1 and 2 do not change from \(\theta\) to \(\phi\), as there is no preference reversal either around \((a,0)\) or around \((a,\epsilon)\), for an arbitrary \(\epsilon\).

4 Implementation on the full domain of preferences and dictatorial rules

In the previous section we have demonstrated that imminent implementation is capable of implementing SCRs which are not implementable in Nash equilibrium and its refinements. We have also observed that whether a SCR is implementable may depend on the particular time preferences: rules may be implementable in one specification but not implementable in another. This observation hints at our next result: on the domain of preferences that
include all possible preference profiles, the full domain of preferences, any unanimous social choice function (SCF), which takes at least three different values, is imminently implementable only if it is “dictatorial;” that is, there is an agent that gets her most preferred outcome whenever she has one.

Before proving this result formally, we remark that the assumption that the designer cannot rule out certain preference profile is hardly realistic, even more so when we allow any time preferences into the full domain. Yet, characterization on the full domain is theoretically important because there are few large domains for which general results exist; thus obtaining results for these domains allows us to compare different modes of implementation.

We first define formally a full domain of preferences. We assume that time is continuous, as defined in section 3.1. It has been established in proposition 1 that for every agent \( i \), \( A \) is partitioned into good, time-neutral and bad outcomes. Call these partitions \( G \), \( N \) and \( B \). We will now define a full domain of preferences on each of these partitions separately. Let \( m = |G| \).

Order pure outcomes in \( G \) so that \( g_1 \) is the (weakly) worst outcome, \( g_2 \) is the (weakly) second-worst and so on. The best outcome is \( g_m \). Formally \( g_k \in G \) is such that \( \forall j > k : g_j \succeq_i g_k \), \( \forall j < k : g_k \succeq g_j \). Any arbitrary preference profile on \( G \) is represented by the Cartesian product of (outcome, delay) pairs, where delay is the time that makes agent \( i \) indifferent between the given outcome and the next-worse one. Formally, the preferences on \( G \) are represented by \( \Pi_{j=1}^{m} (g_j, t_j) \), where \( t_j \in \mathbb{R}_+ \) is defined as \( (g_j, t_j) \sim_i (g_{j-1}, 0) \) for any \( j > 1 \). For \( j = 1 \) we assume \( t_1 = 0 \).

We analogously define an arbitrary preference profile for agent \( i \) on \( B \) as \( \Pi_{j=1}^{m} (b_j, t_j) \). For the outcomes in \( N \), any agent is indifferent between any of these outcomes, so no additional structure is necessary. Therefore, we have the following definition:

**Definition 10.** For a given agent \( i \) consider an arbitrary partition of \( A \) into \( G \), \( N \), and \( B \) and a collection \( C_i = \{ \Pi_{j=1}^{m} (g_j, t_j), N, \Pi_{j=1}^{m} (b_j, t_j) \} \). The domain of preferences on \( A \times \mathbb{R}_+ \) is full if it is represented by a Cartesian product of single collections for agents \( i = 1, \ldots, N \): \( \Pi_{i=1}^{N} C_i \) and include all possible individual \( C_i \).

We now give the definition of a dictatorial rule on \( A \times \mathbb{R}_+ \).

**Definition 11.** A SCF \( f \) is *dictatorial* if there exists an agent \( i \in N \) such that \( \forall \theta \in \Theta \), whenever there exists a top-ranked outcome \( (a, t) : \forall (b, t') \neq (a, t) : (a, t) \succeq_i^\theta (b, t') \), it follows that \( (a, t) \in f(\theta) \).

If an agent \( i \) defined above exists, it is a “dictator”. Note that on the domain \( A \times \mathbb{R}_+ \) there are infinitely many profiles where the definition of a dictator is vacuous. Indeed, if a profile is such that every alternative is *bad*
for a dictator, there is no \((a, t)\) which is better than any other \((b, t')\), as 
\((a, 2t) \succ_{i}^{\theta} (a, t)\). Also note that we require strict relation in the definition of a dictator, thus if the best outcome is time-neutral, the definition is also vacuous.

**Definition 12.** A SCR \(F\) is unanimous if, for a given \(\theta \in \Theta\), whenever 
\(\forall i \in N, \forall (b, t') \in A \times \mathbb{R}_+: (a, t) \succeq_i (b, t')\), it follows that \((a, t) \in F(\theta)\).

**Theorem 3.** Suppose that the domain of preferences over \(A \times \mathbb{R}_+\) is full and the number of pure alternatives \(|A| \geq 3\). Then if a unanimous SCF \(f\) is imminently monotonic, it is dictatorial.

**Proof.** See Appendix.

---

**5 Isomorphism between imminent and restricted virtual environments**

We have established in the previous section that when the environment includes all possible preference profiles, using delay in the mechanism leads to the results analogous to those provided by Nash implementation approach. In contrast, lottery-based “virtual mechanisms” are able to implement almost any SCR. In this section we tackle the question what is responsible for the difference. We restrict virtual mechanism to the lotteries where one of the outcomes is fixed and show that restricted virtual and imminent environments are isomorphic, thus concluding that the ability to “mix” arbitrary outcomes is essential for the permissive results.

Throughout the section the analysis applies to and is identical for every agent and every state. Therefore, with apologies for abuse of notation, we suppress indexing of agents and states.

In virtual implementation, the set of possible outcomes \(A\) is replaced by the set \(\Delta\) of all possible lotteries over \(A\). Formally, \(\Delta = \{(p_1, \ldots, p_k) \in \mathbb{R}_+^k : \sum_{j=1}^k p_j = 1\}\), where \(k = |A|\). For a given SCR \(F : \Theta \to 2^\Delta \setminus \emptyset\), the mechanism is required to implement some approximate \(F^\epsilon\). An approximate SCR prescribes a lottery between an outcome \(a\) that is an original social choice, and some arbitrary outcome \(d \in \Delta\): \(pa + (1 - p)d\); the probability \((1 - p)\) of delivering a “wrong” outcome \(d\) is to be made smaller than any given \(\epsilon\).

Adding lotteries over outcomes and allowing the delivery of a “wrong” outcome with small probability leads to a very permissive result: on the environment that satisfy no total indifference any cardinal rule is virtually implementable.
Theorem (Abreu and Sen (1991), Theorem 5). Let $N \geq 3$. Then any cardinal SCR $F$ is virtually implementable in Nash equilibrium.

As we pointed out in the introduction, an outcome that virtual implementation delivers, albeit with small probability, has nothing to do with the social choice. That outcome is delivered not as a punishment; in fact, it is delivered on the equilibrium path, when all agents “report truthfully.” We therefore argue that mixing may not be a desirable property. Yet, the availability of mixing is what gives the edge to virtual implementation over imminent one. To show this, we restrict the designer to the lotteries between an arbitrary pure outcome $a$ and a fixed outcome $d$. Formally, suppose $d$ is such fixed outcome and denote the probability of $d$ by $p_k$, 

$$\Delta^d = \{(p_1, \ldots, p_k) \in \mathbb{R}_+^k : \exists j, p_j + p_k = 1, \forall i \neq j, k : p_i = 0\}.$$ 

Equivalently, denoting $A \setminus \{d\}$ by $A^d$, we can simply write $\Delta^d = A^d \times [0, 1]$. 

We show that the restricted virtual environment $\Delta^d$ is isomorphic to imminent environment $A^d \times \mathbb{R}_+$. For that, we construct a bijection $\mu$ and show that the assumptions we have made on time preferences hold if and only if the standard assumptions on risk attitudes hold.

**Theorem 4.** $\Delta^d$ is isomorphic to $A^d \times \mathbb{R}_+$. 

**Proof.** The bijection $\mu$ between $\Delta^d$ and $A^d \times \mathbb{R}_+$ are described by the following formula: $\mu(a, t) = (a, \frac{t}{\bar{t}})$ and correspondingly $\mu^{-1}(a, p) = (a, \ln \frac{1}{p})$. We need to show that the preference relations implied by axioms on the lottery space and assumptions on time are preserved under $\mu$.

The continuity at infinity assumption translates into the assumption that if there are two outcomes $(a, 0), (b, 0) \in \Delta^d$ that place zero probability on $a$ and $b$ correspondingly, a person is indifferent between these two alternatives. Note that if $a$ and $b$ happen with zero probability, these lotteries are degenerate $d$. That is, continuity at infinity effectively requires that $d \sim d$ on $\Delta^d$.

The stationarity assumption translates into the standard independence axiom applied to the space of restricted lotteries, $\Delta^d$. Independence axiom is satisfied if the preference relation is such that for all $L_a, L_b, L_d \in \Delta$ and all $\alpha \in [0, 1]$, $L_a \succeq L_b$ if and only if $\alpha L_a + (1 - \alpha) L_d \succeq \alpha L_b + (1 - \alpha) L_d$.

We reformulate the independence axiom on the restricted lottery space, $\Delta^d$. Suppose $L_a = (p a + (1 - p) d) \in \Delta^d$ and $L_b = (q b + (1 - q) d) \in \Delta^d$ and that $a \neq b$ (if $a = b$ the statement of independence axiom is simply that the order on $[0, 1]$ is preserved when two lotteries over the same outcomes $a, d$ are compounded). Then if $L_d \in \Delta^d, L_d \neq d$, either $\alpha L_a + (1 - \alpha) L_d \notin \Delta^d$ or $\alpha L_b + (1 - \alpha) L_d \notin \Delta^d$. Thus, we will only consider $L_d = d$. The compound lottery $\alpha L_a + (1 - \alpha) L_d$ can be therefore written as $\alpha (p \cdot a + (1 - p) \cdot d) + (1 - \alpha) d$ or as $(a, \alpha p) \in \Delta^d$. 

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Therefore, if independence axiom holds on $\Delta$, the corresponding condition on $\Delta^d$ would be: $\forall (a,p), (b,q) \in \Delta^d$ and any $\alpha \in [0,1]$ $(a,p) \succeq (b,q)$ if and only if $(a,\alpha p) \succeq (b,\alpha q)$.

Bijection $\mu^{-1}$ transforms the independence axiom on $\Delta^d$ above into stationarity assumption. Indeed, let $t = \ln \frac{1}{p}, s = \ln \frac{1}{q}, \tau = \ln \frac{1}{\alpha}$, thus $t + \tau = \ln \frac{1}{\alpha p}, s + \tau = \ln \frac{1}{\alpha q}$. Using $\mu^{-1}$ we obtain: for all $(a,t), (b,s) \in A^d \times \mathbb{R}_+$ and $\tau \in \mathbb{R}_+$ $(a,t) \succeq (b,s)$ if any only if $(a, t + \tau) \succeq (b, s + \tau)$. This is our stationarity assumption.

Finally, the continuity assumption is clearly preserved by $\mu$, as $\ln x$ and $e^x$ are continuous functions. We have therefore shown that bijection preserve properties of preferences over lotteries and preferences over time.

We therefore have established the isomorphism between $A^d \times \mathbb{R}_+$ and $\Delta^d$. This implies that, as mathematical objects, the two environments coincide. We note however, that imminent implementability of a SCR is implied by restricted virtual implementation if cardinal preferences over lotteries and the cardinal preferences over time are isomorphic. For an application, this is unlikely to be the case. It is thus possible that a given agent may have reversals in her preferences over time that designer can use but does not have usable reversals in her preferences over lotteries, or vise versa.

6 An Application: Obstruction of Legislation

6.1 Motivation

An important question that implementation theory faces is whether the underlying features of the mechanisms are “realistic.” If a construction depends critically on a feature that cannot be found or conceivably put to work in real life, the viability of such a construction is in question.

We argue that mechanisms that purposely delay an outcome can be observed in real life. Procedural rules used in legislation are an example. In this subsection we show that these rules allow dilatory motions, which only purpose is to delay a vote on a legislation. We also review the existent explanations for the persistence of rules that allow dilatory motions. In the next subsection we present a stylized example to illustrate arguments put forward here.

The tactics available to obstruct a bill are numerous. “Although obstruction in the modern Senate is equated with extended speeches, this is not the only method senators have employed to hinder majorities... [Methods] include motions regarding amendments as well as various procedural motions,
such as motions to adjourn, to recess, to postpone, or to table legislative items. Senators can also suggest the absence of a quorum, forcing a call of the roll” (Wawro and Schickler 2006). A bill may also be referred back to committees, the motion to commit, and it can be kept in committees for a long time without reaching the floor. The mere threat of filibuster may delay the consideration of a bill. We do not distinguish these dilatory tactics in this section. As the filibuster is the most studied obstructionist tactic, we briefly review explanations for its continued existence proposed in the literature.

Filibuster is often treated as a purely supermajority rule (e.g. Krehbiel, 1998)). Currently, 3/5 votes are needed to invoke cloture that stop the discussion. Therefore, a bill needs to muster 3/5 support to pass in the Senate. This explanation, however, have two drawbacks.

First, the cloture motion has been adopted after 1917; no motion could stop a filibuster before that. Thus, supermajority interpretation of filibuster would seem to imply that the Senate should have been a unanimous body (Mayhew 2003), a claim refuted by Mayhew and Wawro and Schickler (2004). Second, “[this theory] has failed to grapple with the simple questions of why senators choose to obstruct rather than simply vote “no” on legislation they oppose or, alternatively, why they would choose not to obstruct legislation that they vote against” (Wawro and Schickler 2006).

More recent theories (Bawn and Koger 2003, Koger 2006, Wawro and Schickler 2006) stress that efforts are required to filibuster the legislation and that costs are incurred while the legislation is filibustered. As efforts cause disutility to the Senators, the bill is filibustered only if the disutility from adopting a piece of legislation is sufficiently large. Therefore, filibuster helps to elicit the intensity of preferences of the senators.

The explanation that the imminent implementation framework would imply is along these lines. We ignore efforts. We assume that the obstruction results in delaying a vote on a bill. That is, we assume that the bill is not withdrawn immediately if it gets obstructed. A delay may have different interpretations for different motions. Filibuster causes delay that forestalls all other agenda. Other motions, such as the motion to commit, or the threat of filibuster postpone consideration of the bill without affecting other

---

8 For example, in 1970s the Senate leaders introduced a “tracking system” that “allowed the majority leader . . . to put aside a filibustered measure and move on to a second “track” of legislation” (Binder, Lawrence, and Smith 2002).

9 In the original citation this statement applies not only to pivotal politics approach, but more broadly to the “extant work on the filibuster.”

10 Wawro and Schickler (2006) also make this assumption in their model.

11 This interpretation is essentially the non-effort cost of filibustering considered by Wawro and Schickler (2006).
agenda. These motions also do not require much efforts, thus the model of Wawro and Schickler cannot be applied to them.

The papers that propose effort-based or cost-based explanations of filibuster consider mechanisms designed to mimic particular Senatorial rules\footnote{For example, Wawro and Schickler (2006) use the fact that senators are typically required to regularly cast votes during a dilatory motion. If one side does not have majority at this particular moment, it loses. We note here that their model is substantially more complex and elaborated since it also explains why senators are not filibustering at the end of the session, where the cost committed and efforts required are small.} in particular, they tend to use complicated sequential games. The mechanism we describe in the next subsection is not tied to particular rules. It is one-shot, thus employs simpler solution concept, Nash equilibrium. It allows us to stress the role of delay in eliciting intensity of preferences. Its disadvantage is the other side of the coin: as it is more general, it is less natural.

The mechanism is a slight modification of a canonical Maskin’s mechanism of implementation theory and is similar to the basic game described in Wawro and Schickler, but instead of being sequential, it is one-shot. A variant of integer game, a game which is often considered to by “unnatural” is present in the mechanism we construct. Curiously, a similar mechanism is also present in Wawro and Schickler’s model in the form of the “game of attrition” and does not appear unnatural there.

6.2 An example

In this subsection we construct an example of a rule that accounts for intensity of preferences and can be implemented by using a majoritarian voting mechanism with an embedded delay.

Suppose that there are three senators, two states, $\theta$ and $\phi$, and two competing pieces of legislation $a$ and $b$. One of them would be adopted at some endogenously determined time $t$ (e.g., $(a, 0)$ denotes immediate passage of legislation $a$). There is an implicit status-quo outcome $q$, which we assume is worse than $a$ and $b$ for any of the senators; it does not play any role in our analysis and, therefore, suppressed.\footnote{If the underlying dilatory motion is the filibuster, $q$ may be interpreted as status-quo once the bill has reached the floor; that is, no other agenda can be considered before the bill is passed or rejected. In that case, $b$ can be interpreted as existing legislation and $a$ as a proposed legislation.} The senators 1 and 3 prefer $a$ to $b$ in both states while the senator 2 prefer $b$ to $a$ in both states. However, the intensity of preferences is different between states. The intensity is measured by the delay senators are willing to tolerate before becoming indifferent between preferred delayed bill and non-preferred bill, accepted immediately.
We summarize these preferences in the following table, where higher outcomes are preferred to the lower outcomes and outcomes on the same row mean indifference.

<table>
<thead>
<tr>
<th>Senator 1</th>
<th>Senator 2</th>
<th>Senator 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\theta)</td>
<td>(\phi)</td>
<td>(\theta)</td>
</tr>
<tr>
<td>((a, 0))</td>
<td>((a, 0))</td>
<td>((b, 0))</td>
</tr>
<tr>
<td>((a, 10))</td>
<td>((a, 1))</td>
<td>((b, 1))</td>
</tr>
</tbody>
</table>

The time preferences are stationary; that is, if \((a, 10) \sim^\theta_i (b, 0)\) (as in state \(\theta\) for senator 1), then \((a, 11) \sim^\theta_i (b, 1)\). We also assume that senators prefer a given bill sooner than later: \(\forall t > 0, \forall i : (a, 0) \succ^\theta_i (a, t), (b, 0) \succ^\theta_i (b, t)\).

We assume that the socially desired bill is \(a\) in state \(\theta\) and \(b\) in state \(\phi\); that is \(F(\theta) = a, F(\phi) = b\).

An “almost-canonical” mechanism implements this SCF. Suppose that the senator takes an action \(m_i \in M_i\), where \(M_i = \{a, b\} \times \{o, n\} \times \mathbb{R}_+\). That is, the senators vote, tell whether they want to obstruct the bill or not and determine how much “resources” \(r \in \mathbb{R}_+\) to use.\(^{14}\)

The mechanism is the following. The alternative which receives the majority of votes is implemented unless it is obstructed. If the bill is obstructed, the delivery of the outcome is delayed by some pre-specified number \(t \in (1, 10)\), and the delivered outcome is determined by the competition in spending the resources on the promotion of the bill: if \(r_{1,3} > r_2\), the resulting outcome is \(a\) and otherwise it is \(b\). We assume that the senators are not competing in spending resources if no one filibusters; this assumption may be justified, for example, if it takes time for resources to alter the outcome.

We do not impose any cost of spending the resource on the senator, so the senator can spend unlimited resources. Imposing such cost will not change our conclusion. Note that competition in spending resources is unwinnable: if someone spends \(r\), the opponent obtains her preferred outcome by spending \(r + \epsilon\). As competition is unwinnable, senators “want” to abstain from using obstruction, but they would do so only if at least two senators vote for \(a\) in state \(\theta\) and if senators vote unanimously for \(b\) in state \(\phi\).

Formally, the outcome function \(g(m)\) is defined by the following two rules:

\(\forall j, m_j = (z, o, r_j)\), where \(z = \{a, b\}\) and \(r_1, r_2, r_3 \in \mathbb{R}_+\) are arbitrary. If two senators vote \(a\) (\(z_i = z_j = a\) for \(i \neq j\)), \(g(m) = (a, 0)\). If two senators vote \(b\), \(g(m) = (b, 0)\).

\(^{14}\)If “resources” delay an outcome further or constitute effort by a senator, our mechanism becomes similar to the “war of attrition”-style mechanism of Wawro and Schickler (2006).
ii If $\exists j, m_j = (z, o, n_j)$, then if $r_1 > r_2$ or $r_3 > r_2$, $g(m) = (a, 5)$, if $r_2 \geq r_1, r_3$, $g(m) = (b, 5)$.

**Proposition 3.** $\Gamma = (M_i, g(m))$ implements $F$.

**Proof.** See Appendix.

7 Conclusion.

In this paper we examine the effect of adding delay to the implementation problem. We allow the designer to delay outcomes infinitesimally on the equilibrium path and to delay them arbitrarily off the equilibrium path. The set of rules implementable if delay is allowed is not a subset of the rules implementable in Nash equilibrium and its refinements: subgame perfect and undominated Nash equilibrium. It is, however, a subset of virtually implementable rules. We identify a restriction on virtual implementation under which the two sets coincide: any lottery in a virtual mechanism should be between two pure outcomes, one of which is fixed for all states. We then argue that such a restriction may be desirable and, if imposed, any virtually implementable rule would be implementable with delay. Delay, in turn, is more typical in real life applications. We argue that voting rules in legislations that allow obstruction and delays may be thought of as an example of a mechanism with delay.

The purpose of our voting in legislations example has been only to show that an explicit delay may be a feature of real-life mechanisms. We think that it would be an interesting generalization of an example to construct a mechanism that match reality closer and to test whether the delaying tactics is indeed used to reveal intensity of agent’s preferences.

An interesting extension of this paper would be to consider sequential mechanisms. Delay is very appealing in a sequential mechanism: it suffices to assume that each stage of the mechanism takes time. That is, an outcome delivered in the second stage is slightly worse then the outcome delivered in the first. This extension may lead not only to the enlargement of the set of implementable SCR and to more transparent conditions that characterize subgame perfect implementation, but also to simpler and “more natural” mechanisms.

Another distinct feature of time is that it is irreversible. If a delay has already occurred, it is impossible to implement an outcome at time zero. In particular, it is impossible to renegotiate time that has already passed.\footnote{Rubinstein and Wolinsky (1992) make this assumption and study a particular – trading – mechanism that incorporates time dimension.}
Thus, adding time to mechanisms may improve their resistance to renegotiation of outcomes. Better renegotiation-resistance may also be achieved if we assume that renegotiation itself takes time.

Finally, the idea of delay can also be used in constructing more natural one-shot mechanisms. In the present paper, we do not explicitly construct mechanisms for Nash implementation, except the mechanism in the last section. The sufficiency is proven by referring to existing mechanisms for Nash implementation (e.g. the canonical mechanism in Maskin, 1999). Thus, any criticism of those mechanisms applies to the current study. One such criticism is that the mechanisms use “unnatural constructions:” the so-called integer games, the constructions that our legislative voting mechanism also use. Another interesting extension of the present paper is to design a mechanism that exploits the structure of the time-augmented outcome space to avoid using these unnatural constructions.

8 Appendix

Proof of proposition 2, “Three cases when monotonicity holds for SCRs that do not prescribe delay”

Proof. Imminent monotonicity handles any \((a,t) \in F(\theta)\). As we only deal with outcomes where \(t = 0\), we re-write this condition with respect to an original outcome \((a,0)\). We first note that part 1 of imminent monotonicity cannot hold because \(\forall \theta \in \Theta, \forall (a,t) \in F(\theta) : t = 0\). Therefore, part 2 should hold. We distinguish two cases, \(t' \leq \bar{t}\) and \(t' \geq \bar{t}\).

\(t' \leq \bar{t}\): Using stationarity, we can re-write imminent monotonicity as: for every \(\theta, \phi \in \Theta\), such that \((a,0) \in F(\theta), (a,0) \notin F(\phi)\) and for any \(\epsilon \in \mathbb{R}^+\) there exist \(i \in N, \bar{t} \in \mathbb{R}^+, t' \in [0,\epsilon]\) such that \((a,0) \succeq^\theta_i (b,\bar{t} - t'), (b,\bar{t} - t') \succ^\phi_i (a,0)\). Note that this preference relation holds for any \(\epsilon\), and for any \(t' < \epsilon\), thus constituting case 1 of the theorem.

\(t' > \bar{t}\): We can re-write the imminent monotonicity as: for every \(\theta, \phi \in \Theta\), such that \((a,0) \in F(\theta), (a,0) \notin F(\phi)\) and for any \(\epsilon > 0\) there exist \(i \in N, \bar{t} \in \mathbb{R}^+, t' \in [0,\epsilon]\) such that

\[
(a, t' - \bar{t}) \succeq^\theta_i (b,0), (b,0) \succ^\phi_i (a, t' - \bar{t}).
\]  

(3)

Note that by our assumption, \(0 < \bar{t} < t'\). Therefore, since \(t' \in [0,\epsilon]\) it follows that \(t' - \bar{t} \in [0,\epsilon]\). As time is continuous, and (3) holds for any \(\epsilon\), (3) imply that \((a,0) \succeq^\theta_i (b,0), (b,0) \succ^\phi_i (a,0)\).

25
There are two cases, \((b, 0) \succeq^\phi_i (a, 0)\) and \((b, 0) \sim^\phi_i (a, 0)\). In the former we immediately establish condition [1] of the theorem.

Suppose that \((b, 0) \sim^\phi_i (a, 0)\). From [3] it follows that \((a, 0) \sim^\phi_i (b, 0) \succ^\phi_i (a, \bar{t} - t')\). We therefore conclude that \(a\) is good in \(\phi\).

Suppose that \(a\) is not good in state \(\theta\); that is, \((a, \bar{t} - t') \succ^\theta_i (a, 0)\). We have just shown that \(a\) is good in state \(\phi\): \((a, 0) \succ^\phi_i (a, \bar{t} - t')\). Therefore, \((a, \bar{t} - t), (a, 0)\) form a test pair required in the condition [3].

If \(a\) is good in state \(\theta\), we have established [2].

\[\square\]

**Proof of theorem 3.** “Only dictatorial rules are implementable on full domain”

We first observe that condition [1] of imminent monotonocity cannot hold when we consider a social choice function. Therefore, we restrict our attention to condition [2] only.

We also note that when we change standing of an outcome, say \((a, t)\), in standing of preferences of agent \(i\), stationarity assumption would imply that every outcome \((a, t')\) would also change its standing correspondingly. That is, if \((a, t) \sim^\theta_i (b, t')\) and \((a, \bar{t}) \sim^\theta_i (b, t'')\) in state \(\theta\), and in state \(\phi\) the position of \(a\) and only \(a\) changes to, for example, \((a, t) \sim^\phi_i (b, t' + 10)\) it would automatically imply that \((a, \bar{t}) \sim^\phi_i (b, t'' + 10)\). We would call the collection of outcomes of the form \((a, t)_{t \in \mathbb{R}^+}\) a tail of outcomes \((a, t)\).

Before we turn to the proof of the theorem, we establish a lemma that rules out the case when the social choice is \((a, t), t > 0\), if \(a\) is good for every agent \(i\).

**Lemma 1.** Suppose that the domain of preferences over \(A \times \mathbb{R}^+\) is full. If a SCF \(f\) is imminently monotonic and unanimous, for any \(t > 0\), any outcome \(a \in A\), which is good for every \(i \in N\), and any \(\theta \in \Theta\), \((a, t) \notin F(\theta)\).

**Proof.** Suppose that \(\exists \theta \in \Theta, \exists a \in A, \forall t > 0 : f(\theta) = (a, t)\). Consider an arbitrary agent \(i\). If \((a, 0)\) is not a top alternative for an agent \(i\), derive profile \(\phi_i\) from \(\theta\) by raising the tail of outcomes \((a, t)\) in her preference profile so that \((a, 0)\) is the top alternative for \(i\). By stationarity, every outcome \((a, t)\) also raises: if \((a, \tau) \succeq^\theta_i (z, \tau'), (a, \tau) \succ^\phi_i (z, \tau')\).

We argue that \(f(\phi_i) = (a, t)\). If \(f(\phi_i) \neq (a, t)\), then, by imminent monotonocity, for any \(\epsilon > 0 \exists (a, t'), t' \in [t, t + \epsilon], (b, \bar{t}) \in A \times \mathbb{R}^+\) such that \((a, t') \succeq^\theta_i (b, \bar{t}), (b, \bar{t}) \succ^\phi_i (a, t')\). However, the only change in preferences from state \(\theta\) to state \(\phi_i\) is that the tail \((a, t)\) has risen: if \((a, t') \succeq^\theta_i (b, \bar{t})\) then \((a, t') \succeq^\phi_i (b, \bar{t})\). Therefore there is no such \((a, t'), (b, \bar{t})\).
Derive profile \( \phi \) by raising the tail \( \{(a, t)\} \) for every agent for whom \((a, 0)\) is not top-ranked in \( \theta \). By the argument above, \( f(\phi) = (a, t) \). However, \((a, 0)\) is the top alternative for every agent and \((a, t)\) is the social choice, a contradiction to unanimity.

Therefore, whenever \( z \) is good, we ignore a possibility that after the change in preferences a social choice is \((z, t), t > 0\) in the proof below.

**Proof.** Let \( 0 < \delta < \frac{1}{2|A|} \), where \(|A|\) is the number of the non-delayed outcomes and let \( 0 < \epsilon < \delta \).

**Step 1.**

In this step we find a pivotal agent, which, as we prove later, is a dictator.

Consider arbitrary allocations \( a, b \in A \) and the preferences in the state \( s^0 \), identical for all \( i \), where each agent is indifferent between a given outcome delayed by one and the next-worse outcome and all outcomes are good; that is, the preferences are:

\[
(a, 0) \succ_i s^0_1 (a, 1) \sim_i s^0_1 (c, 0) \succ_i s^0_1 (c, 1) \ldots (b, 0) \succ_i s^0_1 (b, 1)
\]

By unanimity, \( f(s^0) = (a, 0) \).

Preferences in \( s^1_0 \) are derived from \( s^0 \) by raising the tail of \( \{(b, t)\} \), in the preferences of agent 1, so that \((b, 1) \sim_{j} s^0_{j} (a, 0)\). As \( b \) is good in \( s^0, s^1_1, (b, 0) \) is the top alternative in \( s^0_1 \). To summarize, the preferences of agent 1 are as follows,

\[
(b, 0) \succ_{j} s^0_{j} (b, 1) \sim_{j} s^0_{j} (a, 0) \succ_{j} s^0_{j} (a, 1) \sim_{j} s^0_{j} (c, 0) \succ_{j} s^0_{j} (c, 1) \ldots
\]

and the preferences of other agents are as before.

In \( s^1_1 \), either \( f(s^1_1) = (a, 0) \) or \( f(s^1_1) = (b, 0) \). Suppose to the contrary that \( f(s^1_1) = (d, 0) \), \( d \neq a, b \). The only change in the preferences from \( s^0_1 \) to \( s^0 \) is that the tail \( \{(b, t)\} \) has became worse. That is, if, for some \( \epsilon \), \((d, \epsilon) \succ_{1} (b, t)\), then \((d, \epsilon) \succ_{j} (b, t)\). By imminent monotonicity, \( f(s^0) = (d, 0) \), a contradiction. Therefore, \( f(s^0) \) is either \((a, 0)\) or \((b, 0)\).

Derive \( s^2_1 \) by raising the standing of the tail \( \{(b, t)\} \) in the preferences of agent 2, as for agent 1. Continue with agent 3, and so on.

In the state \( s^0_n \), each agent’s top alternative is \((b, 0)\). By unanimity, \( f(s^0_n) = (b, 0) \). Therefore there is a pivot agent \( j \), such that \( f(s^0_j) = (b, 0) \), \( f(s^0_{j-1}) = (a, 0) \). Denote the profile \( s^0_{j-1} \) by \( \theta \) and \( s^0_j \) by \( \phi \).

**Step 2.**

The profile \( \bar{\theta} \) is derived from profile \( \theta \) by improving all outcomes except \( b \) for agents \( i > j \) (that is, preferences of \( i > j \) become more “condense”) and by substituting the profile for agent \( j \) by a profile where \((a, 1) \sim_j (b, 0)\); the preferences for \( i < j \) are the same. That is, the resulting profiles are:
for $i < j$:
\[(b, 0) \succ_i^\theta (b, 1) \sim_i^\theta (a, 0) \succ_i^\theta (a, 1) \sim_i^\theta (c, 0) \succ_i^\theta (c, 1) \ldots\]

for $j$:
\[(a, 0) \succ_j^\theta (a, 1) \sim_j^\theta (b, 0) \succ_j^\theta (b, 1) \sim_j^\theta (c, 0) \succ_j^\theta (c, 1) \ldots\]

for $i > j$:
\[(a, 0) \succ_i^\theta (a, \delta) \sim_i^\theta (c, 0) \succ_i^\theta (c, \delta) \ldots (b, 0) \succ_i^\theta (b, 1) \ldots (a, \frac{1}{2}) \sim_i^\theta (b, 0)\]

$f(\theta) = (a, 0)$ because preferences of agents $i < j$ have not changed and $(a, 0)$ is the top outcome for $i \geq j$; that is, $\forall (b, t) \in A \times \mathbb{R}_+, b \neq a, (a, \epsilon) \succ_i^\theta (b, t)$, as $\epsilon < \delta$ by assumption. The direction of discounting of $a$ has not changed. Therefore $f(\theta) = (a, 0)$.

**Step 3.**

Profile $\phi'$ is derived from $\phi$ by sending the outcomes \{(a, $t$)\} to the bottom for $i < j$, that is $\forall z \in A : (z, 0) \succ_i^{\phi'} (a, 0)$, and, for $i > j$, second to the bottom, $\forall z \in A, z \neq b : (z, 0) \succ_i^{\phi'} (a, 0)$. Preferences of $j$ do not change. That is, the resulting preference profile for agents is

for $i < j$:
\[(b, 0) \succ_i^{\phi'} (b, 2) \sim_i^{\phi'} (c, 0) \succ_i^{\phi'} (c, 1) \ldots (a, 0) \succ_i^{\phi'} (a, 1) \ldots\]

for $j$:
\[(b, 0) \succ_j^{\phi'} (b, 1) \sim_j^{\phi'} (a, 0) \succ_j^{\phi'} (a, 1) \sim_j^{\phi'} (c, 0) \succ_j^{\phi'} (c, 1) \ldots\]

for $i > j$:
\[(c, 0) \succ_i^{\phi'} (c, 1) \ldots (a, 0) \succ_i^{\phi'} (a, \frac{1}{2}) \sim_i^{\phi'} (b, 0) \succ_i^{\phi'} (b, 1) \ldots\]

$f(\phi') = (b, 0)$ because no outcome $(a, t)$ has became better than $(b, \epsilon)$ for any $\epsilon$ and preferences over other outcomes have not changed.

**Step 4.**

To derive profile $\bar{\phi}$ from profile $\phi'$, send $a$ to the top of $j$’s preferences; the resulting profile is

for $i < j$:
\[(b, 0) \succ_i^{\bar{\phi}} (b, 2) \sim_i^{\bar{\phi}} (c, 0) \succ_i^{\bar{\phi}} (c, 1) \ldots (a, 0) \succ_i^{\bar{\phi}} (a, 1)\]

for $j$:
\[(a, 0) \succ_j^{\bar{\phi}} (a, 1) \sim_j^{\bar{\phi}} (b, 0) \succ_j^{\bar{\phi}} (b, 2) \sim_j^{\bar{\phi}} (c, 0) \succ_j^{\bar{\phi}} (c, 1) \ldots\]

for $i > j$:
\[(c, 0) \succ_i^{\bar{\phi}} (c, 1) \ldots (a, 0) \succ_i^{\bar{\phi}} (a, \frac{1}{2}) \sim_i^{\bar{\phi}} (b, 0) \succ_i^{\bar{\phi}} (b, 1)\]
We argue that \( f(\tilde{\phi}) = (a, 0) \). Consider first the case \( f(\tilde{\phi}) = (d, 0), d \neq a, b \). The change in preferences when we go from \( \tilde{\phi} \) to \( \phi' \) is that the tail \( \{(a, t)\} \) becomes worse for \( j \). That is, if \( (d, \epsilon) \prec_j \tilde{\phi} (a, t), (d, \epsilon) \preceq_j \phi' (a, t) \). By imminent monotonicity, \( f(\phi') = (d, 0) \), a contradiction.

Consider now the case when \( f(\tilde{\phi}) = (b, 0) \). Consider the change in preferences when we go from \( \tilde{\phi} \) to \( \bar{\theta} \), constructed in step 2.

For agents \( i < j \), \( (b, 0) \) is the top alternative in \( \bar{\theta} \). Therefore, there is no \( (z, \tilde{t}) : (z, \tilde{t}) \succ_i \phi' (b, \epsilon) \) as \( \epsilon < \delta \) and the reversal in preferences required by imminent monotonicity cannot happen for agents \( i < j \). For agent \( j \), the preferences around \( (b, t'), t' \in [0, \epsilon] \) do not change as well.

There is also no reversals around \( (b, t'), t' \in [0, \epsilon] \) in preferences of agents \( i > j \) when we go from \( \tilde{\phi} \) to \( \bar{\theta} \). The preferences for \( i > j \) in \( \bar{\theta} \) have been derived from \( \tilde{\phi} \) in the following manner. Worsen the tail of outcomes \( \{(c, t)\} \) so that \( (a, \delta) \sim_i ^{\tilde{\phi}} (c, 0) \). Therefore, if \( (b, \epsilon) \preceq_i ^{\tilde{\phi}} (c, \tilde{t}) \), it follows \( (b, \epsilon) \preceq_i ^{\hat{\phi}} (c, t) \). Continuing with other tails of outcomes \( \{(z, t)\} \) such that \( (z, 0) \succ_i ^{\tilde{\phi}} (a, 0) \) (note that in every step the tail of outcomes \( \{(z, t)\} \) become worse, similarly to \( \{(c, t)\} \)), we construct the profile as in \( \bar{\theta} \) and observe that for every outcome \( (z, \tilde{t}), (b, \epsilon) \succ_i ^{\tilde{\phi}} (z, \tilde{t}) \) implies \( (b, \epsilon) \succ_i ^{\hat{\phi}} (z, \tilde{t}) \).

Therefore, by imminent monotonicity, if \( f(\tilde{\phi}) = (b, 0) \), then \( f(\bar{\theta}) = (b, 0) \), a contradiction. We conclude that \( f(\tilde{\phi}) = (a, 0) \).

\textbf{Step 5.}

Derive profile \( \tilde{\psi} \) from profile \( \tilde{\phi} \) by making the tail \( \{(b, t)\} \) worse so that \( (b, 0) \) is worse than \( (c, 0) \) for agents \( i \leq j \):

\[
\text{for } i < j : \]
\[
(c, 0) \succ_i ^{\tilde{\psi}} (c, \frac{1}{2}) \sim_i ^{\tilde{\psi}} (b, 0) \succ_i ^{\tilde{\psi}} (b, \frac{1}{2}) \ldots (a, 0) \succ_i ^{\tilde{\psi}} (a, 1)
\]

\[
\text{for } j : \]
\[
(a, 0) \succ_j ^{\tilde{\psi}} (a, 3) \sim_j ^{\tilde{\psi}} (c, 0) \succ_j ^{\tilde{\psi}} (c, \frac{1}{2}) \sim_j ^{\tilde{\psi}} (b, 0) \succ_j ^{\tilde{\psi}} (b, \frac{1}{2}) \ldots
\]

\[
\text{for } i > j : \]
\[
(c, 0) \succ_i ^{\tilde{\psi}} (c, 1) \ldots (a, 0) \succ_i ^{\tilde{\psi}} (a, \frac{1}{2}) \sim_i ^{\tilde{\psi}} (b, 0) \succ_i ^{\tilde{\psi}} (b, 1)
\]

\[
f(\tilde{\psi}) = (a, 0) \] because the only change is making the tail \( \{(b, t)\} \) worse; that is, no outcome became better than \( (a, \epsilon) \) and by imminent monotonicity \( f(\tilde{\psi}) = (a, 0) \).

\textbf{Step 6.}

Derive profile \( \psi \) from \( \tilde{\psi} \) by making the tail \( \{(a, t)\} \) worse for agents \( i > j \):
for $i < j$:

$$(c, 0) \succ_i^\psi (c, \frac{1}{2}) \sim_i^\psi (b, 0) \succ_i^\psi (b, \frac{1}{2}) \ldots (a, 0) \succ_i^\psi (a, 1)$$

for $j$:

$$(a, 0) \succ_j^\psi (a, 3) \sim_j^\psi (c, 0) \succ_j^\psi (c, \frac{1}{2}) \sim_j^\psi (b, 0) \succ_j^\psi (b, \frac{1}{2}) \ldots$$

for $i > j$:

$$(c, 0) \succ_i^\psi (c, 1) \ldots (b, 0) \succ_i^\psi (b, 1) \sim_i^\psi (a, 0) \succ_i^\psi (a, 1)$$

We argue that $f(\psi) = (a, 0)$. Suppose $f(\psi) = (d, 0), d \neq a, b$. However, for $i > j$, who are the only agents for whom preferences change, $\forall (a, t) \in A \times \mathbb{R}_+: (d, \epsilon) \succ_i^\psi (a, t)$ as $\epsilon < \delta$ and the change in preferences over over \{$(a, t)\}$ is the only change from $\psi$ to $\bar{\psi}$. Therefore, no outcome $(z, \bar{t})$ exists such that $(d, \epsilon) \succ_i^\psi (z, \bar{t}), (z, \bar{t}) \succ_i^\psi (d, \epsilon)$ and, by imminent monotonicity, it implies that $f(\bar{\psi}) = (d, 0)$, a contradiction.

Suppose that $f(\psi) = (b, 0)$. Worsen the tail \{$(a, t)\}$ for agent $j$, so that $(c, 0)$ is better than $(a, 0)$ for $j$. No outcome has became better than $(b, \epsilon)$ when \{$(a, t)\}$ is worsened, thus by imminent monotonicity the social choice continues to be $(b, 0)$. Yet, $(c, 0)$ is the top alternative for every agent, a contradiction to unanimity.

We therefore conclude that $f(\psi) = (a, 0)$.

**Step 7.**

In this step we show that for any preference profile, where $(a, 0)$ is top ranked for $j$, the social choice is $(a, 0)$. To do so, we show that any profile such that $(a, 0)$ is top-ranked for $j$ can be created from profile $\psi$ by a change in preferences that is imminently monotonic around $(a, 0)$.

Consider an arbitrary profile $\xi$ such that $(a, 0)$ is top-ranked for $j$. Note that $(a, 0)$ is good for $j$, since if $(a, 0)$ is time-neutral or bad, it cannot be top-ranked.

Define for an individual $i \neq j$ the set of good outcomes, $G_i$, the set of time-neutral outcomes $N_i$ and the set of bad outcomes, $B_i$: $G_i = \{z : (z, 0) \succeq_i^\xi (z, 1)\}$, $N_i = \{z : (z, 1) \sim_i^\xi (z, 0)\}$, $B_i = \{z : (z, 1) \succ_i^\xi (z, 0)\}$. $G_i \cup N_i \cup B_i = A$ for any $i$.

Let $(g, 0)$ be the best pure outcome in $G_i$; that is $\forall z \in G_i : (g, 0) \succeq_i^\xi (z, 0)$. Let $(w, 0)$ be the worst pure outcome in $G_i$; that is $\forall z \in G_i : (z, 0) \succeq_i^\xi (w, 0)$. Define $\tau_i$ as a delay such that $(g, \tau_i) \sim_i^\xi (w, 0)$; such $\tau_i$ exists because $(w, 0) \succ_i^\xi (g, \infty)$. We construct a profile $\psi'$ for agent $i$ where $(a, 0)$ is made “very bad”. This will allow the “room” for making other outcomes worse when we will construct the required preference profile. As outcomes will only be getting worse, imminent monotonicity will imply that the social choice outcome does not change.
Formally, \((a, 0) \sim_{i}^{\psi} (g, |G_i|_i)\) and preferences over other alternatives do not change. \(f(\psi') = (a, 0)\). Suppose to the contrary that \(f(\psi) = (z, 0), z \neq a\). Imminent monotonicity and the fact that for any \(z \in A\) and for any \(t \in \mathbb{R}_+\), \((z, \epsilon) \succ_{i}^{\psi} (a, t)\) as \(\epsilon < \delta\), coupled with the fact that preferences over other outcomes have not changed imply that \(f(\psi) = (z, 0)\), a contradiction.

The profile \(\psi^w\) is derived from \(\psi'\) by worsening the tail \{\((w, t)\)\} so that \((w, 0) \sim_{i}^{\psi^w} (a, 0)\). \(f(\psi^w) = (a, 0)\) since no outcome becomes better than \((a, \epsilon)\) when we move from \(\psi'\) to \(\psi^w\). Continue with the second-worst non-delayed outcome \((h, 0)\) by placing it in the state \(\psi^h\) according to \(i\)'s preferences in \(\xi\); that is if \(\tilde{h}\) is such that \((h, \tilde{h}) \sim_{i}^{\xi} (w, 0)\), then \((h, \tilde{h}) \sim_{i}^{\psi^h} (w, 0)\). By the same argument as for \(\psi^w\), \(f(\psi^h) = (a, 0)\). At the last step, if \((a, 0) \succ_{i}^{\psi} (w, 0)\), improve the tail \{\((a, t)\)\} to reconstruct the preference profile \(\xi\) for agent \(i\) on \(G_i \times \mathbb{R}_+\). Denote this preference profile by \(\psi^a\). Since \((a, 0)\) has only improved, \(f(\psi^a) = (a, 0)\). For outcomes in \(N_i, B_i\), we replace preferences over them in \(\psi^a\) by preferences as required by \(\xi_i\). Note that \(\forall \gamma \in \Theta \forall (z, t) \in N_i, B_i : (a, \epsilon) \succ_{i}^{\gamma} (z, t)\). Therefore, \(f(\psi_{-i}, \xi_i) = (a, 0)\).

Following the algorithm for all \(i \neq j\), we conclude that \(f(\psi_j, \xi_{-j}) = (a, 0)\).

Note that both in the profile \(\xi((a, 0)\) is the top-ranked alternative; that is, there exists \(\epsilon\) such that \(\forall b \in A : (a, \epsilon) \succ_{j}^{\xi} (b, 0)\). By monotonicity, \(f(\psi_j, \xi_{-j}) = (a, 0)\) implies \(f(\xi) = (a, 0)\). We have therefore shown that for an arbitrary profile \(\xi\) where \((a, 0)\) is top-ranked, \(f(\xi) = (a, 0)\). That is, we established the existence of the dictator over an arbitrary alternative \(a\); that is the agent who gets \(a\) whenever \(a\) is her top outcome. Denote this agent by \(j_a\).

As \(a\) has been an arbitrary outcome, such a dictator exists for any other \(x \in A\). We will now show that the there could not possibly be two different dictators over outcomes \(a\) and \(x \in \Theta\); that is, \(j_a = j_x\). Suppose to the contrary that \(j_x \neq j_a\). Consider a profile such that \((a, 0)\) is top alternative for \(j_a\) and \((x, 0)\) is top alternative for \(j_x\). For that profile the social choice is both \((a, 0)\) and \((x, 0)\), a contradiction. Therefore, \(j_a\) is the dictator for any alternative \(x \in A\).

\[\square\]

**Proof of proposition 3. “Mechanism with a filibuster”**

**Proof.** Suppose the state is \(\theta\).

1. Two sets of strategies, \(m_1 = (a, n, r_1)\), \(m_2 = (a, n, r_2)\), \(m_3 = (a, n, r_3)\), and \(m_1 = (a, n, r_1)\), \(m_2 = (b, n, r_2)\), \(m_3 = (a, n, r_3)\), where \(r_j\) are arbitrary, are Nash equilibria. Note that all of them result in \(g(m) = (a, 0)\).

The deviation of senator 1 to \(m_i = (b, n, r_i)\) either does not change an outcome, if \(z = a\), or change it to \(b\) thus being not profitable. Deviation to \(m_i = (z', a, r_i)\), for any \(z'\), gives either \((a, t)\) or \((b, t)\) and thus not profitable.
either. The same holds for senator 3.

Senator 2 does not have profitable deviation either. Deviating to \((a, n, r_2)\) does not change an outcome. Deviating to \((z, o, r_2)\) gives at most \((b, t)\), which is not preferred in state \(\theta\).

Thus, two actions above, \((m_1, m_2, m_3)\) and \((m_1, \hat{m}_2, m_3)\) constitute Nash equilibria.

2. There are no other Nash equilibria.

Note first that if, for some \(i\), \(m_i = (z, o, r_i)\), such strategy is not a NE, because, if it results in \((a, t)\), senator 2 can induce \((b, t)\) by increasing her resource \(r_2\); if it results in \((b, t)\), senators 1 or 3 can induce \((a, t)\).

Thus, the only candidates for being a Nash equilibrium are strategies where there is no obstruction. Note that when there is no obstruction, the reported resource is not used in the determination of the outcome. We thus can identify messages of candidates for an equilibrium by just an outcome, \(\{a, b\}\). There are eight combinations and we have already determined that \((a, b, a)\) and \((a, a, a)\) are Nash equilibria. We consider the other six.

- \((a, a, b)\), \((b, a, a)\) are not a NE because senator 2 has profitable deviation to \(m_2 = (b, n, r_2)\)
- \((a, b, b)\), \((b, a, b)\), \((b, b, a)\), \((b, b, b)\) are not NE because senator 1 has profitable deviation to \(m_1 = (a, o, r_1)\), thus inducing \((a, t) \succ_i^\theta (b, 0)\).

Thus, there are no other Nash equilibria in state \(\theta\). Analogously, there is only one class of Nash equilibria in state \(\phi\): \(m_1 = (b, n, r_1)\), \(m_2 = (b, n, r_2)\), \(m_3 = (b, n, r_3)\). The reason for just one class of equilibria is that there are two agents who can profitably deviate if the announcement is not unanimous. Except for this small detail, the proof is the same as for the case when the state is \(\theta\).

\[\square\]

References


\[16\text{Senator 1 has other, more profitable, deviations from some of those strategies, but we are required to find one which is profitable.}\]


