The paper considers a model of competition among firms that produce a homogeneous good in a networked environment. A bipartite graph determines which subset of markets a firm can supply to. Firms compete à la Cournot and decide how to allocate their production output to the markets they are directly connected to. We show that the resulting Cournot game has a unique equilibrium for any given network and provide a characterization of the production quantities at equilibrium. Our results identify a close connection between the equilibrium outcome and supply paths in the underlying network structure. We then proceed to study the impact of changes in the competition structure on firms' profits and consumer welfare. In particular, we examine how profits and welfare are affected by a firm entering a new market or by two firms merging. The analysis points to the fact that some of the insights from studying Cournot competition in a single market do not generalize in a networked environment and one may need to take the entire competition structure into account. Finally, we turn our attention to the operations of a cartel including the entire set of firms. We show that the cartel maximizes its profits by appropriately segmenting the markets among its members so that a firm supplies solely to the ones allocated to it, and we provide an algorithm that computes the optimal production quantities for each firm in the cartel.

Categories and Subject Descriptors: C.2.2 [Computer-Communication Networks]: Network Protocols

General Terms: Cournot competition, Horizontal Mergers, Networks, Cartels

1. INTRODUCTION

Competition models typically feature a number of firms operating in a single, well-defined market. In many settings though, a firm may be participating in a number of markets and in each one of them it may be competing with a different set of firms. For instance, constraints imposed by geographical considerations may give rise to such an environment. The market for natural gas presents such an example. Gas is carried mainly through pipelines since other forms of transportation are not as economical. Thus, the structure of the pipeline network imposes constraints on the set of feasible trades that naturally result in substantial differences between regional prices. As an example, the price per 10^7 Kcals of industrial use natural gas was $74.50 in Kazakhstan and $745.40 in Switzerland in early 2008. In contrast, the market for crude oil enjoys higher liquidity and lower regional price dispersion mainly due to the ease of storage and transportation.

Another example that falls within this framework is the airline industry. Airlines compete with each other in multiple markets, i.e., origin-destination pairs, and each airline may be active only in a subset of those markets. Obviously, prices in each of
those markets crucially depend on the level of competition in the market, which is largely determined by the extent to which airlines participate in the market.

As a last example, we mention the cement industry. Due to its very high weight/price ratio, cement can be economically transported only short distances by land or between export/import terminals by sea. This means a cement plant can supply only to a restricted set of locations, determined by its distance to consumers and access to waterways.

These examples motivate the study of multi-market competition models in which firms may be competing with each other in multiple markets. Our paper presents one step towards this direction. In particular, we consider a model where the competition structure is given by a bipartite graph that represents the set of markets each firm can supply to. Firms compete à la Cournot in each of the markets, i.e., they decide how much to supply to each of the markets they participate in and the price in each market is determined as a function of the aggregate supply.

We begin our analysis by showing that there exists a unique equilibrium in the setting we study. We provide a characterization of the equilibrium flows, i.e., equilibrium production quantities, in terms of the underlying network structure which, for sufficiently sparse networks, can be further simplified and rewritten as a function of supply paths in the network. This characterization highlights the connection between the structure of the network and the extent to which firms’ actions are strategic complements or substitutes.

Armed with a characterization of equilibrium supply decisions, we explore the effect of changes in the network structure on firms’ profits and consumer welfare. First, we study the question of a firm entering a new market. We show that entry may not be beneficial for either the firm or consumers as such a move affects the entire vector of production quantities. The firm might face a more aggressive competition in its original markets due to its entry to a new market. Moreover, the effect on other firms and consumers also depends on their location in the network. Some firms and consumers may benefit and some others may be harmed. This is in stark contrast with standard results in Cournot oligopoly where entry directly implies more competition in the market and thus higher consumer welfare.

Similarly, the effect of a merger between two firms on profits and overall welfare largely depends on the structure of competition in the original Cournot market. In particular, we show that insights from analyzing mergers in a single market do not carry over in a networked environment. Market concentration indices are insufficient to correctly account for the network effect of a merger and one should not restrict attention only in the set of markets that the firms participating in the merger supply to.

Finally, we consider the cartel formation process in such an environment and show that a cartel including all firms maximizes its total profits by segmenting the markets among its members. Each firm then operates only in the set of markets allocated to it. Similar market sharing agreements have been previously studied in Belleflamme and Bloch [2004]. In our model, such market sharing is the outcome of profit maximization by the cartel.

Related Literature. The seminal contribution by Bulow et al. [1985] serves as the main motivation behind our study. They analyze the strategic interactions between two firms that compete in two markets (a monopoly and a duopoly) and show that strategic complementarity and substitutability between the firms’ actions determine the effect of exogenous changes in the markets on profits. We extend their environment by considering an arbitrary network of firms and markets and properly generalize the
notion of strategic complementarity and substitutability to properly account for the network interactions among the firms.

Several papers study trade between agents that are connected through a network. The closest model to ours is the one by Nava [2013]. The motivating question in his work is identifying conditions under which the aggregate behavior in a decentralized oligopolistic market can be well approximated by a centralized Walrasian market. The role of an agent, i.e., whether she is a seller, buyer, or intermediary of a single good, are endogenously determined in his model. He shows that Warasian markets provide close approximations to large decentralized markets when sufficiently many sellers compete directly to supply to any group of buyers. By making stronger functional assumptions in our environment, we are able to provide closed form expressions for the equilibrium production quantities and obtain several results regarding the firms’ profits and consumer welfare as a function of the underlying network structure. Relatedly, Kranton and Minehart [2001] model competition among buyers of a single, indivisible good as ascending price auction and study whether the resulting pattern of trades is efficient. Corominas-Bosch [2004] considers a non-cooperative bargaining game and provides conditions for the equilibrium of the bargaining game to coincide with the Walrasian outcome. Abreu and Manea [2012] and Manea [2011] study decentralized bargaining, relate an agent’s bargaining power to her position in the network that represents the set of feasible trades, and explore efficiency in equilibrium outcomes. Finally, Elliott [2013] studies markets in which there are heterogeneous gains from trade and relationship specific investments which are necessary for trade to take place. In his model, the network of feasible trades is endogenously determined by buyers and sellers investing in enabling bilateral trades. His main goal is to study the resulting inefficiencies that may arise in the network formation process depending on whether investments in enabling trade are negotiated jointly or made separately.

Guzmán [2011] studies Bertrand competition in a networked environment. Most of his results are obtained in a duopoly and a key feature in his model is the presence of locked-in consumers, i.e., consumers that can only purchase from one of the firms. Babaioff et al. [2013] extend his model and provide several results regarding the (non-)existence of pure and mixed Nash pricing equilibria along with structural properties of the sellers’ pricing equilibrium support. Relatedly, Chawla and Roughgarden [2008] and Acemoglu and Ozdaglar [2007] study the extent of inefficiency in Bertrand competition games over networks.

Our analysis of mergers builds on the seminal work of Farrel and Shapiro [1990]. We use their approach of studying a merger between two firms based on the concept of an “infinitesimal merger”, that can be roughly seen as one of the firms buying a small share of the other. They study mergers in a single market, whereas our focus is on the structure of how competition affects the way profits and consumer welfare change after the merger. Importantly, our analysis establishes that the network structure plays a first-order role in determining whether a merger has a positive or negative effect on overall welfare.

Finally, our paper is also related to a recent stream of papers that study games among agents that are embedded on a network structure. Ballester et al. [2006] identify a close relation between an agent’s equilibrium action in a game that features local positive externalities and her position in the network structure as captured by her Katz–Bonacich centrality. Candogan et al. [2012] study the pricing problem of a monopolist that is selling a divisible good to a population of agents and provide a char-

---

1 Several other papers, e.g., Blume et al. [2009], Ostrovsky [2008], and Condorelli and Galeotti [2012] study trading among agents that are connected by a network structure.
acterization of the optimal pricing policies as a function of the social network structure of agents.

2. MODEL

There are $n$ firms $F = \{f_1, \ldots, f_n\}$ competing with each other in $m$ markets $M = \{m_1, \ldots, m_m\}$. A bipartite graph links markets and firms; firms can only supply to the markets they are directly connected to. In particular, bipartite graph $G = (F \cup M, E)$ consists of a set of nodes associated with the markets and firms as well as a set of links $E$ that represents the subset of markets each firm can supply to. We also let $F_i = \{f_i \in M | (f_i, m_k) \in E\}$ denote the set of markets firm $f_i$ can supply to and $M_k = \{f_i \in F | (f_i, m_k) \in E\}$ denote the set of firms that can supply to market $m_k$.

Firms compete a la Cournot in the markets they participate in. We let $q_{ik}$ denote the quantity firm $f_i$ supplies to market $m_k$. We assume that markets have inverse linear demand functions, i.e., the price in market $m_k$ is given by

$$P_k = \alpha_k - \beta_k \cdot C_k,$$

where $\alpha_k, \beta_k > 0$, and $C_k = \sum_{f_i \in M_k} q_{ik}$ denotes the total quantity sold in market $m_k$.

Finally, we assume that firms have quadratic production costs, i.e., firm $f_i$’s total cost of production is given by

$$T_i = c_i \cdot S_i^2,$$

where $S_i = \sum_{m_k \in F_i} q_{ik}$. Hence, firm $f_i$’s profit is given by the following expression

$$\pi_i(q) = \sum_{m_k \in F_i} q_{ik} \cdot P_k - c_i \cdot S_i^2.$$

Note that firm $f_i$’s profit function is not separable in the markets it participates in and, in particular, the marginal profit from $q_{ik}$ depends on markets other than $m_k$.

Given a graph $G$, firm $f_i$ maximizes its profit $\pi_i$ by supplying non-negative quantities to markets in $F_i$. We denote the resulting Cournot game by $CG(\alpha, \beta, c, G)$, where $\alpha = [\alpha_1, \ldots, \alpha_m]^T$, $\beta = [\beta_1, \ldots, \beta_m]^T$, and $c = [c_1, \ldots, c_n]^T$.

3. EQUILIBRIUM ANALYSIS

In this section, we first show that the Cournot game defined above has a unique Nash equilibrium. Then we provide a characterization of the production quantities at equilibrium as a function of the underlying network structure.

**Theorem 3.1.** The Cournot game $CG(\alpha, \beta, c, G)$ has a unique Nash equilibrium.

Theorem 3.1 follows by noting that the first order equilibrium conditions for a Cournot game can be expressed as a linear complementarity problem. Samelson et al. [1958] provide conditions under which such a problem has a unique solution. We show that these conditions hold for the linear complementarity problem defined by the first order equilibrium conditions for the Cournot game. Finally, we verify that the solution is indeed an equilibrium. The details of the proof can be found in the Appendix.

For the remainder of our analysis, we focus on symmetric Cournot games as this enables us to clearly illustrate the effect of the network structure on the equilibrium allocations. Our results readily extend to asymmetric games.

**Definition 3.2.** A Cournot game $CG = (\alpha, \beta, c, G)$ is called symmetric if all the firms have the same production technology, i.e., $c_i = c_j = c$, for all $f_i, f_j \in F$, and markets have the same demand slope, i.e., $\beta_k = \beta_l = \beta$, for all $m_k, m_l \in M$. 

EC’14, June 8–12, 2014, Stanford University, Palo Alto, CA, USA, Vol. , No. , Article 1, Publication date: February 2014.
Next, we provide a characterization of the equilibrium production quantities that illustrates the role of the network structure of competition between the firms. Let \( q_G^* \) denote the column vector of production quantities at equilibrium, where links from firms to markets are ordered lexicographically. Note that a subset of links may carry zero flow, i.e., \( q_{ik}^* = 0 \), even though \((f_i, m_k) \in E\). Our results on the characterization of the equilibrium production quantities are stated for the set of active links, i.e., the subset of links for which the corresponding production quantities are positive at equilibrium.

Given the equilibrium \( q_G^* \) at network \( G \), we use \( Z(q_G^*) \) to denote the set of inactive links at equilibrium, i.e., \( Z(q_G^*) = \{ (f_i, m_k) | q_{ik}^* = 0 \} \). Let \( G^* \) be the graph induced by the set of active links at \( q_G^* \).

Given the equilibrium \( q_G^* \) at network \( G \), we define \( W \) as the following matrix:

\[
w_{i_1,k_1,i_2,k_2} = \begin{cases} 
2c & \text{if } i_1 = i_2, k_1 \neq k_2 \\
\beta & \text{if } i_1 \neq i_2, k_1 = k_2 \\
0 & \text{otherwise.} 
\end{cases}
\] (1)

The columns and the rows of \( W \) correspond to the links in \( G^* \).

Theorem 3.3 provides a concrete characterization of the production quantities at equilibrium as a function of matrix \( W \).

**Theorem 3.3.** The unique Nash equilibrium of Cournot game \( CG(a, \beta, c, G) \) is given by

\[
q^* = [I + aW]^{-1} a\alpha,
\] (2)

where \( a = \frac{1}{2(c + \beta)} \).

Moreover, if \( \lambda_{\text{max}}(aW) < 1 \), Expression (2) can be rewritten as

\[
q^* = \left[ \sum_{k=0}^{\infty} (aW)^{2k} - \sum_{k=0}^{\infty} (aW)^{2k+1} \right] a\alpha.
\] (3)

The following corollary of Theorem 3.3 essentially states that removing inactive links from the original bipartite graph \( G \) is done without any loss of generality since these links are strategically redundant and play no role in determining production quantities at equilibrium.

**Corollary 3.4.** Consider two networks \( G \) and \( G' \) and let \( q^*_G \) and \( q^*_G' \) denote the equilibrium production quantities for the Cournot games \( CG(a, \beta, c, G) \) and \( CG(a, \beta, c, G') \) respectively. If \( G - Z(q^*_G) = G' - Z(q^*_G') \), then

\[
q^*_G - Z(q^*_G) = q^*_G' - Z(q^*_G').
\]

The following corollary provides a necessary and sufficient condition for \( G^* \) to coincide with the original graph \( G \).

**Corollary 3.5.** The unique equilibrium of Cournot game \( CG = (\alpha, \beta, c, G) \) has no inactive links if and only if there does not exist a vector \( x \in \mathbb{R}^{|E|} \) such that

\[
D^T x > 0 \\
\alpha^T x = 0
\]
where $D$ is an $|E| \times |E|$ matrix, where its rows and columns are correspond to links in the network, such that

$$D_{ik,jl} = \begin{cases} 
2(\beta_k + c_i) & \text{if } i = j, k = l \\
2c_i & \text{if } i = j, k \neq l \\
\beta_k & \text{if } i \neq j, k = l \\
0 & \text{otherwise.} 
\end{cases}$$ (4)

To clarify the role of network structure in having inactive links, assume that all markets have the same demand parameter $\alpha$ and matrix $D$ is symmetric i.e. for all $f_i \in F$, $c_i = \frac{1}{2}$ and for all $m_k \in M$, $\beta_k = 1$, then Corollary 3.5 roughly states that the equilibrium features inactive links when the ratio between the number of links which have neighbor links in both of their end-points to the number of links which have neighbor links in just one of their end-points is a small positive number.

The equilibrium characterization in (3) can be interpreted as follows: quantity $q_{ik}$, i.e., the quantity that firm $f_i$ supplies to market $m_k$, is given by a weighted sum of the production quantities associated with other links. The weight that firm $f_i$ puts on production quantity $q_{jl}$ when it determines how much production to supply to market $m_k$, i.e., $q_{ik}$, is increasing (decreasing) with the weights of even (odd) length paths that connect the two links. To make the connection between production quantities at equilibrium and paths between links in the underlying graph structure even more concrete, consider the line graph $L(G^*)$ that is obtained from adding a node in $L(G^*)$ for every link in $G^*$ and connecting two nodes if the corresponding links share an endpoint in $G^*$.

As we show below, the production quantities at equilibrium are closely related to the Katz–Bonacich centrality of the nodes in the line graph $L(G^*)$. The Katz–Bonacich centrality measure is defined as follows.

Definition 3.6. For a weighted adjacency matrix $G$ and for scaler $a > 0$, the Katz–Bonacich centrality vector of nodes in $G$ is given by

$$b(G,a) = \sum_{t=0}^{\infty} (aG)^t \cdot 1.$$ So, one can see the connection between the equilibrium of a Cournot game and the Katz–Bonacich centrality in the line graph $L(G^*)$ but with negative weights $-W$. In particular we can rewrite Equation (3) as follows:

$$q^* = b(-W,a) \circ a\alpha,$$ (5)

where $\circ$ is the entrywise product.

Next, we turn our attention to the condition that leads to the characterization of equilibrium production quantities as weighted sums of paths in the underlying network structure, i.e., $\lambda_{\max}(aW) < 1$. The maximum eigenvalue is generally a measure of the maximum and average degree (density) of the underlying graph, so condition $\lambda_{\max}(aW) < 1$ implies that Equation (3.3) describes equilibrium behavior in a sufficiently sparse network. The following corollary formalizes this intuition.

Corollary 3.7. Consider a symmetric Cournot game. Then, the condition $\lambda_{\max}(aW) < 1$ is satisfied if and only if one of the following two conditions holds:

- The marginal cost of production is sufficiently high, i.e., $2c \geq \beta$, each firm can supply to at most 10 markets, and the total number of supply links connecting firms to markets is bounded above by $4n$. 

The marginal cost of production is sufficiently low, i.e., $2c \leq \beta$, each market can have at most 10 suppliers, and the total number of supply links connecting firms to markets is bounded above by $4m$.

The sparsity conditions required by Corollary 3.7 are natural in several settings of interest. First, consider the case $2c \leq \beta$. This corresponds to a setting where the price in a market is very sensitive to the aggregate production quantity, i.e., competition among firms within a market is intense. In such a setting, we would expect that the number of firms that supply to a market is small. On the other hand, if $2c \geq \beta$, then firms are forced to supply to a small number of markets since doing otherwise is prohibitively costly.

Finally, although when $\lambda_{max}(aW) \geq 1$ we cannot rewrite expression (2) as the more intuitive path-based characterization in (3), Corollary 3.8 below implies that the main insight, i.e., the weight a firm puts on a market is increasing in the number of even paths between the firm and the market and decreasing in the number of odd paths between them, holds for all network structures regardless of the value of $\lambda_{max}(aW)$.

**Corollary 3.8.** Consider the Cournot game $CG(a, \beta, c, G)$. Then, the weight firm $i$ puts on each market $\ell$ when $i$ is considering its production quantity $q_{ik}$ for market $k$ is increasing (decreasing) with the weights of even (odd) paths that start from link $(i,k)$ and end in market $\ell$. More specifically, entry $\psi_{ik,\ell}$ of matrix $\Psi = [I + aW]^{-1}$ is increasing (decreasing) with weights even (odd) paths from $(i,k)$ to link $(j,\ell)$.

We conclude this section with an alternative characterization of the equilibrium production quantities that highlights their dependence on the “importance” of the links in $G$, as captured by the price-impact matrix $\Lambda$ defined below. In particular, $\Lambda$ denotes the $|E| \times m$ matrix

$$
\Lambda_{ik,\ell} = -\beta \sum_{j \in M_1} \frac{\psi_{jl,ik}}{\psi_{ik,ik}}.
$$

As we see shortly in Lemma 4.1, entry $(ik,\ell)$ represents the price change in market $\ell$ that results from a marginal increase in the production of firm $i$ in market $k$. One can view the entries of $\Lambda$ as a measure of the firms’ market power, i.e., the larger their absolute values, the higher market power the corresponding firms have in the underlying networked environment.

**Corollary 3.9.** The equilibrium production quantities can be expressed as

$$
q^* = -V\Lambda \alpha \frac{a}{\beta}.
$$

where $V = Diag(\Psi)$.

Interestingly, since $V \geq 0$ and $\alpha \geq 0$, Corollary 3.9 implies that $q^* \propto -\Lambda$.

**4. Changing the Structure of Competition**

This section explores the effects on the firms’ profits and consumer welfare of changes in the structure of competition among the firms, i.e., changes in graph $G$. In particular, we study changes in welfare when a firm enters a new market as well as when two firms merge and choose their production quantities with the goal of maximizing their joint profit. Our main focus is on highlighting the role of the underlying network structure and identifying how insights derived by the analysis of a single market differ due to the (second order) network effects. The following lemma is central for our analysis. It describes how firms adjust their production quantities in response to an
infinitesimal change in $q_{ik}$, i.e., the production quantity that firm $i$ supplies to market $k$.

**Lemma 4.1.** Consider an exogenous change in the quantity firm $i$ supplies to market $k$. Then, firms adjust their production quantities according to the expression

$$dq_{jl} = \frac{\psi_{jl,ik}}{\psi_{ik,ik}} dq_{ik},$$

(7)

where recall that $\Psi = [I + aW]^{-1}$.

Note that since the diagonal entries of matrix $\Psi$ are positive, Lemma 4.1 suggests that if the sum of the weights of even paths from link $(i, k)$ to $(j, l)$ is greater than the sum of the weights of odd paths, then firms $i$ and $j$ view their actions (production quantities) in markets $k$ and $l$ as complements, otherwise they view them as substitutes. Thus, matrix $\Psi$ not only allows us to determine which actions are complements or substitutes but also how strong the level of complementarity or substitutability is.

The example in Figure 1 clearly demonstrates the second order network effect of a small change in a firm’s output. Consider first the Cournot game defined for the graph in Figure 1(a) and assume that the production quantity corresponding to link $(1,1)$ decreases by $\epsilon_1$ (e.g., due to small changes in market $m_1$’s demand). Then, firm $f_2$ would respond by increasing its output in market $m_1$ by $\epsilon_2$. Next, consider a Cournot game defined over Figure 1(b) and assume again that the production quantity corresponding to link $(1,1)$ decreases by $\epsilon_1$. Then, clearly firm $f_2$ would increase its supply to market $m_1$. One would expect that the increase is smaller than $\epsilon_2$ in this case due to the fact that firm $f_2$ is also supplying to market $m_2$ (and production costs are quadratic). However, this intuition is not true: firm $f_2$ ends up responding more aggressively to the same change in firm $f_1$’s output. This is due to the fact that as firm $f_2$ moves some of its production from market $m_2$ to market $m_1$, the marginal profit of producing for $m_1$ increases, which further results in a more aggressive response from $f_2$.

![Fig. 1](image-url)  
(a) (b)

**Fig. 1.** A change in firm $f_1$’s production quantity leads to a different response from $f_2$.

### 4.1. Entering a New Market

Consider firm $f_i$ entering market $m_k$, which in our setting corresponds to adding link $(i, k)$ to graph $G$. Entry has a direct effect on firm $f_i$’s profit as the firm has to adjust the allocation of its production to the markets it supplies to. In addition, there is a second order effect that relates to how changes in firm $f_i$’s production quantities across its markets affect the actions of its competitors and propagate through the network. In their seminal paper, Bulow et al. [1985] consider a setting with two markets and two firms in which one of the firms supplies to both markets whereas the other supplies only to one and study how changes in a firm’s actions in one market affect its competitors’ actions in a second market and ultimately its own profit. The analysis that follows explores a similar question for a general networked market.
The following example clearly highlights the role of the network structure. Let firm $f_1$ enter market $m_2$ for the two network settings illustrated in Figure 2. Then, for the first case (Figure 2(a)) it is profitable for firm $f_1$ to enter market $m_2$, whereas for the second (Figure 2(b)) it is not. This is due to the fact that when firm $f_1$ enters market $m_2$, it moves some of its production to $m_2$ and thus decreases its supply to market $m_1$. Firm $f_2$, on the other hand, responds to less aggressive competition in $m_1$ by increasing its supply to this market eventually leading to a lower profit for firm $f_1$ in market $m_1$. The increase in $f_2$'s output in $m_1$ depends on the level of complementarity between links $(f_1, m_2)$ and $(f_2, m_1)$. Utilizing the results from Section 3, we obtain that firm $f_2$ responds more aggressively, i.e., increases its supply to market $m_1$ more, in the network of Figure 2(b)) and as a consequence entry is not profitable for $f_1$ in this network. It is important to note that the difference between the two networks depicted in Figure 2 is not local to firm $f_1$. It is not hard to extend this example such that the difference between the two networks is arbitrary far from firm $f_1$, i.e., it involves firms and markets that seemingly should not affect firm $f_1$'s profits. Thus, to determine the effect of entry in a new market on the profits of a firm, one has to take into account the entire network topology.

The remainder of this subsection expands on the discussion above and provides a concrete characterization of the effects on firm $i$'s profits when it enters market $m_k$, i.e., link $(i, k)$ is added to graph $G$. We restrict attention to links $(i, k)$ that have positive marginal profit for firm $i$ in the equilibrium for graph $G^*$. As one would expect, this is without any loss of generality according to Lemma 4.2 below.

**Lemma 4.2.** If link $(i, k)$ has a negative marginal profit in $q^*$, then the equilibrium after entry remains unchanged.

Note that entering a new market, i.e., adding a new link to the competition graph, affects not only the output of links, but also whether competitors view their actions in different markets as complements or substitutes. For the sake of analytical tractability, we assume that production in all links is positive at equilibrium for both network structures. Let $W'$ denote the $(|E|+1) \times (|E|+1)$ weight matrix defined over $G \cup (i, k)$ and $q'$ denote the equilibrium for the new competition structure. Also, let $S'_i$ and $P'_k$ denote the total production of firm $f_i$ and the price in market $m_k$ in the new equilibrium respectively. The relation between $q^*$ and $q'$ is given by the following proposition.

**Proposition 4.3.** Consider firm $f_i$ entering market $m_k$. Then, the production quantities in the new equilibrium are given by

$$q' = q^* + Z \times q_{ik},$$

![Fig. 2](image-url)
where for every link \((i_t, k_t)\) of the original graph \(G\),
\[
z_{i_tk_t} = - \sum_{(j,l) \in E} \psi_{i_tk_t,jl} \cdot w_{j,l,ik},
\]
where we recall that \(\Psi = [I + aW]^{-1}\).

Proposition 4.3 thus implies that when firm \(f_i\) enters market \(m_k\), then the supply of firm \(f_i\) to market \(m_k\) increases (decreases) proportionally to \(q_{ik}\) on the even (odd) paths to link \((f_i,m_k)\).

Finally, proposition 4.4 provides a characterization of the profit change for firm \(i\).

**Proposition 4.4.** Consider firm \(f_i\) entering market \(m_k\) and let \(q_{ik}^{*}\) denote the production quantity that \(f_i\) supplies to \(m_k\) in the resulting equilibrium. Then, the profit of firm \(f_i\) changes as
\[
\Delta \pi_i = q_{ik}^* P_k^* + \sum_{i \in F_i} \left( q_{ik}^* z_i (P_i - \beta q_{ik}^* z_i) - q_{ik}^* \beta q_{ik}^* z_i \right) - c (\Delta S_i^2 + 2 S_i \Delta S_i), \tag{9}
\]
where \(z_i = \sum_{(j,l) \in G \setminus m} z_{jl} \).

The intuition behind the Proposition 4.4 is best understood by considering firm \(f_i\) entering a new market \(m_k\) as a monopoly. Then, Proposition 4.4 states that the effect of entry on firm \(f_i\)'s profits can be decomposed in three terms: First, term \((*)\) illustrates the direct effect of entry as \(q_{ik}^* P_k^*\) is equal to the profit that firm \(f_i\) obtains in market \(m_k\). Term \((\ast)\) captures the network effect, i.e., it is equal to the change in the profit firm \(f_i\) obtains from its operations in markets other than \(m_k\), and finally term \((\dagger)\) is equal to the difference in production costs before and after entry. Note that if \(f_i\)'s competitors do not respond to the entry, then profits for \(f_i\) always increase. However, as we discussed earlier, firms respond to the event of entry by increasing their production supply to the markets they share with firm \(f_i\), which may lead to a decrease in firm \(i\)'s profits in those markets.

The extent to which competitors respond to firm \(f_i\) entering market \(m_k\) depends on the level of strategic complementarity between the new link \((i,k)\) and link \((j,l)\), where \(m_l \in F_i\). From a network perspective, the level of strategic complementarity (and thus the extent of a competitor's strategic response) increases with the number of even paths between \((i,k)\) and \((j,l)\). Equivalently, if link \((i,k)\) corresponds to a node with high centrality in the line graph \(L(G')\), then firm \(f_i\)'s competitors respond aggressively to the event of entry. This further implies that firm \(f_i\)'s profits are adversely affected, thus making entry less profitable or even unprofitable. For instance, entry for firm \(f_1\) to market \(m_2\) is profitable for the example depicted in Figure 2(a) whereas it is unprofitable for the one in Figure 2(b) since edge \((f_2,m_3)\) creates an additional path of even length between links \((f_1,m_2)\) and \((f_2,m_3)\), thus increasing the level of strategic complementarity between them (and leading to a more aggressive response from \(f_2\) in Figure 2(b)).

The following example based on Figure 3 further illustrates the intuition behind Proposition 4.4. In this example, the difference between the two networks is a single link, i.e., the one connecting firm \(f_2\) with market \(m_1\). Link \((f_2,m_1)\) increases the level of strategic complementarity between \((f_2,m_3)\) and \((f_3,m_4)\), however it decreases the

\[\text{Through the path } (f_1,m_2) \rightarrow (f_1,m_1) \rightarrow (f_2,m_1) \rightarrow (f_2,m_3) \rightarrow (f_2,m_1).\]
level of strategic complementarity between \((f_1, m_2)\) and \((f_3, m_4)\). Market \(m_2\) is much “larger” than \(m_3\), and thus the latter effect dominates the former and it is profitable for firm \(f_3\) to enter market \(m_4\) for the setting of Figure 3(b).

![Diagram](image)

Fig. 3. In both figures, \(\alpha = [1, 4, 1, 1.8]^T\), \(\beta = [1, 2, 1, 1]^T\), and \(c = [1, 1, 1]^T\). It is not profitable for firm \(f_3\) to enter market \(m_4\) for the setting in Figure 3(a), whereas it is profitable for the one in Figure 3(b).

Finally, we turn our attention on the effect of entry to the aggregate consumer surplus. Similar to Lemma 5.3, one can show that when firm \(f_i\) enters market \(m_k\), consumer surplus in market \(m_k\) increases (as the price in the market decreases). However, entry has an ambiguous effect to the consumer surplus for the rest of the markets. In particular, let us first define our quantity of interest, the aggregate consumer surplus in the environment.

Definition 4.5. The aggregate consumer surplus for Cournot game \(CG(a, \beta, c, G)\) is defined as the sum of the consumer surplus terms of all markets

\[
CS = \sum_{k=1}^{m} \frac{(\alpha_k - P_k)^2}{2\beta}.
\]

Interestingly, we show below that increasing competition, i.e., adding links, can actually lead to a decrease in the aggregate consumer surplus. This can be roughly explained as follows: a new link \((i, k)\) may “spread” the competition along the network structure, i.e., lead firms move (part of) their production away from a dense area of the network where consumers benefit from intense competition between the firms to an area where competition is significantly less intense. Proposition 4.6 formalizes this intuition. In particular, it states that if we add a link \((i, k)\) to a network and the associated production quantity \(q_{ik}\) is positive in the resulting equilibrium, then the new price in market \(l\) is going to be equal to

\[
P_l' = P_l - \beta z l q_{ik}.
\]

So the price in markets that are connected with paths of odd length to link \((i, k)\) will increase, whereas the price in the rest of the markets will decrease. Finally, note that consumer surplus is a convex function of the price vector.

Proposition 4.6. Adding link \((i, k)\) to network \(G\) results in the following change in the aggregate consumer surplus:

\[
\Delta CS = \sum_{l=1}^{m} \left(\frac{\beta z l q_{ik}^2}{2} - (\alpha_l - P_l) z l q_{ik}' \right).
\]
As a way of gaining intuition on the effect entry to the aggregate consumer surplus, consider the example given in Figure 4.1. Firm $f_4$ faces intense competition in the original environment, which results in low prices for the markets that $f_4$ participates in. Assume now that firm $f_4$ enters market $m_5$, which belongs to a set of markets for which competition is low. Firm $f_4$ in the resulting equilibrium will move most of its production to the new market and aggregate welfare will decrease.

Fig. 4. Let $\alpha = 10$, $\beta = 1$, and $c = 10$. When firm $f_4$ enters market $m_5$, aggregate consumer surplus decreases.

4.2. Horizontal Mergers

This subsection studies horizontal mergers between firms in the networked market environment described in Section 2. As the effect on the profits of the merging firms, the “insiders”\(^4\), should presumably be positive (otherwise, the firms would have no incentive to initiate the merger), the analysis is mostly concerned with the profits of outsiders and the welfare of consumers in the equilibrium that is established after the merger. Farrel and Shapiro [1990] study the same question in a single Cournot market and provide general conditions under which mergers that are profitable for insiders also raise welfare.

Much of the antitrust analysis in real-world markets is centered around changes in the level of concentration in the market that can be attributed to the merger. A reasonable way to extend the analysis to a networked environment is to consider each of the markets in which at least one of the insiders participates and allow mergers when the predicted change in concentration is not too high. However, such an approach would essentially treat each market in isolation and potentially overlook second order network effects. The following example illustrates this pitfall. In the three market setting depicted in Figure 5, considering markets $m_1$ and $m_3$ in isolation would likely lead to a favorable response regarding a potential merger between firms $f_3$ and $f_4$, since the markets they participate in are sufficiently competitive. However, this reasoning is somewhat misleading. Firms $f_5$ and $f_6$ would react to less (more) aggressive competition in markets $m_1$ ($m_3$) respectively and potentially create a captive market in $m_2$. This second order network effect illustrates that considering each market in isolation may be incomplete and motivates our discussion on mergers in a networked environment.

This issue was one of the main concerns of the merger between US Airways and American Airlines:

*The merger, which would result in the creation of the world’s largest airline, would substantially lessen competition for commercial air travel in local markets throughout the United States and result in passengers paying higher airfares and receiving less service.*\(^5\)

\(^4\)We borrow this terminology from Farrel and Shapiro [1990]

\(^5\)http://www.justice.gov/opa/pr/2013/August/13-at-909.html
For this example consider $c = 1$ and $\beta = 1$. Also assume $\alpha_1 = 1$, $\alpha_2 = 0.3$ and $\alpha_3 = 1$. Due to symmetry firms $f_2$ and $f_6$ have the same market share in $m_2$ before the merger. If firms $f_3$ and $f_4$ merge, then (i) their joint production decreases in market $m_1$; (ii) $f_4$ increases its production in $m_3$; (iii) $f_6$ moves a fraction of its production from $m_3$ to $m_2$; (iv) $f_5$ finds market $m_1$ more profitable than $m_2$. Thus, although consumer welfare in markets $m_1$ and $m_3$ does not decrease substantially, competition in market $m_2$ is significantly lower in the post-merger equilibrium and thus the overall effect of the merger on welfare in ambiguous.

As should be evident from the example above, measuring the overall effect of a merger on total welfare may not be as straightforward since one would potentially need to study how changes in firms’ actions propagate across the network. The following lemma serves as a starting point in our approach to study mergers in a networked environment. Following Farrel and Shapiro [1990] we do not impose any assumptions on how the merger affects the insiders’ production costs as these are typically hard to observe or predict. Later in the section we specialize our discussion to a specific form for a merger and provide more detailed results.

Denote the set of merging firms, i.e., the insider, by $I$ and assume that their merger results in a change of their total output in market $m_k$ that is equal to $dQ_{I,k}$. Moreover, let $O$ and $G^O = G \setminus I$ denote the rest of the firms, the outsiders, and their subnetwork respectively. Define $W^{G^O}$ as in equation (1) and let $\Psi^{G^O} = [I + aW^{G^O}]^{-1}$. Then,

**Lemma 4.7.** The change in the production quantity that outsider firm $f_i$ supplies to market $m_k$ is given as a function of the changes in the insiders’ output and the outsiders’ network structure $G^O$ as follows

$$dq_{ik} = \sum_{m_l \in N(I)} \psi^{G^O}_{ik,l} dQ_{I,l},$$

where $\psi^{G^O}_{ik,l} = \sum_{j \in O} \psi^{G^O}_{ik,jl}$.

Lemma 4.7 provides a relation between the post-merger production quantities of the insider firms, i.e., the firms that participate in the merger, and those of the outsider firms. This relation can be helpful for assessing the overall effect of a merger on welfare as it provides a closed form expression for the changes in both prices and market concentration. Concretely, the regulator can use this relation to provide a set of constraints on the post-merger equilibrium supply of insider firms that any merger has to satisfy. For instance, such constraints were imposed in the merger between US-airways and American Airlines in 2013. In particular, the Department of Justice, as a condition for allowing the merger, required that the airlines gave up landing and takeoff slots and gates at “seven key constrained airports.” The slots were to go to low cost airlines, “resulting in more choices and more competitive airfares for consumers.” We strongly believe that Lemma 4.7 can be extremely useful in such a setting as it

---

6For more information, see http://www.justice.gov/opa/pr/2013/November/13-at-1202.html
allows one to quantify the effects of a merger taking also the network interactions into account.

Below we consider a natural form of a merger between two firms: suppose that if firms $f_i$ and $f_j$ merge then they produce so as to maximize their joint profit. Let $\hat{q}$ denote the vector of equilibrium production quantities after the merger. The joint profit of firms $f_i$ and $f_j$ is given by

$$\pi_{ij}(\hat{q}) = \pi_i(\hat{q}) + \pi_j(\hat{q}).$$

Note that since the structure of payoff functions after the merger is different from our original framework, Theorem 3.1 might not hold and in fact there might be cases that we have infinitely many equilibria. However, we can show that the equilibrium always exists and in most cases it is unique. Moreover, when there are multiple equilibria, all equilibria are equivalent in a sense that the total supply of merger and each other firm to each market is same across all equilibriums.

**Definition 4.8.** We say two post merger equilibriums $q$ and $q'$ are equivalent if and only if for every market $k$, $q_{ik} + q_{jk} = q'_{ik} + q'_{jk}$.

Note that since by fixing the supply of merger in different markets the response of outsider firms is unique, the above definition implicitly mean that if two post-merger equilibriums $q_1$ and $q_2$ are equivalent, then for any outsider firm $t$ and every market $k$ we have $q_{tk} = q'_{tk}$.

**Theorem 4.9.** Assume two firms $f_i$ and $f_j$ merge and the joint profit is given by

$$\pi_{ij}(\hat{q}) = \pi_i(\hat{q}) + \pi_j(\hat{q}),$$

then

(i) The post-merger equilibrium always exist.

(ii) If two firms $f_i$ and $f_j$ do not share market then the post-merger equilibrium is unique and is the same as pre-merger equilibrium.

(iii) If $\lambda_{\min}(W) \neq -(2c + \beta)$ the post-merger equilibrium is unique.

(iv) In a case with multiple equilibria, all equilibriums are equivalent.

**Proposition 4.10.** If the merged firms share markets, then the total supply in the markets they share decreases whereas the total supply in the markets in which exactly one of them supplies to, increases. The aggregate production of each one of the firms that participate in the merger decreases.

It is important to note that a merger in a single Cournot market always benefits the entire set of firms and leads to a decrease in consumer welfare. In a networked environment, the effect is not symmetric, and one needs to take the entire network structure into account in order to assess the effect on aggregate welfare.

4.3. Forming a Cartel

We will study the case where a single cartel including all the firms is formed. To focus on the effect of the network structure, we will simplify our model by assuming that all the markets and all the firms are homogenous among themselves. Hence, the profit function of a firm $f_i$ is

$$\pi_i(q) = \alpha \sum_{m_k \in F_i} q_{ik} - cS^2_i - \beta \sum_{m_k \in F_i} q_{ik}\bar{c}_k.$$
Suppose all the firms in the network form a cartel which maximizes the total profit of the firms. Given a supply vector $q$, the profit of the cartel is

$$\Pi(q) = \sum_{f_i \in F} \pi_i(q) = \alpha \sum_{(i,k) \in E} q_{ik} - c \sum_{f_i \in F} S_i^2 - \beta \sum_{m_k \in M} C_k^2.$$ 

First, we will characterize the optimal cartel supply in Proposition 4.11. In a complete bipartite network, due to its symmetry, it is easy to calculate the cartel supply. We next establish that for a class of networks, the cartel supply is equal to those in their completed bipartite graphs (Propositions 4.11 & 4.12). In Proposition 4.14, we provide a network decomposition to calculate the cartel supply.

**Proposition 4.11.** Given a Cournot game $G(\alpha, \beta, c, G)$, the supply vector $q$ maximizes the cartel’s profit if and only if

$$\forall (i,k) \in E \begin{cases} 
\text{if } q_{ik} \neq 0, \text{ then } \alpha = 2c\bar{S}_i + 2\beta\bar{C}_k \\
\text{if } q_{ij} = 0, \text{ then } \alpha < 2c\bar{S}_i + 2\beta\bar{C}_k.
\end{cases}$$

The conditions in Proposition 4.11 are the first order conditions to maximize $\Pi(q)$. Since the profit functions of firms are strictly concave in their supply, the cartel maximizes its profit by distributing the markets among its members as equally as possible within the network $G$. This means smoothing out both the supplies by firms and consumptions in markets. If $q$ is a vector of supplies which maximizes the cartel’s profit, then for a firm $f$, and any two different markets $m_k, m_l \in F_i$,

$$\tilde{q}_{ik}, \tilde{q}_{il} \neq 0 \Rightarrow \tilde{C}_k = \tilde{C}_l \\
\tilde{q}_{ik} = 0 \text{ and } \tilde{q}_{il} \neq 0 \Rightarrow \tilde{C}_k > \tilde{C}_l.$$ 

Similarly, for a market $m_k$ and any two different firms $f_i, f_j \in M_k$,

$$\tilde{q}_{ik}, \tilde{q}_{jk} \neq 0 \Rightarrow \tilde{S}_i = \tilde{S}_j \\
\tilde{q}_{ik} = 0 \text{ and } \tilde{q}_{jk} \neq 0 \Rightarrow \tilde{S}_i > \tilde{S}_j.$$ 

We are not guaranteed a unique solution. Indeed, we will see that, in general, there exists a continuum of solutions to the problem of maximizing the cartel’s profit. But all such supply vectors will lead to the same supply by all firms and the same consumption at each market.

Now we will find a vector of supplies that satisfies the first order conditions. Given a subgraph $G_0 = (F_0 \cup M_0, E_0)$ of $G$, consider the cartel’s profit maximizing supplies and market consumptions in its completed graph $\tilde{G}_0$. Clearly the levels are identical across firms and across markets. Let $\tilde{S}_0$ be the supply by a firm in $\tilde{G}_0$ and $\tilde{C}_0$ the consumption at a market in $\tilde{G}_0$. If $|M_0| = n_0$ and $|F_0| = n_0$, then direct calculation shows that

$$\tilde{S}_0 = \frac{\alpha n_0}{2\beta n_0 + \beta n_0} \quad \text{and} \quad \tilde{C}_0 = \frac{\alpha n_0}{2\beta n_0 + \beta n_0}.$$ 

These values depend only on the market/firm ratio. For two graphs $G_0 = (F_0 \cup M_0, E_0)$ and $G_1 = (F_1 \cup M_1, E_1)$,

$$\frac{|M_0|}{|F_0|} = \frac{|M_1|}{|F_1|} \Rightarrow \tilde{S}_0 = \tilde{S}_1 \quad \text{and} \quad \tilde{C}_0 = \tilde{C}_1.$$ 

We will use the quantities at the complete graph as benchmarks while calculating the amounts at incomplete bipartite graphs.

Given $G$, we say that a supply vector $q$ is feasible if all supplies in $q$ are non-negative. The set of feasible flow vectors in $G_0$ is a subset of the set of feasible flow vectors
in its completed graph $G_0$. Then given the profit maximizing levels of supply $\tilde{S}_0$ and consumption $\tilde{C}_0$ at $G_0$, if these amounts are possible at $G_0$, then they must maximize the cartel’s profit at $G_0$ also.

**Proposition 4.12.** Let $G_0 = (F_0 \cup M_0, E_0)$ be a subgraph of $G$. If the supply of $\tilde{S}_0$ by each firm in $F_0$ is possible without exceeding the consumption $\tilde{C}_0$ in any market in $M_0$, then these levels maximize the cartel’s profit in $G_0$.

To calculate the cartel supply, we introduce two graphical definitions.

**Definition 4.13.** An inclusive subgraph $G_0 = (F_0 \cup M_0, E_0)$ of $G$ is such that $G_0$ is connected and

$$M_0 = \bigcup_{f_i \in F_0} F_i.$$

An inclusive subgraph\(^7\) includes all the markets to which its firms were connected in graph $G$. Let $W(G) = \{G_0 \subseteq G : G_0$ is inclusive $\}$ be the set of inclusive subgraphs in $G$. Since $G$ is an inclusive subgraph of itself, $W(G) \neq \emptyset$. In graph $G_3$ in Figure 6, the subgraph $G_0^3$ that we encircle is inclusive. It includes $f_1$ and all the markets that $f_1$ is connected to.

![Fig. 6. Inclusive subgraph $G_0^3$.](image)

Given a subset of markets $M_0 \subseteq M$ and a subset of firms $F_0 \subseteq F$, $\frac{|M_0|}{|F_0|}$ is the average number of markets per firm. A least inclusive subgraph $\hat{G} = \left(\hat{F} \cup \hat{M}, \hat{E}\right)$ of $G$ is such that

$$\frac{\hat{M}}{\hat{F}} < \frac{|M|}{|F|} \text{ and } \left(\hat{F} \cup \hat{M}, \hat{E}\right) \in \arg\min_{(F_0 \cup M_0, E_0) \in W(G)} \frac{|M_0|}{|F_0|}.$$

The first requirement for $\hat{G}$ to be a least inclusive subgraph of $G$ is for it to have a strictly smaller market/firm ratio than $G$. This means that a graph does not necessarily have a least inclusive subgraph. For example, a complete bipartite graph has no least inclusive subgraphs. The second requirement is for $\hat{G}$ to have the smallest market/firm ratio among the inclusive subgraphs of $G$. A least inclusive subgraph is inclusive and formed by a set of the least connected firms. There should be no firms in $G$ which are strictly worse than them with respect to connectedness.

In Figure 6, the subgraph $G_0^3$ is not least inclusive, because the ratio of markets to firms in it is 1. This ratio for graph $G_3$ is also 1. The subgraph $G_3^1$ of $G_3$, as encircled in

---

\(^7\)See Bochet et al. (2010) for the relationship between inclusive subgraphs and the Gallai–Edmonds decomposition (Ore 1962) of a bipartite graph.
Cournot Competition in Networked Markets

Figure 7 below, is a least inclusive subgraph. Its market/firm ratio is lower than that of $G_3$ and there is no other inclusive subgraph of $G_3$ with a lower ratio.

![Fig. 7](image)

Fig. 7. Least inclusive subgraph $G_1$.

If $\hat{G}$ is a least inclusive subgraph of $G$, then $\hat{G}$ cannot have a least inclusive subgraph of its own. Any inclusive subgraph of $\hat{G}$ is also inclusive in $G$. If $\hat{G}$ had a least inclusive subgraph with a smaller market/firm ratio than $\hat{G}$, this would have contradicted $\hat{G}$ having the smallest market/firm ratio in $G$.

Now we show that if a subgraph $G_0 = (F_0 \cup M_0, E_0)$ of $G$ has no least inclusive subgraph, then the supply of $S_0$ by each firm in $F_0$ is possible without exceeding the consumption $\tilde{C}_0$ in any market in $M_0$.

**Proposition 4.14.** Let $G_0 = (F_0 \cup M_0, E_0)$ of $G$ be an inclusive subgraph. If $G_0$ has no least inclusive subgraph, then the supply of $S_0$ by each firm in $F_0$ is possible without exceeding the consumption $\tilde{C}_0$ in any market in $M_0$.

The result means that if a network has no least inclusive subgraph, it can be treated as a complete network. All the firms are symmetric under efficiency. Hence there is no difference between this problem and the simple Cournot with a single market.

To prove Proposition 4.14, we start with a firm $f_i$ of a graph $G_0$ with no inclusive subgraphs. This firm must be able to supply $S_0$ without exceeding the consumption $\tilde{C}_0$ in any of its markets. If not, that firm with its markets would have formed a least inclusive subgraph in $G_0$. Next, we add a new firm to this subgraph and iteratively show that such supply levels must be possible for all inclusive subgraphs of $G_0$ that contain $f_i$. As $G_0$ is an inclusive subgraph of itself, this proves that such supply levels are possible in $G_0$.

**Decomposing the network.** Now we will break down the network $G$, so that the cartel's optimization problem in each subnetwork is independent from the other ones. We will sequentially cut out least inclusive subgraphs. Hence, they will not have any least inclusive subgraphs of their own. We will continue until we reach a subgraph which has no least inclusive subgraphs. Then in each subgraph, the cartel optimal supplies at each firm and consumptions at each market will be equal to the amounts in their completed graphs. The next result follows from Propositions 4.12 and 4.14.

**Proposition 4.15.**

Given a network of markets and firms $G$, the following algorithm calculates the optimal cartel supply by each firm and consumption at each market.

**Step 1:** Take $G$. Suppose $G = (F \cup M, E)$ has no least inclusive subgraph. Then the supply by a firm $f_i$ and consumption at a market $m_k$ are equal to the levels in a complete bipartite graph with nodes $F \cup M$, and we are done.
Suppose $G = (F \cup M, E)$ has a least inclusive subgraph. Let $G_0 = (F_0 \cup M_0, E_0)$ be the largest least inclusive subgraph$^8$ in $G$. Then, the supply by a firm $f_i \in F_0$ is $\tilde{S}_0$, and the consumption at a market $m_k \in M_0$ is $\tilde{C}_0$.

**Step 2:** Now, for the rest of the firms and markets, apply **Step 1** to $G \setminus G_0$. In this way, we obtain a series of regions out of $\mathfrak{g}$ with a strictly increasing market per firm ratio. In each of them, the supplies would equal to the levels in their respective completed graphs.

So, given a subgraph $G_0 = (F_0 \cup M_0, E_0)$ obtained from the above decomposition, supply by a firm and the consumption at each market in $G_0$ are

$$
\tilde{S}_0 = \frac{\alpha m_0}{2cm_0 + 2\beta n_0}, \quad \tilde{C}_0 = \frac{\alpha m_0}{2cm_0 + 2\beta n_0}.
$$

These levels satisfy the first order conditions within each region. Moreover, less connected firms have lower supplies and less connected markets have lower consumptions. Since there are no flows between different regions, the first order conditions hold for graph $G$ as well.

### 5. **CONCLUSION**

This paper studies a model of competition in a networked environment. A bipartite graph determines the set of potential supply relationships. We provide a characterization of the unique equilibrium that highlights the relation between production quantities and supply paths in the underlying network structure. Using this characterization we study the effect on quantities, prices, and welfare of changes of in the network structure that may be the result of entry or a merger between two firms. Although the network is our study is fixed, our analysis can serve as a starting point for further analysis on strategic network formation in the presence of competition.

### REFERENCES


---

$^8$The ratio $\frac{|N_C(F_0)|}{|F_0|}$ is a submodular function of $F_0$, where $N_C(F_0)$ is the set of markets connected to $F_0$. Then at any graph $G$, there exists a unique largest least inclusive subgraph.
Appendix

Proof of Theorem 3.1

Consider the best response function for firm $f_i$'s supply quantity to market $m_k$. From the first order optimality conditions we obtain:

$$q'_{ik} = \begin{cases} \frac{\alpha_k - 2c_i \sum_{l \in F_i, l \neq k} q_{il} - \beta_k \sum_{j \in M_k, j \neq i} q_{jk}}{2(\beta_k + c_i)} & \text{if } \frac{\partial \pi_i}{\partial q_{ik}} \geq 0 \\ 0 & \text{if } \frac{\partial \pi_i}{\partial q_{ik}} < 0 \end{cases}$$

Also note that the supply at a firm-market pair has to be non-negative at equilibrium

$$q^* \geq 0$$

Finally, the marginal revenue associated with a firm-market pair has to be non-positive at equilibrium

$$\frac{\partial \pi_i}{\partial q_{ik}} |_{q^*_{ik}} = \alpha_k - \beta_k q^*_{ik} + \sum_{j \in M_k} q^*_{jk} - 2c_i \sum_{l \in F_i} q^*_{il} \leq 0$$

This set of equations can be rewritten in matrix form as

$$-\alpha + Dq^* \geq 0$$

where recall that $D$ is a $|E| \times |E|$ matrix defined in Equation 4.

Lastly, we have $\frac{\partial \pi_i}{\partial q_{ik}} |_{q^*_{ik}} = 0$ for every link $(i, k)$ at equilibrium, so if we write $\frac{\partial \pi_i}{\partial q_{ik}} |_{q^*_{ik}} = 0, \forall (i, k) \in E$, in the matrix form we get:

$$q^*(-\alpha + Dq^*) = 0.$$  

The first order equilibrium conditions (11), (12), and (13) constitute a linear complementarity problem $LCP(-\alpha; D)$. According to the results in [Samelson et al. 1958], problem $LCP(-\alpha; D)$ has a unique solution if and only if all the principal minors of $D$ are positive. Positive definite matrices satisfy this condition and thus what remains to be shows is that $D$ is positive definite for any given Cournot game.

One way to show the positive definiteness of $D$ is by showing that there exists a matrix $R$ with independent columns such that $D = RTR$. Let $R$ be a $(|E| + m + n) \times |E|$ matrix which can be written as a block matrix formed by matrices $A$ and $B$ such that

$$R = \begin{bmatrix} A \\ B \end{bmatrix}$$

where $A$ has size $|E| \times |E|$ and $B$ has size $(m + n) \times |E|$.

Matrix $A$ is a diagonal matrix such that both columns and rows of $A$ correspond to the links at $E$ and

$$A_{ik,jl} = \begin{cases} \sqrt{\beta_k} & \text{if } i = j, k = l \\ 0 & \text{otherwise} \end{cases}$$

The columns of matrix $B$ correspond to the links at $E$ and the first $n$ rows correspond to the firms, the last $m$ rows correspond to the markets and

$$B_{t,(i,k)} = \begin{cases} \sqrt{2c_i} & \text{if } t \leq n, t = i \\ \sqrt{\beta_k} & \text{if } t > n, t = n + k \\ 0 & \text{otherwise} \end{cases}$$

9We do not explicitly label the links from 1 to $|E|$ to save notation. It does not matter how the links are labelled as long as a fixed labelling is used for all matrices.
Since \( A \) is a diagonal matrix where all diagonal elements are non-zero, the columns of \( R \) are independent. It is straightforward to check that \( D = R^T R \).

Hence \( D \) is positive-definite and for every given Cournot game the linear complementarity problem \( LCP(-\alpha; D) \) has a unique solution. Finally, we verify that the second order equilibrium conditions are satisfied. We label the links, i.e., firm-market pairs, that correspond to firm \( f_i \) from 1 to \( |F_i| \), i.e., we let \( F_i = \{m_1, \ldots, m_{|F_i|}\} \). Then the Hessian matrix of the profit function \( \pi_i \) is
\[
H = \begin{bmatrix}
 h_{kl} \end{bmatrix}_{|F_i| \times |F_i|}
\]
where
\[
h_{kl} = \begin{cases}
 -2(\beta_k + c_i) & \text{if } k = l \\
 -2c_i & \text{otherwise.}
\end{cases}
\] (14)

Let \( H' = -H \). We can use the same technique applied for \( D \) to show that \( H' \) is positive definite. Hence, \( H \) is negative definite. This establishes that the solution of the linear complementarity problem \( LCP(-\alpha; D) \) is the unique equilibrium of the Cournot game \( CG(a, \beta, c, G) \).

**Proof of Corollary 3.5**

According to the proof of the Theorem 3.1 one can see that the equilibrium does not feature inactive links if and only if we have a vector \( q \) such that
\[
-\alpha + Dq \geq 0 \\
q(-\alpha + Dq) = 0 \\
q > 0
\]

So by using Farka’s lemma one can see that the above is equivalent to, there is no vector \( x \in \mathbb{R}^{|F|} \) such that
\[
D^T x > 0 \\
\alpha x = 0
\]

**Proof of Theorem 3.3**

For every active link \((i, k)\), it should be the case that \( \frac{\partial U_i(a, \beta, c, G)}{\partial q_{ik}} = 0 \), which implies that
\[
q_{ik} = \frac{\alpha_k - 2c \sum_{l \in F_i, l \neq k} q_{il} - \beta \sum_{j \in M_k} q_{jk}}{2(\beta + c)} = \frac{\alpha_k}{2(\beta + c)} - \sum_{(j,l) \in E(G^*)} (aW)_{ik,jl} q_{jl}.
\]

This further implies that
\[
q^* = a\alpha - aW q^* \Rightarrow q^* = [I + aW]^{-1} a\alpha.
\] (15)

In order to show the second part of the theorem note that Expression (15) can be rewritten as:
\[
q^* = [I - (-aW)]^{-1} a\alpha.
\]

Matrix \([I - (-aW)]^{-1}\) can be written as the power series of matrix \((-aW)\) if and only if the spectral radius of \(-aW\) is less than 1, i.e., if and only if \(-1 < \lambda_{\text{min}}(aW) \leq \lambda_{\text{max}}(aW) < 1\).

Finally, we show that for every Cournot game \( \lambda_{\text{min}} > -1 \). Define the \((n + m) \times |E|\) edge incident matrix \( B \) of graph \( G^* \) as follows:
\[ B_{\epsilon,i,k} = \begin{cases} \sqrt{\frac{2c}{2(c+\beta)}} & \text{if } v \leq n, \ i_v = v, \\ \sqrt{\beta} & \text{if } v > n, \ k_v = v - n, \\ 0 & \text{otherwise.} \] 

Then it is easy to see that:

\[ aW = B^T B - \frac{2c + \beta}{2(c + \beta)} I \]

Note that \( B^T B \) is a positive semidefinite matrix and thus all of its eigenvalues are non-negative and since \( \frac{2c + \beta}{2(c + \beta)} < 1 \), we conclude that the following holds:

\[ \lambda_{\text{min}} \geq -\frac{2c + \beta}{2(c + \beta)} \geq -1. \]

So to be able to rewrite \([I - (-aW)]^{-1}\) as a power series, i.e., Expression (3) it has to be the case that \( \lambda_{\text{max}} \leq 1 \).

**Proof of Corollary 3.7**

We provide a proof for the case when \( 2c \geq \beta \). The proof for the case when \( 2\beta \geq c \) is identical. First, remove the links in the line graph \( L(G^*) \) that correspond to markets, i.e., all links with weight \( \beta \), and denote the new weighted matrix by \( W' \). Now since every link in the remaining line graph has weight \( 2c \) we have that:

\[ W' = 2c \times H, \]

where \( H \) is a unweighted adjacency matrix corresponding to \( W' \). Next, note that we know that \( \lambda_{\text{max}}(aW') \leq \lambda_{\text{max}}(aW) \leq 1 \), since removing links always decreases the maximum eigenvalue. So since

\[ \lambda_{\text{max}}(aW') = \frac{2c}{2(c + \beta)} \lambda_{\text{max}}(H) \geq \frac{1}{3} \lambda_{\text{max}}(H), \]

it turns out that \( \lambda_{\text{max}}(H) \leq 3 \). Also note that for every unweighted undirected graph \( H \) we have:

\[ \max\{\sqrt{\deg_{\text{max}}(H)}, \deg(H)\} \leq \lambda_{\text{max}}(H), \]

where \( \deg_{\text{max}}(H) \) and \( \deg(H) \) are the maximum and average degree of graph \( H \) respectively. Thus we have \( \deg_{\text{max}}(H) \leq 9 \) which implies that each firm supplies to at most 10 markets, and also \( \deg(H) \leq 3 \) means that in average in firm supplies to 4 markets. This concludes the proof of the first part of the lemma.

**Proof of Corollary 3.8**

Let \((i_1, k_1), (i_2, k_2), \ldots, (i_t, k_t)\) be the active links in the equilibrium of \( CG(\alpha, \beta, c, G) \). The idea is to reconstruct the matrix \( \Psi = [I + aW]^{-1} \) by adding links \((i_t, k_t)\) one by one. In other words, let \( \Psi(t) \) be the inverse matrix when only the first \( t \) links are available, and so we have \( \Psi = \Psi(t^*) \). Define \( W' = aW \).

We prove the claim by induction on \( t \). Assume that for every \( t' < t \), the entries of \( \Psi(t') \) are increasing with even paths and decreasing with odd paths i.e. each entry \( \psi(t')_{ik,jl} \) is a weighted sum of even paths from \((i, k)\) to \((j, l)\) minus weighted sum of odd paths from \((i, k)\) to \((j, l)\) where paths are defined in a graph induced by first \( t' \) links (i.e. links \((i_1, k_1), (i_2, k_2), \ldots, (i_{t'}, k_{t'})\)). We will show that the entries of \( \Psi(t) \) are increasing.
with even paths and decreasing with odd paths where paths are defined in the graph induced by first \( t \) links. The idea of proof is as follow: We will show that entries of \( \Psi(t) \) can be written as entries of \( \Psi(t-1) \) plus the even paths that use the newly added link \((i_t, k_t)\) minus the odd paths that use the link \((i_t, k_t)\). Assume at \( t = 1 \), we have \( \Psi(1) = 1 \) and the weight matrix is \( W' \) is fully zero. We add the edges one by one and update the weight matrix \( W' \) and \( \Psi(t) \). Note that we have:

\[
\Psi(t) = \left[ 1 + \sum_{l=1}^{t} \left( e_l[w_{l1}', w_{l2}', \ldots, w_{lt}', 0, \ldots, 0]^T + [w_{l1}', w_{l2}', \ldots, w_{lt}', 0, \ldots, 0]^T \right) \right]^{-1} \\
= \left[ \Psi^{-1}(t-1) + e_l[w_{l1}', w_{l2}', \ldots, w_{lt}', 0, \ldots, 0]^T + [w_{l1}', w_{l2}', \ldots, w_{lt}', 0, \ldots, 0]^T \right]^{-1},
\]

(17)

where \( e_l \) is a \((l^* \times 1)\) column vector where only its \( l'\)th entry is 1 and other entries are all zero. Note that for each \( l \), \( e_l[w_{l1}', w_{l2}', \ldots, w_{lt}', 0, \ldots, 0]^T \) corresponds to the weight of edge from link \( l \) to links \( 1, 2, \ldots, l \) in the line graph \( L(G*) \). Similarly \([w_{l1}', w_{l2}', \ldots, w_{lt}', 0, \ldots, 0]^T\) represents the weight of edges from links \( 1, 2, \ldots, l \) to link \( l \) in \( L(G*) \). So when we add \( l'\)th link to the previous links \((i_1, k_1), \ldots, (i_{t-1}, k_{t-1})\), we should add the weights of edges from link \((i_t, k_t)\) to the previous links and also weights of edges from links \((i_1, k_1), \ldots, (i_{t-1}, k_{t-1})\) to the link \((i_t, k_t)\). So \( e_l[w_{l1}', w_{l2}', \ldots, w_{lt}', 0, \ldots, 0]^T + [w_{l1}', w_{l2}', \ldots, w_{lt}', 0, \ldots, 0]^T \) will be added to the weight matrix and thus we can write the equation 17. Note that since

\[
e_l[w_{l1}', w_{l2}', \ldots, w_{lt}', 0, \ldots, 0]^T + [w_{l1}', w_{l2}', \ldots, w_{lt}', 0, \ldots, 0]^T
\]

has rank two, by using two consecutive Sherman-Morrison update on the RHS of equation 17 we get the following relation between \( \Psi(t) \) and \( \Psi(t-1) \):

\[
\Psi(t) = \Psi(t-1) + \Phi,
\]

Where

\[
\Phi = \begin{bmatrix}
  p_{1t} p_{1t} & p_{1t} p_{2t} & \ldots & p_{1t} 0 & 0 \\
p_{2t} p_{1t} & p_{2t} p_{2t} & \ldots & p_{2t} 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
p_{1t} & p_{2t} & \ldots & C 0 & \vdots \\
0 & 0 & \ldots & 0 0 & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 - C & \end{bmatrix}
\]

and for each \( j < t \), \( p_{ij} = -\sum_i w_{ij}'(t-1) \) and \( p_{jt} = -\sum_i \psi_{ij}(t-1)w_{i}'t \). Also \( C = \sum_i w_{ij}'t \).

According to the induction hypothesis each entry of \( \Psi(t-1) \) is increasing with even paths and decreasing with odd paths. So if we can prove that the entries of matrix \( \Phi \) are increasing with even paths and decreasing with odd paths, we are done. First note that for every \( j < t \), according to its definition, \( p_{ij} \) is increasing with even paths and decreasing with odd paths that start from link \((i_t, k_t)\) and end in link \((i_j, k_j)\). Similarly for ever \( j < t \), \( p_{jt} \) is increasing with even paths and decreasing with odd paths that start from link \((i_j, k_j)\) and end in the link \((i_t, k_t)\). Furthermore, according to the definition of \( C \), one can see that \( C \) is a sum of cycles from link \((i_t, k_t)\) to itself where
even cycles have positive coefficients and odd cycles have negative coefficients. Now for a second assume that \( 1 - C \) is a positive number (We will prove it later). So by looking at entries of \( \Psi(t) \) it is easy to see that each entry of \( \Psi(t) \) is a same entry as \( \Psi(t-1) \) plus even paths that use link \((i_t, k_t)\) and minus odd paths that use link \((i_t, k_t)\). So in overall each entry of \( \Psi(t) \) has a positive relation with even paths and negative relation with odd paths. So the only remaining part is to show that \( 1 - C \) is always positive. Note that since \( \psi(t, t) = 1 + C \frac{1}{1-C} = \frac{1}{1-C} \), and if we prove that \( \psi(t, t) > 0 \) we are done. Since \( I + aW \) is a symmetric positive semi-definite matrix, one can conclude that \( \Psi = [I + aW]^{-1} \) is a symmetric positive semidefinite matrix and thus all its diagonal entries are positive.

Proof of Lemma 4.1

The easiest way to see this is to assume that each market \( l \) has a different \( \alpha \) parameter for each firm \( j \) (Note that in the definition of our model we had \( \alpha_{ik} = \alpha_{jk} \forall i, j \in \mathcal{F} \)). The equilibrium definition will be

\[
q_{jl} = \sum_{(f,m) \in q^*} \psi_{jl, fm} \frac{1}{2(c + \beta)} \alpha_{fm},
\]

In this case an infinitesimal change in \( q_{ik} \) can be seen as an infinitesimal change in \( \alpha_{ik} \), thus we have:

\[
\frac{dq_{jl}}{dq_{ik}} = \frac{\frac{dq_{jl}}{d\alpha_{ik}}}{\frac{dq_{ik}}{d\alpha_{ik}}} = \frac{1}{2(\beta + c)} \psi_{jl, ik} \frac{\psi_{jl, ik}}{\psi_{ik, ik}}
\]

Proof of Lemma 4.2

If the marginal profit corresponding to the additional edge for the original equilibrium allocation is negative, all the constraints of the original linear complementarity problem are still satisfied by the original equilibrium allocation. Thus, the equilibrium remains unchanged even after adding this edge.

Proof of Proposition 4.3

We use the same approach that we used for the proof of Corollary 3.8. Let \( W \) be a weighted adjacency matrix of \( L(G^*) \). When edge \((i, k)\) is added to the equilibrium, if \((i, k)\) has a positive flow in it, the equilibrium graph will change. Let \( G' = G^* \cup (i, k) \) be a new equilibrium network and \( W' \) be the weighted adjacency matrix of \( G' \), then we have following relation between \( W \) and \( W' \),

\[
W' = W + e_i \left[ w'_{*,1}, w'_{*,2}, \ldots, w'_{*,I} \right]^T + \left[ w'_{1*,1}, w'_{2*,1}, \ldots, w'_{I*,1} \right]^T e_i^T.
\]
Note that according to Theorem 3.3 the new equilibrium is \( q' = (I + W')^{-1} A' \alpha \), so we have following:

\[
q' = (I + W + e_1 \cdot [w'_{1,1}, \ldots , w'_{1,t}, \ldots ]^T + [w'_{1,t}, \ldots , w'_{t,t}, \ldots ] e_T)^{-1} A' \alpha \\
= \begin{pmatrix}
\Psi + \begin{bmatrix} p_{1t} + p_{1t} \cdot 1 & p_{1t} + p_{1t} \cdot 2 & \ldots & p_{1t} \\
 p_{2t} + p_{2t} \cdot 1 & p_{2t} + p_{2t} \cdot 2 & \ldots & p_{2t} \\
 \vdots & \vdots & \vdots & \vdots \\
p_{1t} + 1 & p_{2t} \cdot 1 & \ldots & 1 \\
\end{bmatrix} \\
1 - C
\end{pmatrix} A' \alpha
\]

Where the last equation comes from using Sherman-Morrison formula. So according to the above, the production in the newly added edge \((i,k)\) is equal to

\[
q'_{ik} = (e_1 \Psi + \begin{bmatrix} p_{1t} + p_{1t} \cdot 1 & p_{1t} + p_{1t} \cdot 2 & \ldots & p_{1t} \\
 p_{2t} + p_{2t} \cdot 1 & p_{2t} + p_{2t} \cdot 2 & \ldots & p_{2t} \\
 \vdots & \vdots & \vdots & \vdots \\
p_{1t} + 1 & p_{2t} \cdot 1 & \ldots & 1 \\
\end{bmatrix} A' \alpha \\
1 - C
\]

\[
= q^*_{ik} + p_{it} q'_{ik}
\]

So if we write the above equation in a matrix form we get the desired result.

**Proof of Proposition 4.4**

According to the Proposition 4.3, the new equilibrium is

\[
q' = q^* + Z \times q'_{ik}
\]

So the new utility of firm \( i \) is

\[
U'_i = q'_{ik} \mathcal{P}'(k) + \sum_{m \in F_i} q^*_{il}(\alpha_i - \beta \sum_{j \in M_l} q^*_{jl}) - c S_i^2
\]

\[
= q'_{ik} \mathcal{P}'(k) + \sum_{m \in F_i} (q^*_{il} + z_i q'_{ik}) \left(\alpha_i - \beta \sum_{j \in M_l} (q^*_{jl} + z_j q^*_{jl})\right) - c(S_i + \Delta S_i)^2
\]

\[
= U_i + q'_{ik} \mathcal{P}'(k) + \sum_{l \in F_i} \left( q'_{ik} z_{il} (\mathcal{P}(l) - \beta q^*_{ik} z_i) - q^*_{il} \beta q^*_{ik} z_i \right) - c \left( \Delta S_i^2 + 2S_i \Delta S_i \right)
\]

and the proof is complete.
Proof of Lemma 4.7
Note that for each outsider link \((j, l)\), according to the first order optimality condition we have

\[
q_{jl} + dq_{jl} = \frac{\alpha_l}{2(\beta + c)} - \sum_{j_1, t_1} (aW)_{j_1, j_1 t_1} (q_{j_1 t_1} + dq_{j_1 t_1}).
\]

If we subtract the equation 5 from the above equation we get the following for changes in the outsiders links output:

\[
dq_{jl} = - \sum_{j_1, t_1} dq_{j_1 t_1}
\]

Now note that for each outsider link \((j_1, l_1)\) in the RHS of the above equation, we can expand it. So the expansion continue until we reach the absorbing links \(dq_{lk}\) where \(f_l \in \mathcal{I}\) (Insiders’ links are absorbing since we do not need to expand them and their change is fixed). So it is easy too see that this expansion is equal to paths from link \((j, l)\) to markets \(N(\mathcal{I})\) in the \(G_c\) network.

Proof of Theorem 4.9
First let prove part \((ii)\). Note that when two firms \(f_i\) and \(f_j\) do not share market then the corresponding \(LCP\) after merger is exactly the same as the \(LCP\) before merger. So in this case the equilibrium is unique and is the same as the pre-merger equilibrium. Now we prove parts \((i)\) and \((iii)\). We follow the same approach that we used to prove the Theorem 3.1. Similar to the proof of Theorem 3.1 we will construct the linear complementary problem \(LCP(-\alpha, D')\) where \(D'\) is a \(|E| \times |E|\) matrix defined as follow:

\[
D'_{i,k,j,l} = \begin{cases} 
2(\beta + c) & \text{if } i_1 = j_1, k = l \\
2c & \text{if } i_1 = j_1, k \neq l \\
\beta & \text{if } i_1 \neq j_1, k = l \text{ and } \{i_1, j_1\} \neq \{i, j\} \text{ or } k \text{ is not shared market} \\
2\beta & \text{if } i_1 \neq j_1, k = l \text{ and } \{i_1, j_1\} = \{i, j\} \text{ and } k \text{ is a shared market} \\
0 & \text{otherwise.}
\end{cases}
\]

(19)

So again if we prove that \(D'\) is positive semi-definite then \(LCP(-\alpha, D')\) has a solution (though may not unique) and if \(D'\) is positive definite matrix then \(LCP(-\alpha, D')\) has a unique solution. Note that we have the following relation between \(D'\) and \(D\):

\[
D' = D + X
\]

where \(X\) is a \(|E| \times |E|\) matrix defined as follow:

\[
X_{i, k, j, l} = \begin{cases} 
\beta & \text{if } i_1 \neq j_1, k = l \text{ and } \{i_1, j_1\} = \{i, j\} \text{ and } k \text{ is a shared market} \\
0 & \text{otherwise.}
\end{cases}
\]

(20)

According to the Wayl’s theorem and noting that \(\lambda_{\min}(X) = -\beta\), we have

\[
\lambda_{\min}(D') \geq \lambda_{\min}(D) + \lambda_{\min}(X) = \lambda_{\min}(D) - \beta
\]

So if we show that \(\lambda_{\min}(D)\) is at least equal to \(\beta\) then matrix \(D'\) is positive semi-definite and thus the equilibrium always exists. Also in those cases where \(\lambda_{\min}(D) > \beta\), \(D'\) is positive definite and thus there will be a unique equilibrium. Note that

\[
D = \frac{1}{a} BB^T + \beta I
\]

EC’14, June 8–12, 2014, Stanford University, Palo Alto, CA, USA, Vol. , No. , Article 1, Publication date: February 2014.
where matrix $B$ is defined in the equation 16. So since matrix $\frac{1}{a}BB^T$ is symmetric positive semi-definite, one can conclude that $\lambda_{\text{min}}(D) \geq \beta$. Also not that we have

$$\frac{1}{a}BB^T = W + (2c + \beta)I$$

So if $\lambda_{\text{min}}(W) \neq (2c + \beta)$ then $\lambda_{\text{min}}(\frac{1}{a}BB^T) > 0$ and thus $D'$ will be positive definite and there is a unique equilibrium.

Next we will prove the last part of Theorem. We say the post merger equilibrium $q$ is balanced if the total supply of firms $f_i$ and $f_j$ is equal. Also we called a post merger equilibrium connected if there is a shared market $k$ such that both firms $f_i$ and $f_j$ have a positive supply to. We first prove following two useful lemmas:

**Lemma 5.1.** All balanced equilibria are equivalent.

**Proof.** Note that

$$\frac{\partial \pi_{ij}}{\partial q_{iv}} = \alpha_v - \beta q_{iv} - \beta \sum_{l \in M_v} q_{lv} - 2c \sum_{l \in F_i} q_{il} \leq 0$$

(21)

We will show that if we add a link $(i, v)$ to the network, $q$ remains a valid equilibrium. The reason is we have

$$\frac{\partial \pi_{ij}}{\partial q_{iv}} = \alpha_v - \beta(q_{iv} + q_{iv}) - \beta \sum_{l \in M_v} q_{lv} - 2c \sum_{l \in F_i} q_{il} \leq 0,$$

(22)

and so firm $f_i$ has no incentive to supply to the market $v$. So given a network $G$ and merged firms $f_i$ and $f_j$, we creat the network $G'$ by adding all the links from $f_j$ to every market in $F_j$ and $f_j$ to every market in $F_i$. So in the network $G'$ firms $f_i$ and $f_j$ share all the markets also $q$ is also a post-merger equilibrium for network $G'$. Now let’s call this merged firm as firm $x$. For a market $k$

$$\frac{\partial \pi_x}{\partial q_{xk}} = \alpha_k - \beta q_{xk} - \beta \sum_{l \in M_k} q_{lk} - 2c \sum_{l \in F_i} q_{il}$$

(23)

Here, it does not matter whether we write the derivative at the cost of $2c \sum_{l \in F_i} q_{il}$ or $2c \sum_{l \in F_j} q_{jl}$, because according to definition in the balance equilibriums, these two quantities are the same at equilibrium. Finally note that since we have $\sum_{l \in F_i} q_{il} = \sum_{l \in F_j} q_{jl}$ we can rewrite the above equation as follow:

$$\frac{\partial \pi_x}{\partial q_{xk}} = \alpha_k - \beta q_{xk} - \beta \sum_{l \in M_k} q_{lk} - c \sum_{l \in F_i} q_{il}$$

This is the derivative we use without any merger for one firm, with the only difference that here $c_x = \frac{c}{2}$. This means, when the two mergers share all their markets, we can transform the merger LCP to the non-merger LCP which according to the Theorem 3.1 has a unique solution. This uniqueness directly implies that in every balanced equilibrium the total supply of merged firms $f_i$ and $f_j$ is constant. So we have shown all connected equilibriums are equivalent.

**Lemma 5.2.** Every connected equilibrium is balanced.

**Proof.** Assume firms $f_i$ and $f_j$ both have positive supply in the market $k$. Then due to the convexity of cost, the total supply of firms $f_i$ and $f_j$ should be exactly the same, since if $S_i < S_j$ then firm $f_j$ may reduce its supply in the market $k$ by $c$ and firm...
Proof of Proposition 4.10

Here we just show the proof for the case where there is only one shared market, but the proof is extended to the general case. Assume that firms $i$ and $j$ are decided to merge and there is a shared market $k$ between them. Assume two firms $i$ and $j$ are not merged yet and we are at the pre-merge equilibrium $q^*$. The marginal profit of firm $f_i$ supplying to market $m_k$ before the merger is

$$\frac{d\pi_i}{dq_{ik}|_{q^*}} = \alpha_k - 2c \sum_{m_l \in F_i} q_{il}^* - \beta \sum_{f_t \in M_k} q_{tk}^* - \beta q_{ik}^* = 0 \quad (24)$$

Once the merger formed, the new marginal profit, calculated at the pre-merger equilibrium is

$$\frac{d\pi_{ij}}{dq_{ik}|_{q^*}} = \alpha_k - 2c \sum_{m_l \in F_i} q_{il}^* - \beta \sum_{f_t \in M_k} q_{tk}^* - \beta q_{ik}^* - \beta q_{jk}^* < 0 \quad (25)$$

hence post-merger marginal profits from supplies to the shared markets from both of the firms are strictly negative at the pre-merger Cournot equilibrium. So firms will decrease their supply to the market $k$ and thus according to the following lemma, in the post-merger equilibrium, the total supply in the market $k$ and the total production of firms $i$ and $j$ will decrease.

**Lemma 5.3.** Consider an exogenous change in firms’s $i$ output in the market $k$, and let other links be adjusted to re-establish a Cournot equilibrium. Then the total supply in market $k$ and the total output of firm $i$ move in the same direction as change in link $(i, k)$, but by less.

**Proof.** Define $u_{f_t}$ as a $l^* \times 1$ column vector, where its $t^{th}$ element is 1 if and only if link $(i_t, k_t)$ belongs to firm $f_t$ i.e. $i_t = i$. Similarly define $u_{m_k}$ as a $l^* \times 1$ column vector whose its $k^{th}$ element is 1 if and only if link $k_t = k$. We have

$$\Psi = [I + aW]^{-1} = \left[ \frac{\beta}{2(\beta + c)} I + \frac{2c}{2(\beta + c)} \sum_{i=1}^n u_{f_i} u_{f_i}^T + \frac{\beta}{2(\beta + c)} \sum_{k=1}^m u_{m_k} u_{m_k}^T \right]^{-1} \quad (26)$$
Now if we define \( l^* \times (m + n) \), matrix, \( M \) as follow
\[
M = [u_{f_1}, \ldots, u_{f_n}, u_{m_1}, \ldots, u_{m_m}],
\]
and also a diagonal matrix \( D = \text{Diag} \left[ \sqrt{\frac{2c}{2(\beta + c)}}, \ldots, \sqrt{\frac{2c}{2(\beta + c)}}, \ldots, \sqrt{\frac{2c}{2(\beta + c)}} \right] \).

Then it is easy to see that we can rewrite the equation 26 as follow:
\[
\Psi = \left[ \frac{\beta}{2(c+\beta)} I + MD^2M^T \right]^{-1}
\]
(27)

Now the lemma basically claim that for any \( i, j \) where \( M_{ij} = 1 \) we must have \( \Psi M > 0 \). By using Woodbury matrix identity, we have:
\[
\Psi M = \left[ \frac{\beta}{2(c+\beta)} I + MD^2M^T \right]^{-1} M
\]
\[
= \left( \frac{2(\beta + c)}{\beta} I - \frac{2(\beta + c)}{\beta} M(D^{-2} + MT2(\beta + c)IM^{-1}MT2(\beta + c)I)M \right)
\]
\[
= \frac{2(\beta + c)}{\beta} IM - M\left( \frac{\beta}{2(\beta + c)} D^{-2} + MT M^{-1} MT \frac{2(\beta + c)}{\beta} M \right)
\]
\[
= \frac{2(\beta + c)}{\beta} \left( M - M\left( \frac{\beta}{2(\beta + c)} D^{-2} + MT M^{-1} \frac{\beta}{2(\beta + c)} D^{-2} + MT M \right) \right)
\]
\[
= \frac{2(\beta + c)}{\beta} \left( M - M\left( I - \frac{\beta}{2(\beta + c)} D^{-2} + MT M^{-1} \frac{\beta}{2(\beta + c)} D^{-2} + MT M \right) \right)
\]
\[
= MD^{-2} \left( \frac{\beta}{2(\beta + c)} D^{-2} + MT M \right)^{-1}
\]
(28)

So according to the above equation it is enough to show that for every entry \( i, j \) where \( M^T_{ij} = 1 \), then \( \left( \frac{\beta}{2(\beta + c)} I + D^2MT M^{-1} MT \right)_{ij} > 0 \). To prove this, note that in every column of \( MT \) there are exactly two non-zero entries. Also define matrix \( P = \left( \frac{\beta}{2(\beta + c)} I + D^2MT M^{-1} MT \right)^{-1} \). Consider the column \( k \) of \( MT \) and assume \( M^T_{ik} \) and \( M^T_{jk} \) are the two non-zero entries. So we want to show two following statements:
\[
(PMT)_{ik} = P_{ii} + P_{ij} > 0
\]
\[
(PMT)_{jk} = P_{jj} + P_{ji} > 0
\]

To show the above statement it is enough to show that for each \( i, j \) we have \( P_{ii} > |P_{ij}| \). To prove it, we use the following lemma showed in Ostrowski [1952]:

**Lemma 5.4. Ostrowski [1952]** If \( A \) is a strictly diagonally dominant matrix with non-negative entries and positive diagonal then \( B = A^{-1} \) always exists and for each \( i, j \) we have:
\[
|B_{ij}| < B_{ii}.
\]
So in order to complete the proof, it is enough to show that the matrix \( \frac{\beta}{2(\beta + c)} I + D^2 M^T M \) is strictly diagonally dominant. To see this, note that

\[
M^T M = \begin{bmatrix}
A_f & B \\
B^T & A_m
\end{bmatrix}
\]

where \( A_f \) and \( A_m \) are diagonal matrices where \( A_{fi} = \| u_{fi} \|_1 \) and \( A_{mi} = \| u_{mi} \|_1 \). Also for every \( i, j \), \( B_{ij} = u_{fi} \cdot u_{mj} \). Thus it is easy to see that for every \( i \) we have:

\[
(M^T M)_{ii} = \sum_{j \neq i} (M^T M)_{ij},
\]

and since \( D \) is a positive diagonal matrix and \( \frac{\beta}{2(\beta + c)} > 0 \), one can conclude that \( \frac{\beta}{2(\beta + c)} I + D^2 M^T M \) is strictly diagonally dominant and proof is complete.

Now consider any market that is connected to exactly one of the firms \( i \) and \( j \). Without loose of generality assume that this market is connected to firm \( f_i \). The total supply of all of this markets cannot be decreased since, the total production of firms \( f_i \) decreased in the post-merger equilibrium and if the price in the market \( l \) in the post-merger equilibrium is higher than pre-merged equilibrium then the marginal profit of merged firm \( f_i + f_j \) is strictly positive and thus the post-merger equilibrium is not stable. Thus in the post-merger equilibrium the price in every market \( l \) that is connected to one of the firm \( f_i \) and \( f_j \) is less than the price in the pre-merger equilibrium and the proof is complete.

**Proof of Proposition 4.11**

\[
\tilde{q} \geq 0
\]

For each link \((i, k) \in G\), at the profit maximizing supply \( \frac{\partial \Pi}{\partial \tilde{q}_{ik}} \big|_{\tilde{q}_{ik}} \leq 0 \). More explicitly

\[
\frac{\partial \Pi}{\partial \tilde{q}_{ik}} \big|_{\tilde{q}_{ik}} = \alpha_k - 2c \sum_{m \in F_i} \tilde{q}_{il} - 2\beta \sum_{j \in M_k} \tilde{q}_{jk} \leq 0
\]

These set of equations can be written in matrix form

\[
-\alpha + B\tilde{q} \geq 0
\]

where \( B \) is a \(|E| \times |E|\) matrix such that

\[
b_{ik, jl} = \begin{cases}
2\beta + 2c, & \text{if } i = j \text{ and } k = l \\
2c & \text{if } i = j \text{ and } k \neq l \\
2\beta & \text{if } i \neq j \text{ and } k \neq l \\
0 & \text{otherwise}
\end{cases}
\]

Lastly, for each link \((i, k) \in G\), at equilibrium \( \frac{\partial \Pi}{\partial \tilde{q}_{ik}} \big|_{\tilde{q}_{ik}} \tilde{q}_{ik} = 0 \). In matrix form

\[
(\tilde{q})^T (-\alpha + B\tilde{q}) = 0
\]

The first order profit maximizing conditions 29, 30 and 31 for the cartel constitute a \( LCP(-\alpha; B) \). We will show that the matrix, \( B \) is positive semi-definite. Hence, \( LCP(\alpha; B) \) has a solution, though not necessarily unique.

We show that for any matrix \( B \) we can find a matrix \( R \) such that \( B = R^T R \).

---

\(^{10}\)This is equivalent to checking that \( B \) is positive semi-definite.
Define \((n + m) \times |E|\) matrix \(R\) as follow:

\[
R_{v,(i_t,k_t)} = \begin{cases} 
    \sqrt{2c} & \text{if } v \leq n \text{ is a firm and } i_t = v, \\
    \sqrt{2\beta} & \text{if } v > n \text{ is a market and } k_t = v-n, \\
    0 & \text{otherwise.}
\end{cases}
\]

The columns of matrix \(R\) correspond to the links at \(E\) and the first \(n\) rows correspond to the firms, the last \(m\) rows correspond to the markets and then clearly \(B = R^T R\). Hence \(B\) is positive semi-definite matrix and \(\alpha, LCP(-\alpha; B)\) has a solution.

The Hessian matrix of \(\Pi\) is \(H_{11} = -B\). Since \(B\) is positive semi-definite, \(H_{11}\) is negative semi-definite. Meaning that any \(\hat{q}\) maximizes \(\Pi\).

**Proof of Proposition 4.12**

We know that the supply of \(\tilde{S}_0\) by each firm and the consumption of \(\tilde{C}_0\) satisfies the first order conditions in \(\tilde{G}_0\). Since \(G_0\) and \(\tilde{G}_0\) have the same set of nodes, they also satisfy the conditions in \(G_0\).

**Proof of Proposition 4.14**

By assumption, \(G_0\) has no least inclusive subgraphs.

Take a firm \(f_t\) in \(G_0\). Let \(f_t\) supply a total of \(\tilde{q}_0\), such that none of the markets consume more than \(\tilde{C}_0\), \(\tilde{S}_0\) and \(\tilde{C}_0\) are functions of the market/firm ratio. If \(f_t\) is not linked to enough markets to achieve such a supply, then firm \(f_t\) and the markets \(F_t\) form a least inclusive subgraph in \(G_0\), which is a contradiction with \(G_0\) having no least inclusive subgraphs.

Now, we are going to show by induction that \(S_0\) supply by a firm in \(G_0\) such that no market consumes more than \(\tilde{C}_0\) is possible in any inclusive subgraph of \(G_0\) that contains \(f_t\). As \(G_0\) is an inclusive subgraph of itself, this will imply that such levels of supply are possible in \(G_0\).

We know that it is possible for the inclusive subgraph with firm \(f_t\) and the markets \(F_t\). Take an inclusive subgraph \(G_{t-1}\) of \(G_0\) that contains \(t-1\) firms including \(f_t\). Suppose that such levels of supply are possible in \(G_{t-1}\). Denote by \(q_{G_{t-1}}\) such a possible amount of flows in \(G_{t-1}\).

Now take an inclusive subgraph \(G_t\) of \(G_0\) that contains \(t\) firms, \(t-1\) which were in \(G_{t-1}\) and a fixed firm \(f_t\) which was not in \(G_{t-1}\).

Assume that in \(G_t\), \(\frac{|M|}{|F_t|} < \frac{|M|}{|F|}\). Then \(G_t\) is a least inclusive subgraph of \(G_0\), which is a contradiction.

Then, \(\frac{|M|}{|F_t|} \geq \frac{|M|}{|F|}\). Take \(q_{G_{t-1}}\) such that each firm supplies \(\tilde{S}_0\) in \(G_{t-1}\). As \(G_t\) contains \(G_{t-1}\) the firms in \(G_{t-1}\) can supply \(\tilde{S}_0\) without exceeding \(\tilde{C}_0\) in any market. Now let \(f_t\) supply through its links such that the consumption at each market in \(F_t\) is \(\tilde{C}_0\). If the total supply of \(f_t\) is at least \(\tilde{S}_0\), then we are done.

If not, denote by \(q^1\) the flow vector for \(G_t\) such that flows for the links which were already in \(G_{t-1}\) equals to \(q_{G_{t-1}}\), and the flows for the links which were not in \(G_{t-1}\) equals to 0. Now, given that \(f_t \notin F_{t-1}\), let\(^{11}\) \(q^2\) be the flow vector for \(G_t\) such that

\[
q^2_{ik} = \tilde{C}_0 - q^1_k, \text{ for } m_k \in F_t \\
q^2_{ik} = q^1_k, \text{ for } i \neq t
\]

\(^{11}\)The subscripts will be used as indices. Hence, for market \(m_k\), \(q^1_k\) will denote its outflow at the vector \(q^1\).

EC’14, June 8–12, 2014, Stanford University, Palo Alto, CA, USA, Vol. , No. , Article 1, Publication date: February 2014.
Since \( \frac{|M_t|}{|F_t|} \geq \frac{|M|}{|F_t|} \), there must be a market \( m_k \) in \( G_t \) not connected to \( f_t \), such that its consumption in \( q^2 \) is strictly less than \( \hat{C}_0 \). Let \( M^\leq_t \) be the set of markets in \( G_t \) which are not connected to \( f_t \) and which have consumption in \( q^2 \) strictly less than \( \hat{C}_0 \).

\[
M^\leq_t = \{ m_k \in M_t : m_k \notin F_t \text{ and } q^2_k < \hat{C}_0 \}
\]

Suppose that for any market \( m_k \in M^\leq_t \) and for all paths

\[
P = \{(m_k, f_1), (f_1, m_1), ..., (f_b, m_b), (m_b, f_t)\}
\]

that connects \( m_k \) with \( f_t \), there exists \( (f_j, m_j) \in P \) such that \( q^2_{jj} = 0 \). Given such a path \( P \), let \( m_P \) denote the market \( m_l \) such that \( (f_l, m_l) \in P \), \( q^2_{ll} = 0 \) and there exists no other market \( m_k \) in \( P \), closer to \( f_t \) than \( m_l \) such that \( (f_k, m_k) \in P \) and \( q^2_{kk} = 0 \). Let \( F_t = \{ f_j \in F_t : \text{there exists a path } P \text{ from } m_k \text{ to } f_t \text{ for some } m_k \in M^\leq_t \} \) and in \( P \), \( f_i \) is between \( m_P \) and \( f_t \). Then the inclusive subgraph with firms \( F_t \cup f_t \) is least inclusive in \( G_t \), which is a contradiction.

Then there exists a market \( m_k \in M^\leq_t \) such that there exists a path

\[
P = \{(m_k, f_1), (f_1, m_1), ..., (f_b, m_b), (m_b, f_t)\}
\]

that connects \( m_k \) with \( f_t \) and \( \min_{(f_j, m_j) \in P} q^2_{jj} \neq 0 \). Let

\[
d = \min_{(f_j, m_j) \in P} \{q^2_{jj}, q^2_k\}
\]

Now, given such a path \( P \), let \( q^3 \) be the flow vector for \( G_t \) such that

\[
q^3_{1k} = q^2_{1k} + d,
q^3_{jj} = q^2_{jj} - d,
q^3_{(j+1)j} = q^2_{(j+1)j} + d
q^3_{tk} = q^2_{tk} + d
q^3_{ll'} = q^2_{ll'}, \text{ for all other links } (l, l')
\]

It is possible to make \( f_t \) supply at least \( \tilde{S}_0 \) by finding such paths from markets in \( M^\leq_t \) to \( f_t \) and changing the flows as explained above for each path from a market in \( M^\leq_t \) to \( f_t \). If after using all such paths, \( f_t \) could still not supply \( \tilde{S}_0 \), then \( G_t \) is a least inclusive subgraph in \( G_0 \), a contradiction.

Then the desired levels of supply are possible in \( G_0 \).