Characterizations of Pareto-efficient, fair, and strategy-proof allocation rules in queueing problems*

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For updated version: http://troi.cc.rochester.edu/~ckyi/KR2006.pdf

This version: November 21, 2006

Abstract

A set of agents with different waiting costs have to receive a service of equal length of time from a single provider which can serve only one agent at a time. One needs to form a queue and set up monetary transfers to compensate the agents who have to wait. We analyze rules that are efficient, fair, and immune to strategic behavior. We prove that no rule is *Pareto-efficient* and *coalitional strategy-proof*. We also show that in combination with *Pareto-efficiency* and *strategy-proofness*, *equal treatment of equals* is equivalent to *no-envy*. We identify the class of rules that satisfy *Pareto-efficiency*, *equal treatment of equals*, and *strategy-proofness*. Among multi-valued rules, there is a unique rule that satisfies *Pareto-efficiency*, *anonymity*, and *strategy-proofness*.

JEL Classification: D63, C72.

Keywords: Queueing problems, efficiency, fairness, strategy-proofness.

*We thank Paulo Barelli, Walter Bossert, Sidartha Gordon, François Maniquet, Manipushpak Mitra, Jay Sethuraman, William Thomson, Gábor Virág, and seminar participants at the University of Maastricht, Social Choice and Welfare Conference 2006, University of Rochester for their helpful discussions and suggestions.

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1 Introduction

A set of agents simultaneously arrive at a service facility that can only serve one agent at a time. Agents require service for the same length of time. The waiting cost may vary from one agent to the other. Each agent is assigned a "consumption bundle" consisting of a position in the queue and a positive or negative transfer. Each agent has quasi-linear preferences over positions and transfers. For such a *queueing problem*, a rule assigns each agent a position in the queue and a positive or negative transfer such that no two agents are assigned the same position, and the sum of transfers is not positive.

Our objective is to identify rules that are well-behaved from the normative and strategic viewpoints. In addition to efficiency, we assess the desirability of a rule from two perspectives: the fairness of the allocations it selects and the incentive it gives to agents to tell the truth about their cost parameters. The first requirement is efficiency. It says that if an allocation is selected, there should be no other feasible allocation that each agent finds at least as desirable and at least one agent prefers. Since preferences are quasi-linear, *Pareto-efficiency* can be decomposed into two axioms: on the one hand, *efficiency of queues*, which says that a queue should minimize the total waiting cost, and on the other hand, *balancedness*, which says that transfers should sum up to zero.

Second is a minimal symmetry requirement: agents with equal waiting costs should be treated equally. As agents cannot be served simultaneously, it is of course impossible for two agents with equal costs to have equal assignments. However, using monetary transfers, we can give them assignments between which they are indifferent. We require *equal treatment of equals in welfare*: agents with equal waiting costs should be indifferent their assignments. It is implied by *no-envy*, which requires that no agent should prefer another agent's assignment to her own.

Third is immunity to strategic behavior. As unit waiting costs may not be known, the rule should provide agents incentive to reveal these costs truthfully. *Strategy-proofness* requires that each agent should find her assignment when she truthfully reveals her unit waiting cost at least as desirable as her assignment when she misrepresents it. We are also concerned about possible manipulations by groups, and consider *coalitional strategy-proofness*: no group of agents should be able to make each of its members at least as well off, and at least one of them better off, by jointly misrepresenting their waiting costs. Finally is *non-bossiness*: if an agent's change in her announcement does not affect her assignment, then it should not affect any other agent's assignment.

We identify the class of rules that satisfy *efficiency of queues* and *strategy-proofness.* We show that a unique allocation rule satisfies *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness.* For each problem, this rule selects a *Pareto-efficient* queue and it sets transfers as follows: consider each pair of agents in turn, make each agent in the pair pay the waiting cost incurred by the other agent in the pair, and distributes the sum of these two payments equally among the others. We refer to this rule as the *Equally Distributed Pairwise Pivotal rule.* As the name indicates, it applies the idea of the well-known Pivotal rule from the class of Groves' rules in public decision-making problems in each pair (Clarke 1971, Groves, 1973). The Equally Distributed Pairwise Pivotal rule also satisfies *no-envy.* Using this result, we also show that in combination with *Pareto-efficiency* and *strategy-proofness*, *equal treatment of equals in welfare* is equivalent to *no-envy.*

We may also be concerned about possible manipulations by groups. However, if we impose the stronger incentive property of *coalitional strategy-proofness*, even with *efficiency of queues*, we have an impossibility result. This result suggests that the previous result is tight.

We then extend the first result to possibly multi-valued rules. First, we consider fairness properties when it is possible to give two agents with equal unit waiting costs same assignments at two different allocations. Then, *symmetry* requires that agents with equal waiting costs should be treated symmetrically, that is, if there is another allocation at which two agents exchange their assignments and the other agents keep theirs, then this allocation should be selected. It is implied by *anonymity*, which requires that agents' names should not matter. However, whereas single- and multi-valued rules may satisfy *equal treatment of equals in welfare*, only multi-valued rules may satisfy *symmetry.* Thus, because agents cannot be served simultaneously, *anonymity* is possible if and only if multi-valuedness is allowed. Second, *strategy-proofness* has to be redefined for multi-valued rules. To compare the welfare levels derived from two sets of feasible allocations, we assume that an agent prefers the former to the latter if and only if for each allocation in the latter, there is an allocation in the former that she finds at least as desirable; and for each allocation in the former, there is an allocation in the latter that she does not prefer.

Next, we define the rule that selects all *Pareto-efficient* queues and for each queue, sets transfers as in the Equally Distributed Pairwise Pivotal rule. We refer to it as the *Largest Equally Distributed Pairwise Pivotal rule.* We prove that a unique allocation rule satisfies *Pareto-efficiency*, *symmetry*, and *strategy-proofness*. Moreover, it is *anonymous*. Also, as *anonymity* implies *symmetry*, and as the Largest Equally Distributed Pairwise Pivotal rule is the union of all the rules that satisfy *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness*, it follows that this rule is the only rule that satisfies *Pareto-efficiency*, *equal treatment of equals in welfare*, *symmetry*, and *strategy-proofness*.

The intuition for the results is simple. Any rule can be described by selecting the queues appropriately and setting each agent's transfer equal to the cost she imposes on the others plus an appropriately chosen amount. By *Pareto-efficiency*, a desirable rule should select *Pareto-efficient* queues and as the costs agents impose on the others are always strictly positive (except for the last agent in the queue), it should redistribute the sum of these costs. By equity, it should select all *Pareto-efficient* queues and it should redistribute this sum fairly. By *strategy-proofness*, it should redistribute this sum in such a way that each agent's share only depends on the others' waiting costs. This is exactly what the Largest Equally Distributed Pairwise Pivotal rule does. It selects all *Pareto-efficient* queues (so it is efficient and fair). It sets each agent's transfer considering each pair of agents in turn, making each agent in the pair pay the cost she imposes on the pair. Then, it distributes the sum of these two payments (so it is efficient) equally (so it is fair) among the others (so it is *strategy-proof*).

Literature Review: Our results provide another example of a situation in which *Pareto-efficiency*, equity axioms such as *equal treatment of equals in welfare* and *symmetry*, and *strategy-proofness* are compatible. For general social choice problems, each equity axiom is incompatible with *strategy-proofness* (Gibbard, 1973 and Satterthwaite, 1975). For the classical problem of distributing of private goods (and even if preferences are homothetic and smooth), *Pareto-efficiency*, *equal treatment of equals*, and *strategy-proofness* are incompatible (Serizawa, 2002). In economies with indivisible goods when monetary compensations are possible, *no-envy* and *strategy-proofness* are incompatible (Alkan, Demange, and Gale, 1991, Tadenuma and Thomson, 1995); moreover, when rules exist that satisfy these axioms on more restricted classes of problems, they violate *Pareto-efficiency*.

There are some exceptions. For the problem of choosing a public good in an interval over which the agents have continuous and single-peaked preferences, *Pareto-efficiency*, *anonymity*, and *strategy-proofness* are compatible (Moulin, 1980). For the problem of distributing an infinitely divisible private good over which the agents have continuous and single-peaked preferences, *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness* are compatible, and so are *Pareto-efficiency*, *anonymity*, and *strategy-proofness* (Sprumont, 1991, Ching, 1994). For the problem of distributing infinitely divisible private goods produced by means of a linear technology, *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness* are compatible (Maniquet and Sprumont, 1999).[1]

The literature on queueing can be organized in two groups of papers. The first group concerns the identification of rules satisfying equity axioms pertaining to changes in the set of agents or in their waiting costs, in addition to the efficiency and equity axioms that we impose too (Maniquet, 2003; Chun, 2004a; Chun, 2004b; Katta and Sethuraman, 2005). Only rules that select *Pareto-efficient* queues and set each agent's transfer in such a way that her welfare is equal to the Shapley value of some associated coalitional game, satisfy these axioms (Maniquet, 200;, Chun, 2004a; Katta and Sethuraman, 2005). However, while there are rules that satisfy *Pareto-efficiency* and *no-envy* (Chun, 2004b; Katta and Sethuraman, 2005), none satisfies the solidarity requirement that if the waiting costs change, then all agents should gain together or lose together (Chun, 2004b). The second group concerns the identification of necessary and sufficient conditions for the existence of rules satisfying *Pareto-efficiency* and *strategy-proofness.* For such problems, like for any public decision-making problem in which agents have additively separable preferences, there are rules that satisfy *efficiency of queues* and *strategy-proofness* (Groves, 1973). Also, like for any public decision-making problem in which preference profiles are convex, only these rules satisfy these properties (Holmström, 1979).[2] However, these rules are not *balanced* (Green and Laffont, 1977). Unless we further restrict the domain, *Pareto-efficiency* and *strategy-proofness* are incompatible. In queueing problems, if preferences are quasi-linear over positions and transfers, there are rules that satisfy *Pareto-efficiency* and *strategy-proofness* (Suijs, 1996, Mitra and Sen, 1998).

In Section 2, we formally introduce the model. In Section 3, we define the axioms on rules. In Section 4, we give the results. In Section 5, we give concluding comments. In Section 6, we provide all proofs.

[1] For an extensive survey on *strategy-proofness*, see Thomson (2006).

[2] In fact, Holmström (1979) shows it that any public decision-making problem in which preference profiles are smoothly connected, i.e., for any profile in the domain, if there is a differentiable deformation of the profile into other then the other profile is also in the domain; only Groves' rules satisfy efficiency of assignment and *strategy-proofness.* This characterization also holds on the universal domain of preferences (Green and Laffont, 1977).

2 Model

There is a finite set of agents N indexed by $i \in N$. Each agent $i \in N$ has to be assigned a *position* $\sigma_i \in \mathbb{N}$ in a queue and may receive a positive or negative monetary *transfer* $t_i \in \mathbb{R}$. Preferences are quasi-linear over $X \equiv \mathbb{N} \times \mathbb{R}$. Let $c_i \in \mathbb{R}_+$ be the *unit waiting cost of* $i \in N$. If i is served σ_i-th, her *total waiting cost* is $(\sigma_i - 1)c_i$. Her preferences can be represented by the function u_i defined as follows: for each $(\sigma_i, t_i) \in X$, $u_i(\sigma_i, t_i) = -(\sigma_i - 1)c_i + t_i$. We use the following notational shortcut. If her waiting cost is c'_i, then her preferences are represented by the function u'_i, defined by $u'_i(\sigma_i, t_i) = -(\sigma_i - 1)c'_i + t_i$; if it is $\tilde{c}_i$, then we use $\tilde{u}_i(\sigma_i, t_i) = -(\sigma_i - 1)\tilde{c}_i + t_i$, and so on. A *queueing problem* is defined as a list $c \equiv (c_i)_{i \in N} \in \mathbb{R}^N_+$. Let $\mathcal{C} \equiv \mathbb{R}^N_+$ be the set of all problems. Let $n = |N|$.

An *allocation for* $c \in \mathcal{C}$ is a pair $(\sigma, t) \equiv (\sigma_i, t_i)_{i \in N} \in X^N$. An allocation $(\sigma, t) \in X^N$ is *feasible for* $c \in \mathcal{C}$ if no two agents are assigned the same position in σ, (i.e., for each $\{i, j\} \subseteq N$ with $i \neq j$, we have $\sigma_i \neq \sigma_j$), and the sum of the coordinates of t is non-positive, (i.e., $\sum_{i \in N} t_i \leq 0$). Let $Z(N)$ be the set of all feasible allocations for $c \in \mathcal{C}$. An (allocation) *rule* φ is a correspondence that associates with each problem $c \in \mathcal{C}$ a non-empty set of feasible allocations $\varphi(c) \subseteq Z(N)$.

Given $c \in \mathcal{C}$ and $S \subseteq N$, $c_S \equiv (c_l)_{l \in S}$ is the restriction of c to S. Given $i \in N$, $c_{-i} \equiv (c_l)_{l \in N \backslash \{i\}}$ is the restriction of c to $N \backslash \{i\}$. Let $(\sigma, t) \in Z(N)$. Given $i \in N$, let $P_i(\sigma) \equiv \{j \in N | \sigma_j < \sigma_i\}$ be the set of agents served before i in σ, (the predecessors), and $F_i(\sigma) \equiv \{j \in N | \sigma_j > \sigma_i\}$ the set of agents served after i in σ, (the followers). Given $\{i, j\} \subseteq N$, let $B_{ij}(\sigma) \equiv \{l \in N | \min\{\sigma_i, \sigma_j\} < \sigma_l < \max\{\sigma_i, \sigma_j\}\}$ be the set of agents served *between* i and j in σ.[3] Given $S \subseteq N$, the *total waiting cost of* S is $\sum_{i \in S}(\sigma_i - 1)c_i$. Given $i \in N$, let σ^{-i} be such that for each $l \in P_i(\sigma)$, we have $\sigma^{-i}_l = \sigma_l$ and for each $l \in F_i(\sigma)$, we have $\sigma^{-i}_l = \sigma_l - 1$. Given $i \in N$, and $S \subseteq N$, the *cost that agent* i *imposes on* S is $\sum_{l \in S \cap F_i(\sigma)} c_l$. Thus, the cost an agent imposes on society is always equal to the sum of the unit waiting costs of her followers in an efficient queue.

3 Properties of rules

In this section, we define properties of rules. Let φ be a rule. First, if an allocation is selected, there should be no other feasible allocation that each agent finds at least as desirable and at least

[3] For each $c \in \mathcal{C}$, each $(\sigma, t) \in Z(N)$, and each $\{i, j\} \subseteq N$, we have $B_{ij}(\sigma) = B_{ji}(\sigma)$.

one agent prefers.

Pareto-efficiency: *For each* $c \in \mathcal{C}$ *and each* $(\sigma, t) \in \varphi(c)$*, if there is no* $(\sigma', t') \in Z(N)$ *such that for each* $i \in N$*,* $u_i(\sigma'_i, t'_i) \geq u_i(\sigma_i, t_i)$ *and for at least one* $j \in N$*, we have* $u_j(\sigma'_j, t'_j) > u_j(\sigma_j, t_j)$*.*

Consider a *Pareto-efficient* allocation for c, any other allocation at which the queue is the same is also *Pareto-efficient*. Therefore, it is meaningful to define *efficiency of queues*. It requires to minimize the total waiting cost. Thus, an allocation (σ, t) is *Pareto-efficient for c* if and only if for each $\sigma' \in \mathbb{N}^N$, we have $\sum_{i \in N}(\sigma'_i - 1)c_i \geq \sum_{i \in N}(\sigma_i - 1)c_i$, i.e., σ is *efficient for c* and $\sum_{i \in N} t_i = 0$, i.e., t is *balanced for c*. Let $Q^*(c)$ be the set of all *efficient* queues for c. For each $c \in \mathcal{C}$ and each $(\sigma, t) \in Z(N)$, we have $\sigma \in Q^*(c)$ if and only if for each $\{i, j\} \subset N$ with $i \neq j$, if $\sigma_i < \sigma_j$, then $c_i \geq c_j$. Thus, up to permutation of agents with equal unit waiting costs, there is only one *efficient* queue.

Summarizing the discussion above, *Pareto-efficiency* can be decomposed into two axioms:

Efficiency of queues: *For each* $c \in \mathcal{C}$ *and each* $(\sigma, t) \in \varphi(c)$*, we have* $\sigma \in Q^*(c)$*.*

Balancedness: *For each* $c \in \mathcal{C}$ *and each* $(\sigma, t) \in \varphi(c)$*, we have* $\sum_{i \in N} t_i = 0$*.*

Equity requires to treat agents with equal unit waiting costs equally. We require that equal agents should have equal welfare.

Equal treatment of equals in welfare: *For each* $c \in \mathcal{C}$*, each* $(\sigma, t) \in \varphi(c)$*, and each* $\{i, j\} \subset N$ *with* $i \neq j$ *and* $c_i = c_j$*, we have* $u_i(\sigma_i, t_i) = u_j(\sigma_j, t_j)$*.*

This requirement is necessary for no agent to prefer another agent's assignment to her own.

No-envy: *For each* $c \in \mathcal{C}$*, each* $(\sigma, t) \in \varphi(c)$*, and each* $i \in N$*, there is no* $j \in N \backslash \{i\}$ *such that* $u_i(\sigma_j, t_j) > u_i(\sigma_i, t_i)$*.*

The last requirements are motivated by strategic considerations. The planner may not know the agents' cost parameters. If agents behave strategically when announcing them, neither efficiency nor equity may be attained. Thus, we require that each agent should find her assignment when she truthfully reveals her unit waiting cost at least as desirable as her assignment when she misrepresents it.

Strategy-proofness: *For each* $c \in \mathcal{C}$*, each* $i \in N$*, and each* $c_i' \in \mathbb{R}_+$*, if* $(\sigma, t) = \varphi(c)$ *and* $(\sigma', t') = \varphi(c_i', c_{-i})$*, then* $u_i(\sigma_i, t_i) \geq u_i(\sigma_i', t_i')$*.*

We also consider the requirement that no group of agents should be able to make each of its members at least as well off, and at least one of them better off, by jointly misrepresenting its members waiting costs.

Coalitional strategy-proofness: *For each* $c \in \mathcal{C}$ *and each* $S \subseteq N$*, there is no* $c_S' \in \mathbb{R}_+^S$ *such that if* $(\sigma, t) = \varphi(c)$ *and* $(\sigma', t') = \varphi(c_S', c_{N \setminus S})$*, then for each* $i \in S$*, we have* $u_i(\sigma_i', t_i') \geq u_i(\sigma_i, t_i)$ *and for some* $j \in S$*, we have* $u_j(\sigma_j', t_j') > u_j(\sigma_j, t_j)$*.*

The next requirement is that if an agent's change in her announcement does not affect her assignment, then it should not affect any other agent's assignment.

Non-bossiness: *For each* $c \in \mathcal{C}$*, each* $i \in N$*, and each* $c_i' \in \mathbb{R}_+$*, if* $\varphi_i(c) = \varphi_i(c_i', c_{-i})$*, then* $\varphi(c) = \varphi(c_i', c_{-i})$*.*

4 Results

In this section, we first characterize the class of *single-valued* rules that satisfy *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness*. We then show that these rules in fact satisfy the stronger fairness property of *no-envy* (Theorem 2). Then, we extend these results to multi-valued rules and we prove that there is a unique rule that satisfies *Pareto-efficiency*, *symmetry*, and *strategy-proofness*. Also, this rule satisfies *anonymity* (Theorem 4).[4]

4.1 Single-valued rules

We first prove that a *single-valued* rule satisfies *efficiency of queues* and *strategy-proofness* if and only if for each problem, it selects an *efficient* queue (of course) and sets each agent's transfer as prescribed in Groves (1973), i.e., equal to the total waiting cost of all other agents plus an amount only depending on these agents' unit waiting costs (Theorem 1). As the domain of preference

[4]By extending Theorem 2 to multi-valued rules, we prove that what holds in the special case of *single-valued* rules still holds in the general case of single- and multi-valued rules. Thus, single-valuedness and Theorem 4 imply Theorem 2.

profiles is convex, it is smoothly connected. Thus, this result follows from Holmström's (1979). However, we are able to give a simpler proof by exploiting the special features of our model. Formally, let $D \equiv \{d|$ for each $c \in \mathcal{C}$, we have $d(c) \in Q^*(c)\}$. Let $H \equiv \{(h_i)_{i\in N}|$ for each $i \in N$, we have $h_i : \mathbb{R}_+^{N\setminus\{i\}} \to \mathbb{R}\}$. A *single-valued* rule φ is a *Groves' rule* if and only if there are $d \in D$ and $h \in H$ such that for each $c \in \mathcal{C}$, $\varphi(c) = (\sigma, t) \in Z(N)$ with $\sigma = d(c)$, and for each $i \in N$, $t_i = -\sum_{l\in N\setminus\{i\}}(\sigma_l - 1)c_l + h_i(c_{-i})$.

Theorem 1. *A* single-valued *rule is a Groves rule if and only if it satisfies* efficiency of queues *and* strategy-proofness.

The class of Groves' rules is large. We distinguish subclasses according to their h function. For instance, the *Pivotal rules* are the Groves' rules associated with $h \in H$ such that for each $c \in \mathcal{C}$, for each $i \in N$, $h_i(c_{-i}) = \sum_{l\in N\setminus\{i\}}(\sigma_l^{-i} - 1)c_l$.[5] By Theorem 1, a *single-valued* rule satisfies *Pareto-efficiency* and *strategy-proofness* if and only if it is a Groves rule and it is *balanced*. However, for two-agent problems, no Groves rule is *balanced* (Suijs, 1996). From now on, we focus on problems with more than two agents.

We now introduce another class of *single-valued* rules. A rule in this class selects for each problem a *Pareto-efficient* queue and sets transfers considering each pair of agents in turn, making each agent in the pair pay what a Pivotal rule recommends for the subproblem consisting of these two agents, and distributing the sum of these two payments equally among the others. Thus, for each problem and each selected queue, each agent's transfer is such that she pays the cost she imposes on the other agent and she receives $\frac{1}{n-2}$-th of the cost each agent imposes on the other agent in the pair that she is not part of.

Equally Distributed Pairwise Pivotal rule, φ^*: For each $c \in \mathcal{C}$, if $(\sigma, t) = \varphi^*(c)$, then $\sigma \in Q^*(c)$ and for each $i \in N$, we have
$t_i = -\sum_{j\in N\setminus\{i\}} \sum_{l\in\{i,j\}\cap F_i(\sigma)} c_l + \frac{1}{(n-2)} \sum_{j\in N\setminus\{i\}} \sum_{k\in N\setminus\{i,j\}} \sum_{l\in\{j,k\}\cap F_j(\sigma)} c_l.$

An example of a problem illustrating the rule:

Let $N = \{1, 2, 3, 4\}$ and $c \in \mathbb{R}_+^N$ such that $c_1 > c_2 > c_3 > c_4$. The *efficiency of queues* implies that agents should be served in decreasing order of their waiting costs. Thus, the efficient queue is

[5] Pivotal rules are also known as *Clarke's rules* (Clarke, 1971).

$\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (1, 2, 3, 4)$. Then, consider each pair of agents, and make each agent in the pair pay the cost that the agent imposes on the other agent. Then, distribute the sum of these two payments equally among the others. The following table shows how payments are calculated. For example, for the pair $\{2, 4\}$, by *Pareto-efficiency* agent 2 should be served before agent 4. The cost agent 2 imposes on agent 4 is c_4 but agent 4 does not impose any cost on agent 2. So, agent 2 pays c_4 and agent 4 pays nothing. The amount collected in total is distributed among agents 1 and 3 equally: each of them receives $c_4/2$.

	1	2	3	4
12	$-c_2$	0	$c_2/2$	$c_2/2$
13	$-c_3$	$c_3/2$	0	$c_3/2$
14	$-c_4$	$c_4/2$	$c_4/2$	0
23	$c_3/2$	$-c_3$	0	$c_3/2$
24	$c_4/2$	$-c_4$	$c_4/2$	0
34	$c_4/2$	$c_4/2$	$-c_4$	0

The final monetary consumption is the sum of all the transfers for each possible pair. Then, $t = (t_1, t_2, t_3, t_4) = (-c_2 - c_3/2, -c_3/2, c_2/2, c_2/2 + c_3)$. The allocation selected by the *Equally Distributed Pairwise Pivotal rule* is *Pareto-efficient.* The rule satisfies *equal treatment of equals*, *no-envy*, and *strategy-proofness.*

As there may be several *Pareto-efficient* queues for a problem, there are several Equally Distributed Pairwise Pivotal rules. Proposition 1 states that for each problem and each *Pareto-efficient* queue, the transfers set by any Equally Distributed Pairwise Pivotal rule can be obtained in three other ways. First, making each agent pay what the Pivotal rule recommends for the problem, giving each agent $\frac{1}{n-2}$-th of what the others pay. Second, giving each agent $\frac{1}{n-2}$-th of her predecessors' total waiting cost and making each agent pay $\frac{1}{n-2}$-th of her followers' gain from not being last (Mitra and Sen, 1998, Mitra, 2001). Third, giving each agent one half of her predecessors' unit waiting cost and making each agent pay one half of her followers' unit waiting cost plus $\frac{1}{2(n-2)}$-th of the difference between two unit waiting costs of any other agent and this agent's predecessors's (Suijs, 1996).

Proposition 1. *Let* φ *be a* single-valued *rule. Then, the following statements are equivalent.*

1. φ *is an Equally Distributed Pairwise Pivotal rule.*

2. φ *is a Groves rule associated with* $h \in H$ *such that for each* $c \in \mathcal{C}$*, if* $(\sigma, t) = \varphi(c)$*, then for each* $i \in N$*,* $h_i(c_{-i}) = \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1)c_l + \frac{1}{(n-2)} \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1)c_l$*.*

3. φ *is such that for each* $c \in \mathcal{C}$*, if* $(\sigma, t) = \varphi(c)$*, then* $\sigma \in Q^*(c)$ *and for each* $i \in N$*,* $t_i = \sum_{l \in P_i(\sigma)} \frac{(\sigma_l - 1)}{(n-2)} c_l - \sum_{l \in F_i(\sigma)} \frac{(n - \sigma_l)}{(n-2)} c_l$*.*

4. φ *is such that for each* $c \in \mathcal{C}$*, if* $(\sigma, t) = \varphi(c)$*, then* $\sigma \in Q^*(c)$ *and for each* $i \in N$*,* $t_i = \sum_{l \in P_i(\sigma)} \frac{c_l}{2} - \sum_{l \in F_i(\sigma)} \frac{c_l}{2} - \sum_{l \in N \setminus \{i\}} \sum_{k \in P_l(\sigma) \setminus \{i\}} \frac{c_k - c_l}{2(n-2)}$ *.*

Next, we prove that requiring *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness* implies choosing an Equally Distributed Pairwise Pivotal rule.

Theorem 2. *A* single-valued *rule satisfies* Pareto-efficiency*,* equal treatment of equals in welfare*, and* strategy-proofness *if and only if it is an Equally Distributed Pairwise Pivotal rule.*

The following paragraphs establish the independence of the axioms in Theorem 2.

(i) Let φ be a rule such that for each $c \in \mathcal{C}$, if $(\sigma, t) = \varphi(c)$, then $\sigma \in Q^*(c)$. Let $i \in N$ if $\sigma_i \neq 1$ and for each $\{j, k\} \subset N$ are such that $\sigma_j = \sigma_i - 1$ and $\sigma_k = \sigma_i + 1$, then $\alpha_i \in [c_j, c_k]$ and $t_i = \sum_{l \in P_i(\sigma) \cup \{i\}} \alpha_l$, and if $\sigma_i = 1$, then $t_i = \alpha_i$ where in each case $\alpha \in \mathbb{R}^N$ is chosen so as to achieve $\sum_{l \in N} t_l = 0$. Any such rule satisfies all the axioms of Theorem 2 but *strategy-proofness* (Chun, 2004b).

(ii) Let φ be a Groves rule associated with $h \in H$ such that for each $c \in \mathcal{C}$ and let $\lambda \in \mathbb{R}$ be such that $\lambda \neq 0$ and $h_1 = \sum_{l \in N \setminus \{1\}} (\sigma_l^{-1} - 1)c_l + \frac{1}{n-2} \sum_{l \in N \setminus \{1\}} (\sigma_l^{-1} - 1)c_l + \lambda$, and for each $i \in N \setminus \{1\}$, we have $h_i = \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1)c_i + \frac{1}{n-2} \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1)c_l - \frac{\lambda}{(n-1)}$. Any such rule satisfies all the axioms of Theorem 2 but *equal treatment of equals in welfare*.

(iii) Let φ be a Groves rule associated with $h \in H$ such that $c \in \mathcal{C}$ and let $\lambda \in \mathbb{R}_+$ be such that for each $i \in N$, $h_i = \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1)c_i + \frac{1}{n-2} \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1)c_l - \lambda$ satisfies all axioms but *Pareto-efficiency*.

Remark 1. *Equally Distributed Pairwise Pivotal rules satisfy* no-envy*. Thus, as* no-envy *implies* equal treatment of equals in welfare*, we prove that only* single-valued *Equally Distributed Pairwise Pivotal rules satisfy* Pareto-efficiency*,* equal treatment of equals in welfare*, and* strategy-proofness*.*

Next, we show that if we impose the stronger condition of *coalitional strategy-proofness*, even with *efficiency of queues*, then we have a negative result.

Theorem 3. *No rule satisfies* efficiency of queues *and* coalitional strategy-proofness*.*

The following paragraphs establish examples of rules that satisfy only one of the axioms in Theorem 3.

(i) Equally Distributed Pairwise Pivotal rule satisfies *efficiency of queues* but not *coalitional strategy-proofness.*

(ii) Any rule that selects the same arbitrary queue and sets the transfer to each agent equal to zero satisfies *coalitional strategy-proofness*, but not *efficiency of queues.*

Theorem 3 implies that no rule satisfies *Pareto-efficiency* and *coalitional strategy-proofness.* It also implies that no rule satisfies *Pareto-efficiency*, *non-bossiness*, and *strategy-proofness.* Since *no-envy* implies *Pareto-efficiency of queues*[6], it follows that no rule satisfies *no-envy*, *non-bossiness*, and *strategy-proofness.*

4.2 Multi-valued rules

Let Φ be a rule. When we allow multi-valuedness of rules, it is possible to give two agents with equal unit waiting costs equal assignments. We require that this be the case: If there is another allocation at which two agents exchange their assignments and the other agents keep theirs, then this allocation should be selected.

Symmetry: *For each $c \in \mathcal{C}$, each $(\sigma, t) \in \Phi(c)$, and each $\{i, j\} \subset N$ with $i \neq j$ and $c_i = c_j$, if $(\sigma', t') \in Z(N)$ such that $(\sigma'_i, t'_i) = (\sigma_j, t_j)$, $(\sigma'_j, t'_j) = (\sigma_i, t_i)$, and for each $l \in N \backslash \{i, j\}$, we have $(\sigma'_l, t'_l) = (\sigma_l, t_l)$, then $(\sigma', t') \in \Phi(c)$.*

[6]Assume that φ satisfies *no-envy.* Let $c \in \mathcal{C}$, $(\sigma, t) = \varphi(c)$, $\{i, j\} \subset N$, with $i \neq j$ be such that $c_i > c_j$ but $\sigma_i > \sigma_j$. By *no-envy*, we have $u_i(\sigma_i, t_i) \geq u_i(\sigma_j, t_j)$ and $u_j(\sigma_j, t_j) \geq u_j(\sigma_i, t_i)$. Then, $(\sigma_i - \sigma_j)c_i + t_j \leq t_i \leq (\sigma_i - \sigma_j)c_i$ that contradicts $c_i > c_j$.

This second requirement is that if we permute agents' unit waiting costs, we should permute the selected assignments accordingly. Formally, let Π be the set of all permutations on N. For each $\pi \in \Pi$ and each $c \in \mathbb{R}_+^N$, let $\pi(c) \equiv (c_{\pi(i)})_{i \in N}$ and $\pi(\sigma, t) \equiv (\sigma_{\pi(i)}, t_{\pi(i)})_{i \in N}$.

Anonymity: *For each $c \in \mathcal{C}$, each $(\sigma, t) \in \Phi(c)$, and each $\pi \in \Pi$, we have $\pi(\sigma, t) \in \Phi(\pi(c))$.*

Single- and multi-valued rules may satisfy *equal treatment of equals in welfare.* However, only multi-valued rules may satisfy *symmetry.* Indeed, *symmetry* is necessary for agents' names not to matter. Thus, the presence of indivisibilities implies that we may require *anonymity* of rules only if we allow multi-valuedness.

For multi-valued rules, *strategy-proofness* has to be redefined. To compare the welfare levels derived from two sets of feasible allocations, we assume that an agent prefers the former to the latter if and only if for each allocation in the latter, there is an allocation in the former that she finds at least as desirable; and for each allocation in the former, there is an allocation in the latter that she does not prefer.[7] Formally, let $\mathcal{X}_i$ be the set of positions and transfers in $\mathbb{N} \times \mathbb{R}$. Given $c_i \in \mathbb{R}_+$, let $R_i(c_i)$ be the preference relation on subsets $\mathcal{X}_i$ defined as follows: for each $\{X_i, X'_i\} \subseteq \mathcal{X}_i$, we have $X_i \; R_i(c_i) \; X'_i$ if and only if $\min_{(\sigma_i,t_i) \in X_i} u_i(\sigma_i, t_i) \geq \min_{(\sigma'_i,t'_i) \in X'_i} u_i(\sigma'_i, t'_i)$ and $\max_{(\sigma_i,t_i) \in X_i} u_i(\sigma_i, t_i) \geq \max_{(\sigma'_i,t'_i) \in X'_i} u_i(\sigma'_i, t'_i)$. Let $\mathcal{Z}$ be the set of all non-empty subsets of $Z(N)$. For each $Z \in \mathcal{Z}$, and each $i \in N$, let $Z_i \equiv \bigcup_{(\sigma,t) \in Z} (\sigma_i, t_i)$.

Strategy-proofness: *For each $c \in \mathcal{C}$, each $i \in N$, and each $c'_i \in \mathbb{R}_+$, if $Z = \Phi(c)$ and $Z' = \Phi(c'_i, c_{-i})$, then $Z_i \; R_i(c_i) \; Z'_i$.*

Thus, as in Pattanaik (1973), Dutta (1977), and Thomson (1979), *strategy-proofness* requires each agent to find the worst assignment she may receive when she reveals her unit waiting cost at least as desirable as the worst assignment she may receive when she misrepresents it. Furthermore, it requires each agent to find the best assignment she may receive when she reveals her unit waiting cost at least as desirable as the best assignment she may receive when she misrepresents it. The second requirement is also implied by further basic incentive compatibility requirements.

[7]Determining how agents rank non-empty sets given their preferences over singletons has been studied in, e.g., Pattanaik (1973), Barberá (1977), Dutta (1977), Kelly (1977), Feldman (1979, 1980), Gärdenfors (1979), Thomson (1979), Ching and Zhou (2000), Duggan and Schwartz (2000), and Barberá, Dutta, and Sen (2001).

In particular, it is a necessary condition for implementation in undominated strategies by bounded mechanisms (Jackson, 1992, Ching and Zhou, 2002).[8]

Next, we show that a unique rule satisfies *Pareto-efficiency*, *symmetry*, and *strategy-proofness* and identify the rule. Moreover, this rule satisfies *no-envy* and *anonymity*. For each problem, the rule selects all *Pareto-efficient* queues and for each queue, it sets transfers as in the Equally Distributed Pairwise Pivotal rule. Thus, it is the union of the desirable rules introduced in the previous subsection. Formally,

The Largest Equally Distributed Pairwise Pivotal rule, Φ^*: For each $c \in \mathcal{C}$, we have $(\sigma, t) \in \Phi^*(c)$ if and only if $\sigma \in Q^*(c)$ and for each $i \in N$, we have $t_i = -\sum_{j \in N\setminus\{i\}} \sum_{l \in \{i,j\}\cap F_i(\sigma)} c_l + \frac{1}{(n-2)} \sum_{j \in N\setminus\{i\}} \sum_{k \in N\setminus\{i,j\}} \sum_{l \in \{j,k\}\cap F_j(\sigma)} c_l$.

For each problem and each *Pareto-efficient* queue, the transfers set by the Largest Equally Distributed Pairwise Pivotal rule can be obtained as the transfers set by any rule described in Proposition 1. Thus, for each $c \in \mathcal{C}$, each $(\sigma, t) \in \Phi^*(c)$, and each $i \in N$, we have

$$t_i = -\sum_{l \in N\setminus\{i\}} (\sigma_l - 1) c_l + \sum_{l \in N\setminus\{i\}} (\sigma_l^{-i} - 1) c_l + \frac{1}{(n-2)} \sum_{l \in N\setminus\{i\}} (\sigma_l^{-i} - 1) c_l$$
$$= \sum_{l \in P_i(\sigma)} \frac{(\sigma_l - 1)}{(n-2)} c_l - \sum_{l \in F_i(\sigma)} \frac{(n - \sigma_l)}{(n-2)} c_l$$
$$= \sum_{l \in P_i(\sigma)} \frac{c_l}{2} - \sum_{l \in F_i(\sigma)} \frac{c_l}{2} - \sum_{l \in N\setminus\{i\}} \sum_{k \in P_l(\sigma)\setminus\{i\}} \frac{c_k - c_l}{2(n-2)}.$$

Furthermore, Theorem 4 states that only subcorrespondences of the Largest Equally Distributed Pairwise Pivotal rule can satisfy *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness*. They also satisfy *no-envy*. Then, we prove that the Largest Equally Distributed Pairwise Pivotal rule is the only rule that satisfies *Pareto-efficiency*, *symmetry*, and *strategy-proofness*.

Theorem 4.

1. *A rule satisfies* Pareto-efficiency, equal treatment of equals in welfare*, and* strategy-proofness *if and only if it is a subcorrespondence of the Largest Equally Distributed Pairwise Pivotal rule.*

[8] Formally, an agent does not find misrepresenting her unit waiting cost more desirable as revealing it if there is no $c \in \mathcal{C}$, each $i \in N$, and each $c_i' \in \mathbb{R}_+$ such that for $(\sigma', t') \in \Phi(c_i', c_{-i})\setminus\Phi(c)$, we have $u_i(\sigma_i', t_i') > \min_{(\sigma,t)\in\Phi(c)} u_i(\sigma_i, t_i)$ or for $(\sigma, t) \in \Phi(c)\setminus\Phi(c_i', c_{-i})$, we have $max_{(\sigma',t')\in\Phi(c_i',c_{-i})} u_i(\sigma_i', t_i') < u_i(\sigma_i, t_i)$.

2. *A rule satisfies* Pareto-efficiency*,* symmetry*, and* strategy-proofness *if and only if it is the Largest Equally Distributed Pairwise Pivotal rule.*

The following paragraphs establish the independence of axioms in the second statement in Theorem 4.

(i) Consider a rule that selects all *Pareto-efficient* queues and sets each agent's transfer equal to the Shapley value of the associated coalitional game, where the worth of a coalition is the minimum possible sum of its members waiting costs (Maniquet, 2003). Such a rule satisfies all the axioms of the second statement of Theorem 4 but *strategy-proofness*.

(ii) Consider any proper subcorrespondence of a rule that is the union of all the *single-valued* rules that are Groves' rules associated with $h \in H$ and satisfy *balancedness*. Such a rule satisfies all the axioms of the second statement of Theorem 4 but *symmetry*.

(iii) Consider a rule such that for each queueing problem and each $\lambda \in \mathbb{R}_+$, it selects a fixed queue and sets each agent's transfer equal to $-\lambda$. Such a rule satisfies all the axioms of the second statement of Theorem 4 but *Pareto-efficiency*.

Remark 2. *The Largest Equally Distributed Pairwise Pivotal rule also satisfies* anonymity*. Since* anonymity *implies* symmetry *and the Largest Equally Distributed Pairwise Pivotal rule is the union of all the rules that satisfy* Pareto-efficiency*,* equal treatment of equals in welfare*, and* strategy-proofness*, it follows that this rule is the only rule that satisfies* Pareto-efficiency*,* equal treatment of equals in welfare*,* symmetry*, and* strategy-proofness*.*

5 Concluding comments

Our objective was to identify allocation rules for queueing problems that satisfy efficiency, equity, and incentive requirements simultaneously on the domain of quasi-linear preferences in positions and transfers. We proved that the Largest Equally Distributed Pairwise Pivotal rule is the only such rule. It is the only rule, together with any of its subcorrespondences, that satisfies *Pareto-efficiency*, *equal treatment of equals in welfare*, and *strategy-proofness*. It is the only rule that satisfies *Pareto-efficiency*, *symmetry*, and *strategy-proofness*. As any of it subcorrespondences, it satisfies *no-envy*. Furthermore, it satisfies *anonymity*.

We draw three lessons from these results. First and foremost, queueing problems are among the few problems in which *Pareto-efficiency*, a weak equity axiom such as *equal treatment of equals in welfare* or *symmetry*, and *strategy-proofness*, are compatible. The natural next step is to determine if this compatibility extends to other queueing problems, in particular to ones in which agents have different processing times. However, in queueing problems in which waiting costs vary non-linearly across positions, no rule satisfies *Pareto-efficiency* and *strategy-proofness* (Mitra, 2002).

Second, while *Pareto-efficiency* and *strategy-proofness* leave us with a large class of *single-valued* rules, adding as weak equity axiom as *equal treatment of equals in welfare* imposes a unique way of setting transfers. The open question is to determine the class of multi-valued rules that satisfy *Pareto-efficiency* and *strategy-proofness.*

Finally, in the queueing problems we studied, simply requiring *equal treatment of equals* in addition to *Pareto-efficiency* and *strategy-proofness*, guarantees further basic fairness requirements. First, it prevents agents from envying one another. In allocation problems of private goods, *equal treatment of equals in welfare* and *coalition strategy-proofness* together imply *no-envy* (Moulin, 1993). In general public decision-making problems in which the domain of preferences is strictly monotonically closed, *equal treatment of equals in welfare*, *strategy-proofness*, and *non-bossiness* together imply *no-envy* (Fleurbaey and Maniquet, 1997). However, these results do not apply to the problems we studied. Indeed, here no rule satisfies *Pareto-efficiency* and *coalition strategy-proofness.* Also, as preferences are quasi-linear in positions and transfers, they are not monotonically closed. In fact, no rule satisfies *Pareto-efficiency*, *non-bossiness*, and *strategy-proofness.* Second, it guarantees that agents' names do not matter. Finally, it guarantees each agent a minimal welfare level. Indeed, in allocation problems of at most one indivisible private good per agent, *no-envy* implies the *identical-preferences lower bound*, i.e., each agent should find her assignment at least as desirable as any assignment recommended by *Pareto-efficiency* and *equal treatment of equals in welfare* when the other agents have her preferences (Bevia, 1996).

6 Appendix

Proof of Theorem 1.

Let φ be a *single-valued* rule. Then,

If part:

Efficiency of queues: Let φ be a Groves rule. Let $c \in \mathcal{C}$ and $(\sigma, t) = \varphi(c)$. Then, by definition of a Groves rule, there is $d \in D$ such that $\sigma = d(c) \in Q^*(c)$.

Strategy-proofness: Let φ be a Groves rule. Let $c \in \mathcal{C}$, $i \in N$, $c'_i \in \mathbb{R}_+$, $(\sigma, t) = \varphi(c)$, and $(\sigma', t') = \varphi(c'_i, c_{-i})$. By definition of a Groves rule, there is $d \in D$ such that $\sigma = d(c) \in Q^*(c)$. Also, there is $h \in H$ such that $t_i = -\sum_{l \in N \setminus \{i\}} (\sigma_l - 1) c_l + h_i(c_{-i})$ and $t'_i = -\sum_{l \in N \setminus \{i\}} (\sigma'_l - 1) c_l + h_i(c_{-i})$. By contradiction, suppose $u_i(\sigma'_i, t'_i) > u_i(\sigma_i, t_i)$. Then, $-(\sigma'_i - 1)c_i - \sum_{l \in N \setminus \{i\}} (\sigma'_l - 1) c_l + h_i(c_{-i}) > -(\sigma_i - 1)c_i - \sum_{l \in N \setminus \{i\}} (\sigma_l - 1) c_l + h_i(c_{-i})$. Thus, $-\sum_{l \in N} (\sigma'_l - 1) c_l > -\sum_{l \in N} (\sigma_l - 1) c_l$, contradicting $\sigma \in Q^*(c)$.

Only if part: Let φ be a rule satisfying *efficiency of queues* and *strategy-proofness*. Then, by *efficiency of queues*, for each $c \in \mathcal{C}$, if $(\sigma, t) = \varphi(c)$, then $\sigma \in Q^*(c)$. Thus, there is $d \in D$ such that for each $c \in \mathcal{C}$, if $(\sigma, t) = \varphi(c)$, then $\sigma = d(c)$. In what follows, we prove that there is $h \in H$ such that for each $c \in \mathcal{C}$, if $(\sigma, t) = \varphi(c)$, then for each $i \in N$, $t_i = -\sum_{l \in N \setminus \{i\}} (\sigma_l - 1) c_l + h_i(c_{-i})$.

Let $c \in \mathcal{C}$, $i \in N$, and $g_i : \mathbb{R}^N_+ \to \mathbb{R}$ be a real-valued function such that ***(i)*** if $(\sigma, t) = \varphi(c)$, then $t_i = -\sum_{l \in N \setminus \{i\}} (\sigma_l - 1) c_l + g_i(c)$. By contradiction, suppose that $c'_i \in \mathbb{R}_+$, we have ***(ii)*** $g_i(c) - g_i(c'_i, c_{-i}) > 0$. (The symmetric case is immediate.) Let $(\sigma, t) = \varphi(c)$ and $(\sigma', t') = \varphi(c'_i, c_{-i})$. By *strategy-proofness*, the following inequalities hold:

- $u_i(\sigma_i, t_i) - u_i(\sigma'_i, t'_i) \geq 0$.
- $u'_i(\sigma'_i, t'_i) - u'_i(\sigma_i, t_i) \geq 0$.

By ***(i)***,
$[-(\sigma_i - 1)c_i - \sum_{l \in N \setminus \{i\}} (\sigma_l - 1) c_l + g_i(c)] - [-(\sigma'_i - 1)c_i - \sum_{l \in N \setminus \{i\}} (\sigma'_l - 1) c_l + g_i(c'_i, c_{-i})] \geq 0$.
Thus, $g_i(c) - g_i(c'_i, c_{-i}) \geq (\sigma_i - \sigma'_i) c_i + \sum_{l \in N \setminus \{i\}} (\sigma_l - \sigma'_l) c_l$.
By ***(i)***,
$[-(\sigma'_i - 1)c'_i - \sum_{l \in N \setminus \{i\}} (\sigma'_l - 1) c_l + g_i(c'_i, c_{-i})] - [-(\sigma_i - 1)c'_i - \sum_{l \in N \setminus \{i\}} (\sigma_l - 1) c_l + g_i(c)] \geq 0$.
Thus, $g_i(c'_i, c_{-i}) - g_i(c) \geq (\sigma'_i - \sigma_i) c'_i + \sum_{l \in N \setminus \{i\}} (\sigma'_l - \sigma_l) c_l$.
Altogether,
(iii) $(\sigma_i - \sigma'_i) c'_i + \sum_{l \in N \setminus \{i\}} (\sigma_l - \sigma'_l) c_l \geq g_i(c) - g_i(c'_i, c_{-i}) \geq (\sigma_i - \sigma'_i) c_i + \sum_{l \in N \setminus \{i\}} (\sigma_l - \sigma'_l) c_l$.
Let us rewrite this expression. By *efficiency of queues*, for each $S \subseteq N$, if for each $\{k, k'\} \subseteq S$ with $k \neq k'$, we have $c_k = c_{k'}$ and there is no $k'' \in N \setminus S$ such that $k'' \in B_{kk'}(\sigma) \cup B_{kk'}(\sigma')$,

then $-\sum_{l\in S}(\sigma_l - 1)c_l = -\sum_{l\in S}(\sigma'_l - 1)c_l$. Also, there is $j \in N$ such that $\sigma_j = \sigma'_i$. Thus, $\sum_{l\in N\setminus\{i\}}(\sigma_l - \sigma'_l)c_l = -sign(\sigma_i - \sigma'_i)\sum_{l\in B_{ij}(\sigma)\cup\{j\}} c_l$.[9] Thus, we may rewrite ***(iii)*** as

(iv) $(\sigma_i - \sigma'_i)c'_i - sign(\sigma_i - \sigma'_i)\sum_{l\in B_{ij}(\sigma)\cup\{j\}} c_l \geq g_i(c) - g_i(c'_i, c_{-i}) \geq (\sigma_i - \sigma'_i)c_i - sign(\sigma_i - \sigma'_i)\sum_{l\in B_{ij}(\sigma)\cup\{j\}} c_l$.

Then, we distinguish three cases:

Case 1: $(\sigma_i - \sigma'_i) = 0$. Then, $-sign(\sigma_i - \sigma'_i)\sum_{l\in B_{ij}(\sigma)\cup\{j\}} c_l = 0$. Thus, by ***(iv)***, $g_i(c) - g_i(c'_i, c_{-i}) = 0$ contradicting ***(ii)***.

Case 2: $|\sigma_i - \sigma'_i| = 1$. Suppose $c'_i > c_i$. (The symmetric case is immediate.) Then, $(\sigma_i - \sigma'_i) = 1$ and $-sign(\sigma_i - \sigma'_i)\sum_{l\in B_{ij}(\sigma)\cup\{j\}} c_l = -c_j$. Thus, by ***(iv)***, $c'_i - c_j \geq g_i(c) - g_i(c'_i, c_{-i}) \geq c_i - c_j$. Thus, as $c'_i > c_i$, either $c'_i - c_j > g_i(c) - g_i(c'_i, c_{-i})$ or $g_i(c) - g_i(c'_i, c_{-i}) > c_i - c_j$. Suppose $g_i(c) - g_i(c'_i, c_{-i}) > c_i - c_j$. (The other case is also immediate.) Let $c''_i \in \mathbb{R}_+$ be such that ***(v)***$g_i(c) - g_i(c'_i, c_{-i}) > c''_i - c_j > 0$. Let $(\sigma'', t'') = \varphi(c''_i, c_{-i})$. By ***(iv)*** and ***(v)***, $c'_i > c''_i > c_j > c_i$. Thus, by *efficiency of queues*, $\sigma''_i = \sigma'_i$. Thus, $(\sigma_i - \sigma''_i) = (\sigma_i - \sigma'_i) = 1$ and $\sum_{l\in N\setminus\{i\}}(\sigma_l - \sigma''_l)c_l = \sum_{l\in N\setminus\{i\}}(\sigma_l - \sigma'_l)c_l = -c_j$. Also, by the logic of Case 1, $g_i(c''_i, c_{-i}) = g_i(c'_i, c_{-i})$, implying $g_i(c) - g_i(c''_i, c_{-i}) = g_i(c) - g_i(c'_i, c_{-i})$. Thus, by ***(v)***, $g_i(c) - g_i(c''_i, c_{-i}) > (\sigma_i - \sigma''_i)c''_i + \sum_{l\in N\setminus\{i\}}(\sigma_l - \sigma''_l)c_l$. Thus, $-(\sigma_i - 1)c''_i - \sum_{l\in N\setminus\{i\}}(\sigma_l - 1)c_l + g_i(c) > -(\sigma''_i - 1)c''_i - \sum_{l\in N\setminus\{i\}}(\sigma''_l - 1)c_l + g_i(c''_i, c_{-i})$. Thus, by ***(i)***, $u''_i(\sigma_i, t_i) > u''_i(\sigma''_i, t''_i)$, contradicting *strategy-proofness*.

Case 3: $|\sigma_i - \sigma'_i| > 1$. By the logic of Case 2, starting from σ'_i, we can find $\tilde{c}_i$ such that $\tilde{\sigma}_i$ is one position closer to σ_i. We continue by one position at a time and at each step we obtain $g_i(c) = g_i(\tilde{c}_i, c_{-i})$. Thus, $g_i(c) = g_i(c'_i, c_{-i})$ contradicting ***(ii)***. $\square$

Proof of Proposition 1.

Let φ be a *single-valued* rule. Let $c \in \mathcal{C}$, $(\sigma, t) = \varphi(c)$, and $i \in N$. Let $h \in H$ be as in Statement 2. Then,

$$\begin{aligned}
t_i &= -\sum_{j\in N\setminus\{i\}}\sum_{l\in\{i,j\}\cap F_i(\sigma)} c_l + \tfrac{1}{(n-2)}\sum_{j\in N\setminus\{i\}}\sum_{k\in N\setminus\{i,j\}}\sum_{l\in\{j,k\}\cap F_j(\sigma)} c_l \\
&= -\sum_{l\in F_i(\sigma)} c_l + \tfrac{1}{(n-2)}\sum_{j\in N\setminus\{i\}}\sum_{l\in F_j(\sigma^{-i})} c_l \\
&= -\sum_{l\in F_i(\sigma)} c_l + \tfrac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\sigma^{-i}_l - 1)c_l \\
&= -\sum_{l\in N\setminus\{i\}}(\sigma_l - 1)c_l + \sum_{l\in N\setminus\{i\}}(\sigma^{-i}_l - 1)c_l + \tfrac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\sigma^{-i}_l - 1)c_l \text{ (a Groves rule)} \\
&= -\sum_{l\in F_i\sigma} c_l + \tfrac{1}{n-2}\sum_{l\in P_i(\sigma)}(\sigma_l - 1)c_l + \tfrac{1}{(n-2)}\sum_{l\in F_i(\sigma)}(\sigma_l - 2)c_l \\
&= \sum_{l\in P_i(\sigma)} \tfrac{(\sigma_l-1)}{(n-2)}c_l + \sum_{l\in F_i(\sigma)} \tfrac{(\sigma_l-2)-(n-2)}{(n-2)}c_l
\end{aligned}$$

[9] For each $a \in \mathbb{R}$, let $sign(a) = -1$ if and only if $a < 0$, $sign(a) = 0$ if and only if $a = 0$, and $sign(a) = 1$ if and only if $a > 0$.

$= \sum_{l\in P_i(\sigma)} \frac{(\sigma_l-1)}{(n-2)} c_l - \sum_{l\in F_i(\sigma)} \frac{(n-\sigma_l)}{(n-2)} c_l$ (rule in Mitra and Sen, 1998, and Mitra, 2001)
$= \sum_{l\in P_i(\sigma)} \frac{c_l}{2} - \sum_{l\in F_i(\sigma)} \frac{c_l}{2} - \sum_{l\in N\backslash\{i\}} \frac{(n-2\sigma_l)c_l}{2(n-2)}$
$= \sum_{l\in P_i(\sigma)} \frac{c_l}{2} - \sum_{l\in F_i(\sigma)} \frac{c_l}{2} - [\sum_{l\in N\backslash\{i\}} \frac{(n-\sigma_l-1)c_l}{2(n-2)} - \sum_{l\in N\backslash\{i\}} \frac{(\sigma_l-1)c_l}{2(n-2)}]$
$= \sum_{l\in P_i(\sigma)} \frac{c_l}{2} - \sum_{l\in F_i(\sigma)} \frac{c_l}{2} - \sum_{l\in N\backslash\{i\}} \sum_{k\in P_l(\sigma)\backslash\{i\}} \frac{c_k-c_l}{2(n-2)}$ (rule in Suijs, 1996). □

Proof of Theorem 2.

Let φ be a *single-valued* rule. Then,

If part: Let φ be a rule satisfying the axioms in the first statement of Theorem 2. Let $c \in \mathcal{C}$ and $(\sigma, t) = \varphi(c)$. Then, by *Pareto-efficiency*, $\sigma \in Q^*(c)$. By Theorem 1, *Pareto-efficiency* and *strategy-proofness* imply that φ is a Groves rule, i.e., there is $(h_i)_{i\in N} \in H$ such that for each $i \in N$, $t_i = -\sum_{l\in N\backslash\{i\}}(\sigma_l - 1)c_l + h_i(c_{-i})$. For $(\gamma_i)_{i\in N} \in H$ such that $t_i = -\sum_{l\in F_i(\sigma)} c_l + \gamma_i(c_{-i})$. In what follows, we prove by induction that $\gamma_i(c_{-i}) = \frac{1}{(n-2)} \sum_{l\in N\backslash\{i\}}(\sigma_l^{-i} - 1)c_l$. Then, $t_i = -\sum_{l\in N\backslash\{i\}}(\sigma_l - 1)c_l + \sum_{l\in N\backslash\{i\}}(\sigma_l^{-i} - 1)c_l + \frac{1}{(n-2)} \sum_{l\in N\backslash\{i\}}(\sigma_l^{-i} - 1)c_l$. Thus, by Proposition 1, $t_i = -\sum_{j\in N\backslash\{i\}} \sum_{l\in\{i,j\}\cap F_i(\sigma)} c_l + \frac{1}{(n-2)} \sum_{j\in N\backslash\{i\}} \sum_{k\in N\backslash\{i,j\}} \sum_{l\in\{j,k\}\cap F_j(\sigma)} c_l$.
Without loss of generality, suppose $N = \{1, 2, ..., n\}$ and $c_1 \geq c_2 \geq ... \geq c_n$. Let $i \in N$. Then,

Basis Step: $c = (c_n, ..., c_n)$.
By *Pareto-efficiency*, $\gamma_1(c_n, ..., c_n) + ... + \gamma_n(c_n, ..., c_n) = \frac{n(n-1)}{2} c_n$. By *equal treatment of equals in welfare*, $\gamma_1(c_n, ..., c_n) = ... = \gamma_n(c_n, ..., c_n)$. Thus, for each $j \in N$, $\gamma_j(c_n, ..., c_n) = \frac{(n-1)}{2} c_n$.

Step 1: $c = (c_1, c_n, ..., c_n)$.
By *Pareto-efficiency*, $\gamma_1(c_n, ..., c_n) + \gamma_2(c_1, c_n, ..., c_n) + ... + \gamma_n(c_1, c_n, ..., c_n) = \frac{(n-1)n}{2} c_n$. By the basis step, $\gamma_1(c_n, ..., c_n) = \frac{(n-1)}{2} c_n$. By *equal treatment of equals in welfare*, $\gamma_2(c_1, c_n, ..., c_n) = ... = \gamma_n(c_1, c_n, ..., c_n)$. Thus, for each $j \in N\backslash\{1\}$, we have $\gamma_j(c_1, c_n, ..., c_n) = \frac{(n-1)}{2} c_n$. This holds for each $k \in N\backslash\{n\}$. Thus, for each $j \in N$:

- if $j = k$, then $\gamma_j(c_n, ..., c_n) = \frac{(n-1)}{2} c_n$;
- if $j \in N\backslash\{k\}$, then $\gamma_j(c_k, c_n, ..., c_n) = \frac{(n-1)}{2} c_n$.

⋮

Step s: (Induction step) $c = (c_1, c_2, ..., c_s, c_n, ..., c_n)$.
By *Pareto-efficiency*, $\gamma_1(c_2, c_3, ..., c_s, c_n, ..., c_n) + \gamma_2(c_1, c_3, ..., c_s, c_n, ..., c_n) + ... + \gamma_n(c_1, c_2, ..., c_s, c_n, ..., c_n)$
$= \sum_{l\in\{1,2,...,s\}}(\sigma_l - 1)c_l + \frac{(n-s)(n+s+1)}{2} c_n$.

By Step $s-1$, for $j \in \{1,2,...,s\}$, we have
$\gamma_j(c_1, c_2, ..., c_s, c_n, ..., c_n) = \sum_{l \in \{1,2,...,s\} \setminus \{j\}} \frac{(\sigma_l^{\{1,2,...,s\} \setminus \{j\}} - 1)}{(n-2)} c_l + \frac{(n-1-(s-1))(n-2+(s-1))}{2(n-2)} c_n$.
By *equal treatment of equals in welfare*, $\gamma_{s+1}(c_1, c_2, ..., c_s, c_n, ..., c_n) = ... = \gamma_n(c_1, c_2, ..., c_s, c_n, ..., c_n)$.
Thus, for each $j \in N \setminus \{1,2,...,s\}$, we have $\gamma_j(c_1, c_2, ..., c_s, c_n, ..., c_n) = \sum_{l \in \{1,2,...,s\}} \frac{(\sigma_l^{\{1,2,...,s\}} - 1)}{(n-2)} c_l + \frac{(n-1-(s))(n-2+(s))}{2(n-2)} c_n$.
This holds for each $S \subset N \setminus \{n\}$ with $|S| = s$. Thus, for each $j \in N$:

- if $j \in S$, then $\gamma_j(c_{S \setminus \{j\}}, c_n, ..., c_n) = \sum_{l \in S \setminus \{j\}} \frac{(\sigma_l^{S \setminus \{j\}} - 1)}{(n-2)} c_l + \frac{(n-1-|S \setminus \{j\}|)(n-2+|S \setminus \{j\}|)}{2(n-2)} c_n$;
- if $j \in N \setminus S$, then $\gamma_j(c_S, c_n, ..., c_n) = \sum_{l \in S} \frac{(\sigma_l^S - 1)}{(n-2)} c_l + \frac{(n-1-|S|)(n-2+|S|)}{2(n-2)} c_n$.

$\vdots$

Step $n-1$: $c = (c_1, c_2, ..., c_{n-1}, c_n)$.
By *Pareto-efficiency*, $\gamma_1(c_2, c_3, ..., c_{n-1}, c_n) + \gamma_2(c_1, c_3, ..., c_{n-1}, c_n) + ... + \gamma_n(c_1, c_2, ..., c_{n-1})$
$= \sum_{l \in \{1,2,...,n-1\}} (\sigma_l - 1) c_l$.
By Step $n-2$, for $j \in \{1,2,...,n-1\}$, we have
$\gamma_i(c_1, c_2, ..., c_{n-1}, c_n) = \sum_{l \in \{1,2,...,n-1\} \setminus \{i\}} \frac{(\sigma_l^{\{1,2,...,n-1\} \setminus \{i\}} - 1)}{(n-2)} c_l + c_n$.
Thus, $\gamma_n(c_1, c_2, ..., c_{n-1}) = \sum_{l \in \{1,2,...,n-1\}} \frac{(\sigma_l^{\{1,2,...,n-1\}} - 1)}{(n-2)} c_l$. Thus, we have
$\gamma_i(c_{-i}) = \sum_{l \in N \setminus \{i\}} \frac{(\sigma_l^{N \setminus \{i\}} - 1)}{(n-2)} c_l = \sum_{l \in N \setminus \{i\}} \frac{(\sigma_l^{-i} - 1)}{(n-2)} c_l$.

Only if part:

Pareto-efficiency: Let $c \in \mathcal{C}$ and $(\sigma, t) = \varphi^*(c)$. By definition of φ^* rule, $\sigma \in Q^*(c)$ and by Proposition 1, for each $i \in N$,
$t_i = -\sum_{l \in N \setminus \{i\}} (\sigma_l - 1) c_l + \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1) c_l + \frac{1}{(n-2)} \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1) c_l$. Thus,

$$\begin{aligned}
\sum_{i \in N} t_i &= \sum_{i \in N} [-\sum_{l \in N \setminus \{i\}} (\sigma_l - 1) c_l + \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1) c_l + \frac{1}{(n-2)} \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1) c_l] \\
&= \sum_{i \in N} [-\sum_{l \in F_i(\sigma)} c_l + \frac{1}{(n-2)} \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1) c_l] \\
&= -\sum_{i \in N} \sum_{l \in F_i(\sigma)} c_l + \frac{1}{(n-2)} \sum_{i \in N} \sum_{l \in N \setminus \{i\}} (\sigma_l^{-i} - 1) c_l \\
&= -\sum_{i \in N} (\sigma_i - 1) c_i + \frac{1}{(n-2)} \sum_{i \in N} (n-2)(\sigma_i - 1) c_i \\
&= 0.
\end{aligned}$$

Strategy-proofness: By Proposition 1, φ^* is a Groves rule. Thus, by Theorem 1, φ^* is *strategy-proof.*
□

Proof of Remark 1. *No-envy:*[10] Let $c \in \mathcal{C}$, $(\sigma, t) = \varphi^*(c)$, and $\{i, j\} \subset N$ with $i \neq j$. Then, by definition of φ^*, $\sigma \in Q^*(c)$ and by Proposition 1,
$t_i = \sum_{l \in P_i(\sigma)} \frac{(\sigma_l - 1)}{(n-2)} c_l - \sum_{l \in F_i(\sigma)} \frac{(n-\sigma_l)}{(n-2)} c_l$ and $t_j = \sum_{l \in P_j(\sigma)} \frac{(\sigma_l - 1)}{(n-2)} c_l - \sum_{l \in F_j(\sigma)} \frac{(n-\sigma_l)}{(n-2)} c_l$.
Then, we distinguish two cases:

Case 1: $\sigma_i < \sigma_j$. Let $d \in \mathbb{N}$ be such that $\sigma_j = \sigma_i + d$. Then, as, by assumption, $1 \leq \sigma_i < \sigma_j \leq n$, we have $d \leq n - \sigma_i$. Also, as $\sigma \in Q^*(c)$, for each $l \in B_{ij}(\sigma)$, we have $c_i \geq c_l \geq c_j$. Thus,

$$\begin{aligned} u_i(\sigma_i, t_i) - u_i(\sigma_j, t_j) &= (-(\sigma_i - 1)c_i - \textstyle\sum_{l \in B_{ij}(\sigma)} \frac{(n-\sigma_l)}{(n-2)} c_l - \frac{(n-\sigma_j)}{(n-2)} c_j) \\ &\quad - (-(\sigma_j - 1)c_i + \tfrac{(\sigma_i - 1)}{(n-2)} c_i + \textstyle\sum_{l \in B_{ij}(\sigma)} \frac{(\sigma_l - 1)}{(n-2)} c_l) \\ &= \tfrac{(n-2)d - (\sigma_i - 1)}{(n-2)} c_i - \tfrac{(n-1)}{(n-2)} \textstyle\sum_{l \in B_{ij}(\sigma)} c_l - \frac{(n-\sigma_i - d)}{(n-2)} c_j \\ &\geq (\tfrac{(n-2)d - (\sigma_i - 1) - (n-1)(d-1) - (n-\sigma_i - d)}{(n-2)}) c_i \\ &= 0. \end{aligned}$$

Case 2: $\sigma_i > \sigma_j$. Let $d \in \mathbb{N}$ be such that $\sigma_i = \sigma_j + d$. Then, as, by assumption, $n \geq \sigma_i > \sigma_j \geq 1$. Also, as $\sigma \in Q^*(c)$, for each $l \in B_{ji}(\sigma)$, we have $c_i \leq c_l \leq c_j$. Thus,

$$\begin{aligned} u_i(\sigma_i, t_i) - u_i(\sigma_j, t_j) &= (-(\sigma_i - 1)c_i + \tfrac{(\sigma_j - 1)}{(n-2)} c_j + \textstyle\sum_{l \in B_{ji}(\sigma)} \frac{(\sigma_l - 1)}{(n-2)} c_l) \\ &\quad - (-(\sigma_j - 1)c_i - \textstyle\sum_{l \in B_{ji}(\sigma)} \frac{(n-\sigma_l)}{(n-2)} c_l - \frac{(n-\sigma_i)}{(n-2)} c_i) \\ &= \tfrac{-(n-2)d + (n-\sigma_j - d)}{(n-2)} c_i + \tfrac{(n-1)}{(n-2)} \textstyle\sum_{l \in B_{ji}(\sigma)} c_l + \frac{(\sigma_j - 1)}{(n-2)} c_j \\ &\geq (\tfrac{-(n-2)d + (n-\sigma_j - d) + (n-1)(d-1) + (\sigma_j - 1)}{(n-2)}) c_i \\ &= 0. \ \square \end{aligned}$$

Proof of Theorem 3. By contradiction, let φ be a rule satisfying the axioms of Theorem 3. We show that φ satisfies *non-bossiness*. Let $c \in \mathcal{C}$, $i \in N$, $c'_i \in \mathbb{R}_+$, $(\sigma, t) = \varphi(c)$, and $(\sigma', t') = \varphi(c'_i, c_{-i})$ be such that $(\sigma_i, t_i) = (\sigma'_i, t'_i)$. Suppose that there is $j \in N$ such that $(\sigma_j, t_j) \neq (\sigma'_j, t'_j)$. Since $(\sigma_i, t_i) = (\sigma'_i, t'_i)$, we have $u_i(\sigma_i, t_i) = u_i(\sigma', t'_i)$. By *efficiency of queues*, $\sigma_j = \sigma'_j$. Since $(\sigma_j, t_j) \neq (\sigma'_j, t'_j)$, we have $t_j \neq t'_j$. First, suppose $t_j > t'_j$. Then, $u_j(\sigma_j, t_j) > u_j(\sigma'_j, t'_j)$. Then, there is $(c'_i, c_j) \in \mathbb{R}_+^{\{i,j\}}$ such that $u_i(\sigma_i, t_i) = u_i(\sigma'_i, t'_i)$ and $u_j(\sigma_j, t_j) > u_j(\sigma'_j, t'_j)$ contradicting *coalitional strategy-proofness*. Second, suppose $t_j < t'_j$. Then, $u_j(\sigma_j, t_j) < u_j(\sigma'_j, t'_j)$. Then, there is $(c_i, c_j) \in \mathbb{R}_+^{\{i,j\}}$ such that $u'_i(\sigma_i, t_i) = u'_i(\sigma'_i, t'_i)$ $u_j(\sigma'_j, t'_j) > u_j(\sigma_j, t_j)$ contradicting *coalitional strategy-proofness*.

[10] Chun (2004b) provides a necessary and sufficient condition for a rule φ to satisfy *Pareto-efficiency* and *no-envy*: For each $c \in \mathcal{N} \times \mathbb{R}_+^N$ and each $(\sigma, t) \in \varphi(c)$, we have $\sigma \in Q^*(c)$, $\sum_{i \in N} t_i = 0$, and for each $\{i, j\} \subset N$, if $\sigma_j = \sigma_i + 1$, then $c_i \geq t_j - t_i \geq c_j$. An alternative proof thus consists in proving that φ^* satisfies this condition. In fact, rules in Suijs (1996) satisfy this condition (Katta and Sethuraman, 2005). Thus, by Proposition 1, φ^* satisfies this condition.

Now, we establish two claims:

Claim 1: For each $c \in \mathbb{R}^N_+$, each $i \in N$, and each $c'_i \in \mathbb{R}_+$, if $(\sigma, t) = \varphi(c)$ and $(\sigma', t') = \varphi(c'_i, c_{-i})$ are such that $\sigma_i = \sigma'_i$, then $(\sigma, t) = (\sigma', t)$.

Let $c \in \mathbb{R}^N_+$, $i \in N$, $c'_i \in \mathbb{R}_+$, $(\sigma, t) = \varphi(c)$, and $(\sigma', t') = \varphi(c'_i, c_{-i})$ be such that $\sigma_i = \sigma'_i$. By *strategy-proofness*, $-(\sigma_i - 1)c_i + t_i \geq -(\sigma'_i - 1)c_i + t'_i$ and $-(\sigma - 1)c'_i + t_i \leq -(\sigma'_i - 1)c'_i + t'_i$. Thus, as $\sigma_i = \sigma'_i$, we have $t_i = t'_i$. By *non-bossiness*, $(\sigma, t) = (\sigma', t')$.

Claim 2: For each $c \in \mathbb{R}^N_+$ such that for each $\{j, k\} \subseteq N$, we have $c_j \neq c_k$ if and only if $j \neq k$, for each $i \in N$, and each $c'_i \in \mathbb{R}_+$ such that for each $j \in N \backslash \{i\}$, we have $c'_i > c_j$ if and only if $c_i > c_j$, if $(\sigma, t) = \varphi(c)$, then $(\sigma, t) = \varphi(c'_i, c_{-i})$.

Let $c \in \mathbb{R}^N_+$, $i \in N$, $c'_i \in \mathbb{R}_+$ be such that for each $j \in N \backslash \{i\}$, we have $c'_i \neq c_j$ and $c'_i > c_j$ if and only if $c_i > c_j$, and $(\sigma, t) = \varphi(c)$, $(\sigma', t') = \varphi(c'_i, c_{-i})$. By *efficiency of queues*, we have $\sigma'_i = \sigma_i$. By Claim 1, $(\sigma, t) = \varphi(c'_i, c_{-i})$.

Claims 1 and 2 being proved, we now come to a contradiction. Without loss of generality, suppose $N = \{1, 2, ..., n\}$. Let $\{c, c'\} \subseteq \mathbb{R}^N_+$ be such that

(i) $c_1 > c_2 > c_3 ... > c_n$,

(ii) $c'_2 > c'_1 > c'_3 > ... > c'_n$, and

(iii) for each $i \in N \backslash \{1\}$, $c'_i = c_i$.

Let $(\sigma, t) = \varphi(c)$ and $(\sigma', t') = \varphi(c')$. By *efficiency of queues*, for each $i \in N$, we have $\sigma_i = i$, whereas $\sigma'_1 = 2$, $\sigma'_2 = 1$, and for each $i \in N \backslash \{1, 2\}$, we have $\sigma_i = \sigma'_i = i$. Thus, $(\sigma, t) \neq (\sigma', t')$. By *strategy-proofness*, $u_1(\sigma_1, t_1) = t_1 \geq -c_1 + t'_1 = u_1(\sigma'_1, t'_1)$ and $u'_1(\sigma'_1, t'_1) = -c'_1 + t'_1 \geq t_1 = u'_1(\sigma_1, t_1)$. That is, $t'_1 \in [t_1 + c'_1, t_1 + c_1]$. Thus, agent 1's transfer depends either on a constant, i.e., $\bar{t}_1 = t_1 + c$ with $c \in [c_1, c'_1]$, or on its own announcement, i.e., $\bar{t}_1 = t_1 + f(c'_1, c_1)$ with $f(c'_1, c_1) \in [c_1, c'_1]$. Clearly, this contradicts *strategy-proofness*. □

Proof of Statement 1 in Theorem 4. Let φ be a rule. Then,

If Part Let φ be a rule satisfying the axioms of Theorem 4.1. Let $c \in \mathcal{C}$ and $(\sigma, t) \in \varphi(c)$. By *Pareto-efficiency*, $\sigma \in Q^*(c)$. The Claims 1 and 2 state that *Pareto-efficiency* and *strategy-proofness* imply that there is $\{\underline{h}, \overline{h}\} \subseteq H$ such that for each $i \in N$,

- if $(\underline{\sigma}, \underline{t}) \in \arg\min_{(\sigma,t) \in \varphi(c)} u_i(\sigma_i, t_i)$, then $\underline{t}_i = -\sum_{l \in N \backslash \{i\}} (\underline{\sigma}_l - 1)c_l + \underline{h}_i(c_{-i})$ and

- if $(\overline{\sigma}, \overline{t}) \in \arg\max_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$, then $\overline{t}_i = -\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l - 1)c_l + \overline{h}_i(c_{-i})$.

Thus, repeating the proof by induction of Theorem 2, for each $i \in N$,

- $\underline{t}_i = -\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)c_l + \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l^{-i} - 1)c_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l^{-i} - 1)c_l$ and

- $\overline{t}_i = -\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l - 1)c_l + \sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l^{-i} - 1)c_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l^{-i} - 1)c_l$.

By *Pareto-efficiency*, for each $i \in N$, $u_i(\underline{\sigma}, \underline{t}) = u_i(\overline{\sigma}, \overline{t})$. Thus, for each $i \in N$, $t_i = -\sum_{l\in N\setminus\{i\}}(\sigma_l - 1)c_l + \sum_{l\in N\setminus\{i\}}(\sigma_l^{-i} - 1)c_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\sigma_l^{-i} - 1)c_l$. Thus, by Proposition 1, for each $i \in N$, $t_i = -\sum_{j\in N\setminus\{i\}}\sum_{l\in\{i,j\}\cap F_i(\sigma)} c_l + \frac{1}{(n-2)}\sum_{j\in N\setminus\{i\}}\sum_{k\in N\setminus\{i,j\}}\sum_{l\in\{j,k\}\cap F_j(\sigma)} c_l$.

Claim 1: There is $\overline{h} \in H$ such that for each $c \in \mathcal{C}$ and each $i \in N$, if $(\overline{\sigma}, \overline{t}) \in \arg\max_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$, then $\overline{t}_i = -\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l - 1)c_l + \overline{h}_i(c_{-i})$.

For each $i \in N$, let $\overline{g}_i : \mathbb{R}^N_+ \to \mathbb{R}$ be a function such that ***(i)*** for each $c \in \mathbb{R}^N_+$ if $(\underline{\sigma}, \underline{t}) \in \arg\max_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$, then $\overline{t}_i = -\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l - 1)c_l + \overline{g}_i(c)$. By contradiction, suppose that for $c \in \mathbb{R}^N_+$ and $c'_i \in \mathbb{R}$, we have ***(ii)*** $\overline{g}_i(c) - \overline{g}_i(c'_i, c_{-i}) \neq 0$. Let $(\overline{\sigma}, \overline{t}) \in \arg\max_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$ and $(\overline{\overline{\sigma}}, \overline{\overline{t}}) \in \arg\max_{(\sigma,t)\in\varphi(c'_i,c_{-i})} u'_i(\sigma_i, t_i)$. Then, by *strategy-proofness*,

- $u_i(\overline{\sigma}_i, \overline{t}_i) \geq \max_{(\sigma',t')\in\varphi(c'_i,c_{-i})} u_i(\sigma'_i, t'_i)$,

- $\max_{(\sigma,t)\in\varphi(c)} u'_i(\sigma_i, t_i) \leq u'_i(\overline{\overline{\sigma}}_i, \overline{\overline{t}}_i)$,

- $max_{(\sigma',t')\in\varphi(c'_i,c_{-i})} u_i(\sigma'_i, t'_i) \geq u_i(\overline{\overline{\sigma}}_i, \overline{\overline{t}}_i)$,

- $u'_i(\overline{\sigma}_i, \overline{t}_i) \leq max_{(\sigma,t)\in\varphi(c)} u'_i(\sigma_i, t_i)$.

Thus, ***(iii)*** $u_i(\overline{\sigma}_i, \overline{t}_i) - u_i(\overline{\overline{\sigma}}_i, \overline{\overline{t}}_i) \geq 0$ and $u'_i(\overline{\overline{\sigma}}_i, \overline{\overline{t}}_i) - u'_i(\overline{\sigma}_i, \overline{t}_i) \geq 0$. By the logic of Theorem 1, ***(i)***, ***(ii)***, and ***(iii)*** together imply a contradiction.

Claim 2: There is $\underline{h} \in H$ such that for each $c \in \mathcal{C}$ and each $i \in N$, if $(\underline{\sigma}, \underline{t}) \in \arg\min_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$, then $\underline{t}_i = -\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)c_l + \underline{h}_i(c_{-i})$.
Let $c \in \mathbb{R}^N_+$, $i \in N$, $c'_i \in \mathbb{R}_+$, $(\underline{\sigma}, \underline{t}) \in \arg\min_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$ and
$(\underline{\underline{\sigma}}, \underline{\underline{t}}) \in \arg\min_{(\sigma,t)\in\varphi(c'_i,c_{-i})} u'_i(\sigma_i, t_i)$. Let $\underline{g}_i : \mathbb{R}^N_+ \to \mathbb{R}$ be a function such that

- $\underline{t}_i = -\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)c_l + \underline{g}_i(c)$ and

- $\underline{\underline{t}}_i = -\sum_{l\in N\setminus\{i\}}(\underline{\underline{\sigma}}_l - 1)c_l + \underline{g}_i(c'_i, c_{-i})$.

In what follows, we prove that there is $\underline{h}_i : \mathbb{R}_+^{N\setminus\{i\}} \to \mathbb{R}$ such that $\underline{g}_i(c) = \underline{h}_i(c_{-i})$ and $\underline{g}_i(c'_i, c_{-i}) = \underline{h}_i(c_{-i})$. Thus, $\underline{g}_i(c) = \underline{g}_i(c'_i, c_{-i})$.

First, there is $\underline{h}_i : \mathbb{R}_+^{N\setminus\{i\}} \to \mathbb{R}$ such that

- if $(\sigma^*, t^*) \in \arg\min_{(\sigma,t)\in\varphi(c)} u'_i(\sigma_i, t_i)$ then $t^*_i = -\sum_{l\in N\setminus\{i\}}(\sigma^*_l - 1)c_l + \underline{h}_i(c_{-i})$ and

- if $(\sigma^{**}, t^{**}) \in \arg\min_{(\sigma,t)\in\varphi(c'_i,c_{-i})} u_i(\sigma_i, t_i)$, then $t^{**}_i = -\sum_{l\in N\setminus\{i\}}(\sigma^{**}_l - 1)c_l + \underline{h}_i(c_{-i})$.

Let $(\sigma^*, t^*) \in \arg\min_{(\sigma,t)\in\varphi(c)} u'_i(\sigma_i, t_i)$ and $(\sigma^{**}, t^{**}) \in \arg\min_{(\sigma,t)\in\varphi(c'_i,c_{-i})} u_i(\sigma_i, t_i)$. Let $g^*_i : \mathbb{R}^N_+ \to \mathbb{R}$ be a function such that by choosing g^*_i appropriately,,

(i) $t^*_i = -\sum_{l\in N\setminus\{i\}}(\sigma^*_l - 1)c_l + g^*_i(c)$ and $t^{**}_i = -\sum_{l\in N\setminus\{i\}}(\sigma^{**}_l - 1)c_l + g^*_i(c'_i, c_{-i})$.

By contradiction, suppose

(ii) $g^*_i(c) - g^*_i(c'_i, c_{-i}) \neq 0$.

Then, by *strategy-proofness*, $u_i(\underline{\sigma}_i, \underline{t}_i) \geq u_i(\sigma^{**}_i, t^{**}_i)$ and $u'_i(\sigma^*_i, t^*_i) \leq u'_i(\underline{\underline{\sigma}}_i, \underline{\underline{t}}_i)$. By assumption, $u_i(\sigma^*_i, t^*_i) \geq u_i(\underline{\sigma}_i, \underline{t}_i)$ and $u'_i(\underline{\underline{\sigma}}_i, \underline{\underline{t}}_i) \leq u'_i(\sigma^{**}_i, t^{**}_i)$. Thus,

(iii) $u_i(\sigma^*_i, t^*_i) - u_i(\sigma^{**}_i, t^{**}_i) \geq 0$ and $u'_i(\sigma^{**}_i, t^{**}_i) - u'_i(\sigma^*_i, t^*_i) \geq 0$.

By the logic of Theorem 1, ***(i)***, ***(ii)***, and ***(iii)*** together imply a contradiction. This holds for each $c'_i \in \mathbb{R}_+$.

Second, $\underline{g}_i(c) = \underline{h}_i(c_{-i})$ and $\underline{g}_i(c'_i, c_{-i}) = \underline{h}_i(c_{-i})$. By contradiction, suppose $\underline{g}_i(c) - \underline{h}_i(c_{-i}) \neq 0$. (The other case is immediate.) First, by assumption, $u_i(\sigma^*_i, t^*_i) \geq u_i(\underline{\sigma}_i, \underline{t}_i)$. Thus, $-(\sigma^*_i - 1)c_i - \sum_{l\in N\setminus\{i\}}(\sigma^*_l - 1)c_l + \underline{h}_i(c_{-i}) \geq -(\underline{\sigma}_i - 1)c_i - \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)c_l + \underline{g}_i(c)$. Thus, $-\sum_{l\in N}(\sigma^*_l - 1)c_l + \underline{h}_i(c_{-i}) \geq -\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)c_l + \underline{g}_i(c)$. Thus, by *Pareto-efficiency*, $\underline{h}_i(c_{-i}) \geq \underline{g}_i(c)$. Second, by *strategy-proofness*, $u_i(\underline{\sigma}_i, \underline{t}_i) \geq u_i(\sigma^{**}_i, t^{**}_i)$. Thus, $-(\underline{\sigma}_i - 1)c_i - \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)c_l + \underline{g}_i(c) \geq -(\sigma^{**}_i - 1)c_i - \sum_{l\in N\setminus\{i\}}(\sigma^{**}_l - 1)c_l + \underline{h}_i(c_{-i})$. Thus, $\underline{g}_i(c) \geq (\underline{\sigma}_i - \sigma^{**}_i)c_i + \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - \sigma^{**}_l)c_l + \underline{h}_i(c_{-i})$. Altogether,

(iv) $\underline{h}_i(c_{-i}) \geq \underline{g}_i(c) \geq (\underline{\sigma}_i - \sigma^{**}_i)c_i + \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - \sigma^{**}_l)c_l + \underline{h}_i(c_{-i})$.

By *Pareto-efficiency*, for each $S \subseteq N$, if for each $\{k, k'\} \subseteq S$ with $k \neq k'$, we have $c_k = c_{k'}$ and there is no $k'' \in N\setminus S$ such that $k'' \in B_{kk'}(\sigma) \cup B_{kk'}(\sigma')$, then $\sum_{l\in S} -(\sigma_l - 1)c_l = \sum_{l\in S} -(\sigma'_l - 1)c_l$. Also, there is $j \in N$ such that $\underline{\sigma}_j = \sigma^{**}_i$. Thus, $\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - \sigma^{**}_l)c_l = -sign(\underline{\sigma}_i - \sigma^{**}_i)\sum_{l\in B_{ij}(\sigma)\cup\{j\}} c_l$. Thus, we may rewrite ***(iv)*** as

(v) $\underline{h}_i(c_{-i}) \geq \underline{g}_i(c) \geq (\underline{\sigma}_i - \sigma^{**}_i)c_i - sign(\underline{\sigma}_i - \sigma^{**}_i)\sum_{l\in B_{ij}(\sigma)\cup\{j\}} c_l + \underline{h}_i(c_{-i})$.

We distinguish three cases:

Case 1: $(\underline{\sigma}_i - \sigma_i^{**}) = 0$. Then, $-sign(\underline{\sigma}_i - \sigma_i^{**})\sum_{l\in B_{ij}(\sigma)\cup\{j\}} c_l = 0$. Thus, by ***(v)***, $\underline{g}_i(c) = \underline{h}_i(c_{-i})$ contradicting $\underline{g}_i(c) - \underline{h}_i(c_{-i}) \neq 0$.

Case 2: $|\underline{\sigma}_i - \sigma_i^{**}| = 1$. Suppose $c_i' > c_i$. (The symmetric case is immediate.) Then, $(\underline{\sigma}_i - \sigma_i^{**}) = 1$ and $-sign(\underline{\sigma}_i - \sigma_i^{**})\sum_{l\in B_{ij}(\sigma)\cup\{j\}} c_l = -c_j$. Thus, by ***(v)***, $\underline{h}_i(c_{-i}) \geq \underline{g}_i(c) \geq (c_i - c_j) + \underline{h}_i(c_{-i})$. Let $c_i'' \in \mathbb{R}_+$ be such that ***(vi)*** $\underline{g}_i(c) > (c_i'' - c_j) + \underline{h}_i(c_i)$ and $c_i' > c_i'' > c_i$. Let $(\sigma^{***}, t^{***}) \in \arg\min_{(\sigma,t)\in\varphi(c_i'',c_{-i})} u_i(\sigma_i, t_i)$. Then, by *Pareto-efficiency of queues*, $\sigma_i^{***} = \sigma_i^{**}$. Then,
$(\underline{\sigma}_i - \sigma_i^{***}) = (\underline{\sigma}_i - \sigma_i^{**}) = 1$ and $\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - \sigma_l^{***})c_l = \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - \sigma_l')c_l = -c_j$. By ***(vi)***,
$\underline{g}_i(c) > (\underline{\sigma}_i - \sigma_i^{***})c_i'' + \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - \sigma_l^{***})c_l + \underline{h}_i(c_i)$.
Then, $-(\underline{\sigma}_i - 1)c_i'' - \sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)c_l + \underline{g}_i(c) > -(\sigma_i^{***} - 1)c_i'' - \sum_{l\in N\setminus\{i\}}(\sigma_l^{***} - 1)c_l + \underline{h}_i(c_{-i})$.
Thus, $u_i''(\underline{\sigma}_i, \underline{t}_i) > u_i''(\sigma_i^{***}, t_i^{***})$. Also, $u_i''(\sigma_i^{***}, t_i^{***}) \geq min_{(\sigma,t)\in\varphi(c_i'',c_{-i})} u_i''(\sigma_i, t_i)$. Therefore, $u_i''(\underline{\sigma}_i, \underline{t}_i) > min_{(\sigma,t)\in\varphi(c_i'',c_{-i})} u_i''(\sigma_i, t_i)$ contradicting *strategy-proofness.*

Case 3: $|\underline{\sigma}_i - \sigma_i^{**}| > 1$. By the logic of Case 2, starting from σ_i^{**}, we can find $\tilde{c}_i$ such that $\tilde{\sigma}_i$ is one position closer to $\underline{\sigma}_i$. We continue by one position at a time and at each step we obtain $\underline{g}_i(\tilde{c}_i, c_{-i}) = \underline{h}_i(c_{-i})$. Thus, $\underline{g}_i(c) = \underline{h}_i(c_{-i})$ contradicting $\underline{g}_i(c) - \underline{h}_i(c_{-i}) \neq 0$.

Only if part:

Pareto-efficiency: Straightforward from Theorem 2.

No-envy: Straightforward from Theorem 2.

Strategy-proofness: Let $c \in \mathcal{C}$, $i \in N$, $c_i' \in \mathbb{R}_+$, $(\sigma, t) \in \Phi^*(c)$, and $(\sigma', t') \in \Phi^*(c_i', c_{-i})$. Then, by definition of Φ^*, $\sigma \in Q^*(c)$ and by Proposition 1, there is $h \in H$ such that for each $i \in N$, $h_i(c_{-i}) = \sum_{l\in N\setminus\{i\}}(\sigma_l^{-i} - 1)c_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\sigma_l^{-i} - 1)c_l = \sum_{l\in N\setminus\{i\}}(\sigma_l'^{-i} - 1)c_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\sigma_l'^{-i} - 1)c_l$ and $t_i = -\sum_{l\in N\setminus\{i\}}(\sigma_l - 1)c_l + h_i(c_{-i})$ and $t_i' = -\sum_{l\in N\setminus\{i\}}(\sigma_l' - 1)c_l + h_i(c_{-i})$. Suppose $u_i(\sigma_i', t_i') > u_i(\sigma_i, t_i)$. Thus, $-(\sigma_i' - 1)c_i - \sum_{l\in N\setminus\{i\}}(\sigma_l' - 1)c_l + h_i(c_{-i}) > -(\sigma_i - 1)c_i - \sum_{l\in N\setminus\{i\}}(\sigma_l - 1)c_l + h_i(c_{-i})$. Thus, $-\sum_{l\in N}(\sigma_l' - 1)c_l > -\sum_{l\in N}(\sigma_l - 1)c_l$ contradicting $\sigma \in Q^*(c)$. Thus, $u_i(\sigma_i', t_i') \leq u_i(\sigma_i, t_i)$. This holds for each $(\sigma, t) \in \varphi(c)$ and each $(\sigma', t') \in \varphi(c_i', c_{-i})$. Thus, if $Z = \varphi(c)$ and $Z' = \varphi(c_i', c_{-i})$, then $Z_i\ R_i(c_i)\ Z_i'$.

Proof of Statement 2 in Theorem 4.

If Part: Let φ be a rule satisfying the axioms of the third statement of Theorem 4. Let $c \in \mathcal{C}$ and $(\sigma, t) \in \varphi(c)$. Then, by *Pareto-efficiency*, $\sigma \in Q^*(c)$. By Statement 1, *Pareto-efficiency* and *strategy-proofness* imply that there is $\{\underline{h}, \overline{h}\} \subseteq H$ such that for each $i \in N$,

- if $(\underline{\sigma}, \underline{t}) \in \arg\min_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$, then $\underline{t}_i = -\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l - 1)c_l + \underline{h}_i(c_{-i})$,

- if $(\overline{\sigma}, \overline{t}) \in \arg\max_{(\sigma,t)\in\varphi(c)} u_i(\sigma_i, t_i)$, then $\overline{t}_i = -\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l - 1)c_l + \overline{h}_i(c_{-i})$.

By *symmetry*, for each $\{i, j\} \subset N$, if $c_{-i} = c_{-j}$, then $\underline{h}_i(c_{-i}) = \underline{h}_j(c_{-j})$ and $\overline{h}_i(c_{-i}) = \overline{h}_j(c_{-j})$. Thus, for each $\{i, j\} \subset N$, if $c_i = c_j$, then $\underline{h}_i(c_{-i}) = \underline{h}_j(c_{-j})$ and $\overline{h}_i(c_{-i}) = \overline{h}_j(c_{-j})$. This is true for each $c \in \mathbb{R}_+$. Thus, repeating the proof by induction of Theorem 2, for each $i \in N$, we have $\underline{t}_i = -\sum_{l\in F_i(\underline{\sigma})} c_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\underline{\sigma}_l^{-i} - 1)c_l$ and $\overline{t}_i = -\sum_{l\in F_i(\overline{\sigma})} c_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\overline{\sigma}_l^{-i} - 1)c_l$. Thus, by *Pareto-efficiency*, for each $i \in N$, $u_i(\underline{\sigma}, \underline{t}) = u_i(\overline{\sigma}, \overline{t})$. Thus, for each $i \in N$, we have $t_i = -\sum_{l\in F_i(\sigma)} c_l + \frac{1}{(n-2)}\sum_{l\in N\setminus\{i\}}(\sigma_l^{-i} - 1)c_l$. Thus, by Proposition 1, $t_i = -\sum_{j\in N\setminus\{i\}}\sum_{l\in\{i,j\}\cap F_i(\sigma)} c_l + \frac{1}{(n-2)}\sum_{j\in N\setminus\{i\}}\sum_{k\in N\setminus\{i,j\}}\sum_{l\in\{j,k\}\cap F_j(\sigma)} c_l$. Thus, for each $c \in \mathcal{C}$, we have $\varphi(c) \subseteq \Phi^*(c)$. Thus, by *symmetry*, $\varphi(c) = \Phi^*(c)$.

Only if part: Suppose that for each $c \in \mathcal{C}$, we have $\varphi(c) = \Phi^*(c)$. By the second statement of Theorem 4, φ satisfies *Pareto-efficiency* and *strategy-proofness*. Also, φ does not depend on agents' names. In particular, t_i has the same structure for each $i \in N$. Thus, φ satisfies *anonymity*. □

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